On the logic resolution of the wave particle duality paradox in quantum mechanics (Extended abstract)

M.A. Nait Abdallah

On the logic resolution of the wave particle duality paradox in quantum mechanics (Extended abstract)

M.A. Nait Abdallah

UWO, London, Canada and INRIA, Paris, France

Abstract. In this paper, we consider three landmark experiments of quantum physics and discuss the related wave particle duality paradox. We present a formalization, in terms of formal logic, of single photon self-interference. We show that the wave particle duality paradox appears, from the logic point of view, in the form of two distinct fallacies: the *hard information fallacy* and the *exhaustive disjunction fallacy.* We show that each fallacy points out some fundamental aspect of quantum physical systems and we present a logic solution to the paradox.

1 Introduction

It is a common view among physicists to see Quantum Mechanics simply as a calculation method which yields results that are in astonishing accordance with all experimental observations performed so far. Under such a view, the mathematical theory of Quantum Mechanics does not operate on mathematical objects directly representing quantum physical objects, but rather on probabilities yielding useful pragmatic results.

The question arises as to whether a more analytic mathematical approach to the study of quantum mechanics would be of some use, or even at all possible. In this paper, we explore, from the point of view of mathematics, the logical content of three major experiments due to Grangier, Roger and Aspect [11] and Scully, Englert and Walther [14], and draw from this analysis a logic-based solution to the wave particle duality paradox in Quantum Mechanics.

1.1 Three quantum physics experiments

In reference [11] the authors describe two experiments that indisputably show that the light quantum, or *photon,* is both a wave and a particle. Such a simultaneous double identity is very difficult to grasp within a classical logic conceptual framework, where it indeed leads to a paradox.

The setup of Grangier *et al.* experiment #1 (Figure 1) has a light source that emits single photons, one by one, well-separated in time, and sends them through a beam splitter (half-silvered mirror). The emitted photons all have exactly the same physical properties. The intensity of light (luminous flow) thus emitted is evenly split by the half-slivered mirror, between a vertically reflected portion and a horizontally transmitted portion. One positions a detector on each of the two exit channels, as well as a joint detection device at the end of both channels.

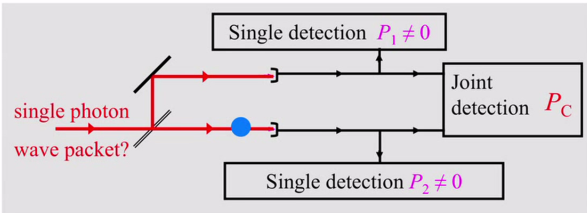

Fig. 1. Grangier *et al.* First experiment

Experiment # 2 setup (Figure 2) recombines both beams of light with a second beam splitter, thus yielding a Mach-Zehnder interferometer, where M_1, M_2

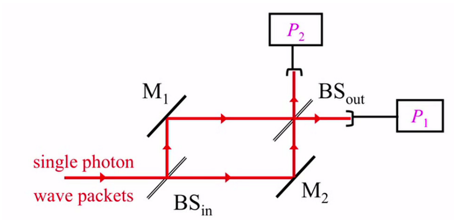

Fig. 2. Grangier *et al.* Second experiment

are mirrors and BS_{in}, B_{out} are beam splitters. Again, the intensity of light traveling the lower (resp. upper) path into the second beam splitter BS_{out} is evenly split by the half-slivered mirror, between a horizontally reflected (resp. transmitted) portion and a vertically transmitted (resp. reflected) portion.

The first experiment (Figure 1) shows that every single photon is detected by a single detector on a single path. The photons are not split on a beam splitter, each photon is insecable and can take only one of either path. Whence each single photon obviously behaves as a particle.

In the second experiment (Figure 2), every single photon is detected as a single particle by a single end detector. However, every photon registers at horizontal end detector C, and none registers at vertical end detector D, thus displaying an interference fringe. Thus, light emitted by the single photon source, at discrete points in time, behaves like a wave. The electromagnetic field is coherently split on a beam splitter, where "coherent" means that the two subwaves

(Huygens principle) have a constant phase difference (equal to π in this case.) One observes an interference fringe *i.e.,* the photon behaves like a wave.

Since at each beam splitter the photon has an equal chance to be reflected or transmitted, it should randomly register with equal probability at one end detector or the other. On the contrary, Grangier *et al.* experiment #2 demonstrates that probabilities associated with each path do not add up, all photons go to one detector only. From the point of view of probabilities, the quantum phenomenon does not decompose into a statistical distribution in a space of classical phenomena as in *e.g.,* thermodynamics.

Thus, the very same photon, upon traversing the first beam-splitter, behaves as a particle or as a wave, depending on which devices are to be found long *after* it has traveled through this beam splitter. How can the single photon be both split (*i.e.,* a wave) and insecable (*i.e.,* a particle) at the same time? This seemingly contradictory situation is the Wave Particle Duality Paradox and is one of the fundamental problems of Quantum Mechanics.

One problem with the photon is that as a light quantum it vanishes upon detection. An third experiment, clarifying this matter, is Scully *et al.* experiment [14] which, instead of photons, uses excited atoms as particles. A microwave cavity, finely tuned on an atomic transition, is positioned on each channel of a Young slit interference experimental device. Atoms are first brought to an excited state using a laser beam, and then sent through the setup. Upon traveling through the cavities, the excited atom returns to its ground state, emitting one single photon. That photon remains trapped in the cavity. The presence of the photon in one of the cavities characterizes the path taken by the atom, thus providing which-way information and at the same time erasing the interference fringes *i.e.,* making the wave-like behaviour of the atom disappear. Scully *et al.* experiment shows that the wave behaviour and the particle behaviour of the atom cannot be observed simultaneously: the physical availability of which-path information (particle behaviour) and the occurrence of interference (wave behaviour) are mutually exclusive.

According to Feynman [10], p. 1-1, the wave particle duality phenomenon "*has in it the heart of quantum mechanics; in reality, it contains the only mystery*" of the theory "*which cannot be explained in any classical way.*"

1.2 Notations

Before we get to the heart of the matter, let us define some notations. The basic experimental setup of Grangier *et al.* experiment #2 is a Mach-Zehnder interferometer. It is constituted (Figure 3) of a single photon light source e, two half-silvered mirrors (beam splitters) B_1, B_2, two mirrors M_1, M_2, and two detectors C, D positioned on a rectangle as shown. The mirrors and beam splitters are oriented diagonally, the first beam splitter silvered face upward, the second beam splitter face down.

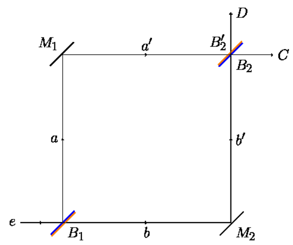

Fig. 3. Mach-Zehnder interferometer

Each photon emitted at source e registers either at detector C or at detector D. Experiment shows that detection frequencies P_C and P_D are given by

$$P_C = 1 \qquad\qquad P_D = 0 \tag{1}$$

Every photon emitted at e registers at detector C (constructive interference) and none at D (destructive interference).

Define propositional formulae:
$a :=$ "the photon takes the upper path a,"
$b :=$ "the photon takes the lower path b,"
$i :=$ "there is interference of probability waves."
Such definitions implicitly assume that formulae $a, b, i, \ldots$ *i.e.,* phrases such as "the photon takes the upper path" have a clear and unambiguous meaning. Given the propositional nature of the alphabet $\{a, b, i, \ldots\}$ thus chosen, we shall use propositional logic throughout this paper.

Using these definitions, the experimental results [11, 14] reviewed above read as follows. (For reasons that will become clear later, the corresponding propositional formulae are labeled using object variables belonging to some set $V = \{z, t, f, g, \ldots\}$.)

– *Grangier* et al. *experiment #1:* Upon leaving the first beam splitter, the photon traverses the a path or the b path

$$z : a \vee b \tag{2}$$

and there is never joint detection.
In (2) expression $z : a \vee b$ means "z is a label of formula $a \vee b$." In that expression, $z \in V$ is a variable used as a label for bookkeeping purposes, colon ":" is the labeling operation, and a, b are defined above. Labeling conventions are the same in (3) through (5) below. A deeper meaning of this seemingly innocuous labeling device will be revealed by Curry-Howard correspondence in Section 2.2, p. 10, where $z : a \vee b$ will become an "inhabitation claim" and z an "inhabitant" of formula $a \vee b$. For the time being, we can safely ignore this, and see it as a simple bookkeeping notational convention.

– *Grangier* et al. *experiment #2:* There is interference

$$t : i \tag{3}$$

where variable $t \in V$ is a label for i, and i is defined above.

– *Scully* et al. *experiment:* There is interference if and only if no which-way information is physically available

$$f : a \to \neg i \qquad\qquad g : b \to \neg i \tag{4}$$

To these we add the obvious observation that absence of light (photon) on both channels implies absence of interference, since there is nothing to interfere with

$$h : \neg a \to \neg b \to \neg i. \tag{5}$$

Each of properties (2) through (4) is an experimental fact that holds true of the photon; property (5) is commonsense: if there is no light, there is no interference.

2 The wave particle duality paradox in Quantum Mechanics

2.1 Paradox

A naive understanding of the light quantum leads one to assume that the photon simultaneously verifies all properties (2) through (4) revealed by experiment, as well as property (5). The ensuing logic paradox, first outlined in Section 1.1, comes in the form of two distinct fallacies which we call the *hard information fallacy* and the *exhaustive disjunction fallacy.* (A fallacy is an incorrect result coupled with an apparently logical explanation of why the result is correct.) As we shall see, each points out some fundamental aspect of quantum physical systems, and this will explain our terminology.

The hard information fallacy Scully *et al.* experiment [14] establishes that the physical availability of which-path information and the occurrence of interference are mutually exclusive. This implies that if the photon is detected on channel a, then one observes no interference. Similarly for b. Hence, by contraposition, in the presence of interference, the photon determinately traverses neither channel a nor b. However, in the absence of photon on both channels, there is no interference. Let us assume that there is interference. By the first implication, one deduces that the photon traverses neither channel; by the second implication one concludes that there is no interference. By *reductio ad absurdum,* we conclude that there is no interference. This logical conclusion is refuted by Grangier *et al.* experiment #2.

The exhaustive disjunction fallacy Grangier *et al.* experiment #1 establishes that the photon is a particle taking the a path or the b path. Scully *et al.* experiment establishes that if the photon takes either path, then there is no interference. By reasoning by cases, one concludes that no interference takes place. This logical conclusion is refuted by Grangier *et al.* experiment #2.

2.2 Ways out of paradox

The wave particle duality paradox has aroused many responses, ranging from philosophy to physics. An obviously adequate solution is the one offered by the Quantum Mechanics mathematical formalism itself, but which makes little logical sense, as far as classical logic is concerned, suggesting, given other "strange properties" of the quantum physical world, that understanding quantum mechanics might be beyond the ability of the human mind. R. Feynman writes [9], p. 129

> *I think I can safely say that nobody understands quantum mechanics.*

Physics based solutions Major solution proposals to the wave particle duality paradox include Bohr's complementarity principle, de Broglie's pilot wave model in Bohm's theory [3] and Everett's relative-state presentation of Quantum Mechanics [7]. Bohr's complementarity principle states that some questions cannot be asked simultaneously; thus here one must choose the question to be asked: Which path does the photon travel? Is there interference? Both are relevant to the physical reality of the quantum physical system considered, but they are mutually exclusive.

Bohr's solution is part of the Copenhagen interpretation of Quantum Mechanics and is the most widely accepted one among practicing physicists.

The meta-physical approach The question arises: Which fundamental mathematical properties of the quantum physical world are pointed out by the wave particle duality paradox? In this work, we aim at exploring an alternative solution to the paradox, based on mathematical logic. To this end, we draw on two theoretical tools: (*i*) Wheeler's dichotomy between phenomenon and phenomenon-to-be [16, 17] and (*ii*) the logic of partial information [1].

(One observes in passing that probability theory, which constitutes a significant part of Quantum Mechanics, is silent about the hard information fallacy, and the exhaustive disjunction fallacy.)

Hard vs. *soft information in logic* The logic partial information is an extension of classical logic, inspired from the philosophical logic of scientific discovery of Popper and Lakatos [13], for the purpose of reasoning with partial and tentative information, and geared towards the needs of practicing computer scientists. In contrast with classical logic, which reasons on the basis of total, certain information, the logic of partial information separates knowledge in two broad categories, hard knowledge and soft knowledge. Hard knowledge corresponds to total, certain information, whereas soft knowledge statements correspond to plausible, hypothetical information *awaiting confirmation.* (An ansatz is an instance of soft knowledge in physics methodology.) Available and non-available information are both relevant in the reasoning process in that non-available information has a direct, "positive" impact on the conclusions that can be drawn. Previously established soft theorems may be withdrawn in the face of new hard information to the contrary. As an example, if Tweety is a bird and birds fly, then one concludes that Tweety flies. However, upon learning that Tweety is a penguin and penguins don't fly, one withdraws the conclusion that Tweety flies. Thus the hard *vs.* soft dichotomy is actually a hierarchy where soft knowledge yields to hard knowledge.

Phenomenon vs. *phenomenon-to-be in physics* In Quantum Mechanics, experiments conducted by Scully *et al.* and other experiments show that whether some information is physically available has a direct impact on which physical processes take place, *e.g.,* which-way information precludes interference. Physically available and non-available information are both relevant to the reasoning and physical process. Along the same line of thought, Bohr [4, 5] and Wheeler [16] establish a sharp distinction between phenomena (*i.e.,* observed phenomena) and phenomena-to-be. According to Wheeler [16] p. 189, 202:

> *Until the act of detection the phenomenon-to-be is not yet a phenomenon . . . No elementary phenomenon is a phenomenon until it is an observed phenomenon.*

Also, in Wheeler [17]:

> *No elementary quantum phenomenon is a phenomenon until, as Bohr puts it [4], "It has been brought to a close" by "an irreversible act of amplification." . . .*
> *We know perfectly well that the photon existed neither before the emission nor after the detection. However, we also have to recognize that any talk of the photon "existing" during the intermediate period is only a blown-up version of the raw fact, a count.*

Quantum mechanics vs. *logic of partial information* Each of the two theories "quantum mechanics" and "logic of partial information" has two levels distinguished by "quality of information":

1. in quantum mechanics: phenomenon *vs.* phenomenon-to-be,
2. in the logic of partial information: hard knowledge *vs.* soft knowledge.

For each level in both theories, we see here a formal analogy:

(*i*) between hard knowledge in the logic of partial information, and phenomena (*i.e.,* observed phenomena) in the sense of Bohr and Wheeler,
and
(*ii*) between soft knowledge and its treatment in the sense of the logic of partial information on one hand, and phenomena-to-be in the sense of Wheeler and their functioning in physics on the other.

Just as in the logic of partial information, a phenomenon-to-be in the sense of Wheeler might be seen as a phenomenon *awaiting confirmation* in the sense of the logic of partial information. If one were to equate "knowledge" and "hard knowledge” in a way similar to Bohr’s and Wheeler’s equation of “phenomenon” and “observed phenomenon,” then just as in quantum mechanics, soft knowledge might be seen as *knowledge-to-be* in the sense of Wheeler.

To further explore this analogy, and make it into a correspondence

(observed) phenomenon as hard knowledge

and

phenomenon-to-be as soft knowledge

between quantum mechanics and the logic of partial information, as in Figure 4, an algebraic tool for tracking information is needed. In formal logic,

$$\begin{tikzcd}
 & \text{phenomenon} \arrow[r, dash] & \text{hard knowledge} & \\
\text{QM} & & & \text{LPI} \\
 & \text{phenomenon-to-be} \arrow[r, dash] \arrow[uu] & \text{soft knowledge} \arrow[uu] &
\end{tikzcd}$$

Fig. 4. Correspondence between Quantum Mechanics and the Logic of Partial Information

Curry-Howard correspondence [6, 12] offers a means to track proof-theoretic information in the reasoning process. This means is λ-calculus [15]. The question then arises: using Curry-Howard correspondence and the logic of partial information, can we formalize and/or learn more about fundamental concepts of Quantum Mechanics?

In this paper, we consider a restricted form of this question, namely single photon self-interference and the wave particle duality paradox in quantum mechanics. More precisely, we provide a formal logic description and analysis of a specific physical phenomenon, the fact that a quantum particle can interfere with itself. Such an approach is akin to Boolean logic providing a description and analysis of an electric circuit. As a consequence, we obtain a logic solution to the wave particle duality paradox.

For this purpose, we proceed as follows. The articulation point between Bohr’s and Wheeler’s phenomenon dichotomy and the logic of partial information is a generalization of Curry-Howard correspondence which will allow us to express quantum physics processes as programs (in the sense of computer science) written in a generalized λ-calculus which we now outline. In some sense, the general idea here is that of an attempt at providing an abstraction of Quantum Mechanics in terms of a (generalized) λ-calculus, in the same sense as, in computer science, λ-calculus provides a complete abstraction of actual computer programs.

λ-calculus is a universal programming language (*i.e.,* one which has an expressive power equivalent to that of Turing machines) with the simplest possible syntax. Its programs, called λ-terms, are defined as follows. Given a set $V = \{x, y, \ldots\}$ of variables,

1. any variable $x \in V$ is a λ-term.
2. application rule: If M, N are λ-terms, then so is (MN).
3. abstraction rule: If $x \in V$ is a variable, and M is a λ-term, then $(\lambda x.M)$ is a λ-term.

Intuitively, application (MN) designates the result of applying function M to argument N, and λ-abstraction $(\lambda x.M)$ designates function $x \mapsto M$ taking x as an argument and returning M as a result. This definition of λ-terms may be abbreviated using the following grammar

$$\begin{aligned} \Lambda &::= V \mid (\Lambda\Lambda) \mid (\lambda V.\Lambda) \\ V &::= x \mid y \mid \ldots \end{aligned} \tag{6}$$

This grammar has two rules, each rule defining a subset of λ-terms. In each rule, relation ::= indicates that the lefthand side, the *definiendum,* is defined in terms of the righthand side, the *definiens.* The *definiendum* is Λ in the first rule, and V in the second rule. Vertical stroke | indicates alternative choices. The first rule reads as follows: a Λ (*i.e.,* a λ-term) is defined as being either a V (*i.e.,* a variable) or a $(\Lambda\Lambda)$ (*i.e.,* a λ-term M applied to a λ-term N), or a $(\lambda V.\Lambda)$ (*i.e.,* is some $(\lambda x.M)$ obtained by abstracting variable $x \in V$ in some λ-term M). The second rule reads: a V (*i.e.,* a variable) is either x or y or *etc.*

Even though language Λ thus defined is universal, to facilitate expression in this "assembly language of formal thought," one introduces some higher level constructs (data and control structures), namely: pairing $\langle M, N\rangle$, projections $\pi_i(M)$, $i = 1, 2$, injections $\mathrm{in}_i(M)$, $i = 1, 2$ and case analysis $\mathrm{case}(L\ ;\ M\ ;\ N)$. This yields the enriched language of λ-terms defined by modified grammar

$$\begin{aligned} \Lambda ::= {} & V \mid (\lambda V.\Lambda) \mid (\Lambda\Lambda) \mid \langle\Lambda,\Lambda\rangle \mid \pi_1(\Lambda) \mid \pi_2(\Lambda) \\ & \mid \mathrm{in}_1(\Lambda) \mid \mathrm{in}_2(\Lambda) \mid \mathrm{case}(\Lambda\ ;\ \lambda V.\Lambda\ ;\ \lambda V.\Lambda) \\ V ::= {} & x \mid y \mid \ldots \end{aligned} \tag{7}$$

λ-terms may be typed, just as expressions and functions in Fortran are typed (*e.g.,* as in type declaration "complex: x" which in Fortran 77 declares variable x to be of type complex.)

Our set of logic formulae will be the customary set of propositional formulae, defined by grammar

$$\mathcal{F} ::= P \mid \mathsf{F} \mid (\mathcal{F} \wedge \mathcal{F}) \mid (\mathcal{F} \vee \mathcal{F}) \mid (\mathcal{F} \to \mathcal{F}) \tag{8}$$

where P is the set of propositional variables. We code negation of a formula φ by implication $\varphi \to \mathsf{F}$, where F stands for "falsehood"; $\wedge$, $\vee$, $\to$ are the common logic connectives.

A central connection between λ-terms and (propositional) formulae is provided by Curry-Howard correspondence [6, 12, 15]. Curry-Howard correspondence establishes a one-to-one proofs-as-terms/formulae-as-types correspondence between formal proofs in logic using natural deduction of some formula φ, and typed λ-terms of the type corresponding to formula φ. The suitable type structure is given here by

$$\mathcal{T} ::= P \mid \mathbf{empty} \mid (\mathcal{T} \times \mathcal{T}) \mid (\mathcal{T} + \mathcal{T}) \mid (\mathcal{T} \to \mathcal{T}) \tag{9}$$

where P is the set of base types (*e.g.,* integer, real in Fortran), and $\times$ (cross product), $+$ (direct sum of types) and $\to$ (function type) are the customary operations on types. One sees that each formula may be read as a type, and conversely. In case some term t is of some type T (or alternatively term t codes a proof of formula T), we write $t : T$ and we say that t is an *inhabitant* of T, and we call the expression $t : T$ an *inhabitation claim.* Such a notation was informally introduced in Section 1.2 above. For example in (5), inhabitation claim

$$h : \neg a \to \neg b \to \neg i$$

may be read as a curried (in the sense of Currying) version of a two-argument function h' of type $(\neg a) \times (\neg b) \to (\neg i)$:

$$h' : (\neg a) \times (\neg b) \to (\neg i), \quad (u, v) \mapsto h'(u, v)$$

taking an element (proof term) u of type $\neg a$ and an element (proof term) v of type $\neg b$ and yielding result $h'(u, v) = (hu)v$ of type $\neg i$, *i.e.,* a proof term inhabiting $\neg i$.

In our particular case, we show that the wave particle duality paradox corresponds to the following two distinct typed λ-terms (10), (12) proving that no interference takes place. These λ-terms correspond to the two fallacies (hard information and exhaustive disjunction) already discussed in Section 2.1. First we have the *hard information fallacy*

$$h(\lambda x.fxt)(\lambda y.gyt) : \neg i \tag{10}$$

which says that no interference takes place, and can be re-expressed as

$$h(\lambda x.fxt)(\lambda y.gyt)t : \mathsf{F} \tag{11}$$

Second, we have

$$\mathrm{case}(z\ ;\ \lambda x.fx\ ;\ \lambda y.gy) : \neg i \tag{12}$$

corresponding to the *exhaustive disjunction fallacy,* which can be re-expressed as

$$(\mathrm{case}(z\ ;\ \lambda x.fx\ ;\ \lambda y.gy))t : \mathsf{F} \tag{13}$$

Conclusion $\neg i$ in the two inhabitation claims (10) and (12), stating that there is no interference, is refuted by Grangier *et al.* experiment #2, whose corresponding inhabitation claim is $t : i$. This refutation is at the heart of the wave particle duality paradox.

3 Formalism of quantum logic of partial information

In the face of this wave particle duality paradox, something has got to change. What can be done in logic? Following the direction suggested by Bohr and Wheeler, and the logic of partial information (Figure 4), we proceed as follows. We first enrich the set of formulae, the set of λ-terms, and we lay down tools for mirroring Feynman's path integral [8, 10] and the no-joint-detection property demonstrated by Grangier *et al.* experiment #1. We then analyze the three experiments in the next Section. We finally show how to eliminate the fallacies.

3.1 Logic formulae and soft knowledge

First, to express the notion of "property-to-be," we extend our set of logic formulae with a new unary connective denoted by $\star$ (star) :

If φ is a formula, then so is $\varphi^\star$.

where the intended meaning of $\varphi^\star$ is "*It is not physically excluded that φ be the case, awaiting confirmation.*" Whence grammar

$$\mathcal{F}_\star ::= P \mid \mathsf{F} \mid (\mathcal{F}_\star \wedge \mathcal{F}_\star) \mid (\mathcal{F}_\star \vee \mathcal{F}_\star) \mid (\mathcal{F}_\star \rightarrow \mathcal{F}_\star) \mid (\mathcal{F}_\star)^\star \tag{14}$$

defining the new set of propositional formulae replacing $\mathcal{F}$.

3.2 λ-terms and soft information

By definition, star formulae $\varphi^\star$ are tentative hypotheses conveying only soft, defeasible information; as a result, the existence of the corresponding proof λ-terms is far from assured, and such terms may not exist at all. We therefore introduce an additional set Ξ of parameters to serve as place-holders for possible, albeit not sure inhabitants of such formulae $\varphi^\star$. As in the logic of partial information, the value of such parameters will depend on the context in which the reasoning takes place. Intuitively, parameters $\xi \in \Xi$ may be seen as meta-variables whose possible values range over the set of ordinary, "hard" λ-terms. This addition step means extending the grammar definition of Λ as follows:

$$\begin{aligned} \Lambda &::= V \mid \Xi \mid (\lambda V.\Lambda) \mid (\Lambda\Lambda) \mid \langle \Lambda, \Lambda \rangle \mid \pi_1(\Lambda) \mid \pi_2(\Lambda) \\ &\qquad \mid \mathrm{in}_1(\Lambda) \mid \mathrm{in}_2(\Lambda) \mid \mathrm{case}(\Lambda\ ;\ \lambda V.\Lambda\ ;\ \lambda V.\Lambda) \\ V &::= x \mid y \mid \ldots \\ \Xi &::= \xi \mid \rho \mid \ldots \end{aligned} \tag{15}$$

Observe that, in contrast with variables $\in V$, parameters $\in \Xi$ are never abstracted over. They are always used as place-holders for missing information, never for transmitting arguments of functions.

Elements of Λ containing no occurrences of parameters $\in \Xi$ are called *hard terms,* and correspond to hard, certain information. The other elements correspond to soft information *i.e.,* in a Wheeler's style terminology, to (hard information)-to-be, and are called *soft terms.*

To tackle the Quantum Mechanics wave particle duality paradox, such an extension of Λ is not sufficient, however, due to the interference phenomenon pointed out by Grangier *et al.* experiment #2. We cannot express interference using $\mathcal{F}_\star$ and Λ thus extended.

In physical optics, to account for light interference, one records the phase changes as light propagates. In the case of Mach-Zehnder interferometer, according to Fresnel rules, phase changes occur when light is reflected by the metallized face of a mirror or beam splitter. The corresponding change is always equal to $+\pi$ *i.e.,* corresponds to multiplying the current amplitude by $e^{i\pi} = -1$. To simplify the exposition, since distances traveled on both paths of our rectangle (Figure 3) are equal and introduce no optical path difference, we shall, without loss of generality, ignore phase shifts undergone while the photon travels through physical space.

In the current setting, we formalize light propagation as proof steps in natural deduction. Whence, from the point of view of physics, when recording such proof steps, one must take into account the phase shifts undergone by the light beam. Since we are using Curry-Howard correspondence to record proofs as λ-terms, the λ-terms used in natural deduction need to be generalized by the introduction of phase shifts [2]. In the case considered here, it is enough to observe that all phase changes are equal to π, whence correspond to a "multiplication by -1" of the physical amplitude. No other phase shift values need to be considered for the solution of the logic problem considered here.

This leads to the introduction of signed λ-terms, which are defined as follows. A *atomic signed λ-term* is either a λ-term or a λ-term affected by a minus sign, and one closes the set of signed λ-terms under the usual grammatical constructions.

Putting all these elements together yields the following definition of *signed partial information λ-terms for quantum mechanics:*

$$\begin{aligned}
\Lambda_s &::= \Lambda \mid -\Lambda \\
&\qquad \mid (\lambda V.\Lambda_s) \mid (\Lambda_s \Lambda_s) \mid \langle \Lambda_s, \Lambda_s \rangle \mid \pi_1(\Lambda_s) \mid \pi_2(\Lambda_s) \\
&\qquad \mid \mathrm{in}_1(\Lambda_s) \mid \mathrm{in}_2(\Lambda_s) \mid \mathrm{case}(\Lambda_s \; ; \; \lambda V.\Lambda_s \; ; \; \lambda V.\Lambda_s) \\
\Lambda &::= V \mid \Xi \mid (\lambda V.\Lambda) \mid (\Lambda\Lambda) \mid \langle \Lambda, \Lambda \rangle \mid \pi_1(\Lambda) \mid \pi_2(\Lambda) \\
&\qquad \mid \mathrm{in}_1(\Lambda) \mid \mathrm{in}_2(\Lambda) \mid \mathrm{case}(\Lambda \; ; \; \lambda V.\Lambda \; ; \; \lambda V.\Lambda) \\
V &::= x \mid y \mid \ldots \\
\Xi &::= \xi \mid \rho \mid \ldots
\end{aligned} \tag{16}$$

Again, elements of Λ_s containing no occurrences of parameters $\in \Xi$ are called *hard terms,* and correspond to certain information. The other elements correspond to soft information and are called *soft terms.*

The rules of natural deduction are generalized accordingly and in addition, for dealing with phases, one has equational rules on terms:

$$(-M)N = -(MN), \qquad (M(-N)) = -(MN), \qquad (-M)(-N) = (MN) \tag{17}$$

3.3 Integral proofs

In Feynman's presentation of quantum mechanics [8], the probability amplitude to go to a given state is equal to the sum of all the amplitudes for all the evolution paths leading to that state.

Let us define a *context* Γ as being a finite collection of

(i) variable inhabitation claims of the form $x : \varphi$, where x is a possibly signed variable $\in V$ and $\varphi \in \mathcal{F}_\star$, providing the available hard information for the problem at hand, together with

(ii) corresponding auxiliary tentative inhabitation claims $\xi : \psi^\star$, where $\xi \in \Xi$, for star formulae $\psi^\star \in \mathcal{F}_\star$ providing soft information.

We define the integral proof of a formula φ as being the formal sum of all (signed partial information) λ-terms proving φ, namely, s is the integral proof of φ in context Γ, written

$$\Gamma \Vdash s : \varphi$$

if and only if s is the formal sum of all terms t such that t is a proof of φ in context Γ, *i.e.,*

$$s = \sum\{t \mid \Gamma \vdash t : \varphi\}.$$

We denote by 0 the empty formal sum.

3.4 Orthogonality

Finally, given a quantum physical system, let Γ be the context providing the available information about that system.

By definition, we say that two formulae φ_1, φ_2 are orthogonal in context Γ if and only if for any variables $u, v \in V$ not occurring in Γ the following integral judgments hold

$$\Gamma, u : \varphi_1 \Vdash 0 : \varphi_2 \qquad\qquad \Gamma, v : \varphi_2 \Vdash 0 : \varphi_1 \tag{18}$$

in addition to those obtained through natural deduction, where $\Gamma, u : \varphi_1$ is context Γ augemented with inhabitation claim $u : \varphi_1$, *et cetera.*

From the point of view of physics, the first judgment in this definition means that *e.g.,* whenever there is definite information u (obtained *e.g.,* via some observation) that φ_1 is the case, then φ_2 is not being observed, since the the sum of its inhabitants is zero. In other words, intuitively, "$\Gamma, u : \varphi_1 \Vdash 0 : \varphi_2$" says that whenever φ_1 has a hard inhabitant *i.e.,* φ_1 is a phenomenon, this implies that the integral proof supporting (orthogonal) claim φ_2 is vacuous, which means that φ_2 is neither a phenomenon nor a phenomenon-to-be, it is never observed *i.e.,* φ_2 is just not there. Similarly for the second judgment.

We are now in a position to summarize our main results. They consist of formal proofs of the experimentally observed properties of the Mach-Zehnder interferometer (Section 1.1), and of a logic solution to the wave particle duality paradox in quantum mechanics. We present these in turn.

4 A logical analysis of the experiments

The Mach-Zehnder interferometer setup (Section 1.1) may be formalized as follows. Using the proposition notation given earlier (Section 1.2) the corresponding context Γ is given by the following inhabitation claims (clauses):

$$\begin{array}{ll} x : e & \\ -P : e \to (e \to a)^\star \to a & Q : e \to (e \to b)^\star \to b \\ -J : a \to a' & -J' : b \to b' \\ -P' : b' \to (b' \to c)^\star \to c & Q' : b' \to (b' \to d)^\star \to d \\ P'' : a' \to (a' \to d)^\star \to d & Q'' : a' \to (a' \to c)^\star \to c \end{array} \tag{19}$$

together with the following tentative inhabitation claims, where parameters ρ (resp. τ) stand for reflection (resp. transmission)

$$\begin{array}{lll} \rho_1 : (e \to a)^\star & \rho_2 : (b' \to c)^\star & \rho_2' : (a' \to d)^\star \\ \tau_1 : (e \to b)^\star & \tau_2 : (b' \to d)^\star & \tau_2' : (a' \to c)^\star \end{array}$$

In addition to context Γ, we add the requirement that formulae a, b be orthogonal. This is to express the "no joint detection rule" from Grangier *et al.* experiment #1. For our current purpose, this (incomplete) formalization will be sufficient.

Context Γ yields the following two integral inhabitation claims

$$\begin{array}{l} \Gamma \Vdash Q''(J(Px\rho_1))\tau_2' + P'(J'(Qx\tau_1))\rho_2 : c \\ \Gamma \Vdash P''(J(Px\rho_1))\rho_2' + (-Q'(J'(Qx\tau_1))\tau_2) : d \end{array} \tag{20}$$

which give a logic formalization of the transit of the photon through the interferometer, given emission $x : e$.

Using notation convention $M - N := M + (-N)$, this simplifies into

$$\begin{array}{l} \Gamma \Vdash Q''(J(Px\rho_1))\tau_2' + P'(J'(Qx\tau_1))\rho_2 : c \\ \Gamma \Vdash P''(J(Px\rho_1))\rho_2' - Q'(J'(Qx\tau_1))\tau_2 : d \end{array} \tag{21}$$

We need to define what it means to have interference fringes in the current setting. We first define a standard interpretation I from signed λ-terms to real numbers and real mappings of real variables; this definition follows from the intended interpretation of our extended λ-calculus.

For any context Γ' containing context Γ defined in (19), we say that a pair of integral inhabitation claims $\begin{pmatrix} \Gamma' \Vdash t_1 : c \\ \Gamma' \Vdash t_2 : d \end{pmatrix}$ *defines an interference pattern* under interpretation I with domain set $\mathbb{R}$ of real numbers, if and only if $I(t_1) \neq 0$ and $I(t_2) = 0$.

One then proves the following theorems.

Theorem 1. ***(Interference)*** *Interference occurs in Mach-Zehnder interferometer* i.e., *pair of integral inhabitation claims (21)*

$$\Gamma \Vdash Q''(J(Px\rho_1))\tau_2' + P'(J'(Qx\tau_1))\rho_2 : c$$
$$\Gamma \Vdash P''(J(Px\rho_1))\rho_2' - Q'(J'(Qx\tau_1))\tau_2 : d$$

defines an interference pattern under the standard interpretation, where Γ is the context defined above (19). □

Theorem 2. ***(No joint detection)*** *Upon being detected along one of channels a, b, the photon is an insecable particle which is never detected on the other channel.* □

Theorem 3. ***(Which-way information)*** *If hard information that the photon travels the a path (resp. the b path) is added to context Γ, then no interference takes place.* □

Define a formula as being valid in context Γ if and only if it has a hard inhabitant in Γ. Then one proves:

Theorem 4. ***(No exhaustive disjunction)*** *Neither formula $a \wedge b$ nor $a \vee b$ is valid in context Γ (19) formalizing the Mach-Zehnder interferometer.* □

Observe that the classical logic context brought to light earlier in Section 1.2

$$\{z : a \vee b, \qquad f : a \to \neg i, \qquad g : b \to \neg i, \qquad h : \neg a \to \neg b \to \neg i, \qquad t : i\}$$

including the results of the three experiments, yielded two fallacies

$$h(\lambda x.fxt)(\lambda y.gyt)t : \mathsf{F}$$

and

$$(\mathrm{case}(z\ ;\ \lambda x.fx\ ;\ \lambda y.gy))t : \mathsf{F}$$

in contrast with quantum logic of partial information context Γ (19).

5 Elimination of the wave particle duality paradox in quantum mechanics

We now show how to eliminate the two fallacies. This elimination then resolves the wave particle duality paradox.

5.1 Hard information fallacy

Recall that the hard information fallacy proceeded as follows in classical logic:

1. If the photon is absent on both channels, then neither a nor b is inhabited.
2. If neither a nor b is inhabited, then there is no interference.
3. If a (resp. b) is inhabited, then there is no interference.

4. If there is interference then neither a nor b is inhabited. (This is by contraposition of the third step 3.)
5. By putting together implications 4 and 2, one concludes that if there is interference, then there is no interference.
6. We deduce, by *reductio ad absurdum,* that there is no interference.
7. This conclusion is refuted by Grangier *et al.* experiment #2.

In the new quantum logic of partial information setting we have outlined in this paper, and using the theorems that have been established, we reason as follows.

(i) If the photon is absent on both channels, then a has no hard inhabitant, and b has no hard inhabitant. By absent on both channels, we mean $\Gamma \Vdash 0 : a$, and $\Gamma \Vdash 0 : b$.

(ii) If the photon is absent on both channels a, b, then there is no interference (by definition of interference).

(iii) If there is interference, then a has no hard inhabitant, and b has no hard inhabitant (by Which-way information theorem).

(iv) However, if a has no *hard* inhabitant, and b has no *hard* inhabitant, then it is not necessarily the case that a has no inhabitant and b has no inhabitant *i.e.,* that there is no interference. Indeed, in the above context Γ (19) formalizing Mach-Zehnder interferometer, neither a nor b has a hard inhabitant, since

$$\Gamma \Vdash -(Px\rho_1) : a \qquad\qquad \Gamma \Vdash Qx\tau_1 : b \tag{22}$$

and by Theorem 1 (Interference), one concludes that there *is* interference (as confirmed and demonstrated by Grangier *et al.* experiment #2). Therefore, due to the presence of soft inhabitants $-(Px\rho_1)$ and $Qx\tau_1$, we cannot apply a *reductio ad absurdum* argument by combining (ii) and (iii), since (ii) requires absence of inhabitant. The hard information fallacy vanishes.

The hard information fallacy stems from the implicit assumption that *all information is hard,* and hence that absence of hard information is equivalent to absence of any information. This implicit assumption produces here the confusion between *e.g.,* "a has no inhabitant" and "a has no *hard* inhabitant" in steps (2) and (3,4) of the hard information fallacy above.

Implicit assumption "*All information is hard*" is a "physical" fallacy, it is refuted by Grangier *et al.* experiment #2, which demonstrates the occurrence of interference. In terms of physics, the fallacy amounts to saying that "*Every phenomenon-to-be is a phenomenon.*" This proves the logical necessity of the sharp distinction between phenomenon and phenomenon-to-be pointed out by Bohr and Wheeler, if one is to correctly reason about the quantum physical system at hand. Using typed λ-term

$$h(\lambda x.fxt)(\lambda y.gyt)t : \mathsf{F}$$

we have provided a formal logic form to an instance of this distinction being violated.

This solves the hard information fallacy. The hard information fallacy underlines the fundamental character of the distinction between phenomenon and phenomenon-to-be *i.e.,* in other words, between hard and soft information in quantum mechanics (Figure 4).

5.2 Exhaustive disjunction fallacy

The exhaustive disjunction fallacy is expressed by inhabitation claim

$$(\mathrm{case}(z\ ;\ \lambda x.fx\ ;\ \lambda y.gy))t : \mathsf{F}$$

where $z : a \vee b$. "Experimentally observed facts" $f : a \to \neg i$ and $g : b \to \neg i$ in the old syntax are expressed in our new setting by Theorem 3. The claim, inferred from Grangier *et al.* experiment #1, that

"there is some hard information that disjunction $a \vee b$ is the case"

is not valid, since by reasoning by cases, the *availability* of which-way information, even if it is unknown or left unrecorded, makes the interference disappear, thus contradicting Grangier *et al.* experiment #2. Whence, in the framework of Grangier *et al.* experiment #2, corresponding to Mach-Zehnder interferometer (Figure 2) and formalized by context Γ (19), where interference occurs, *i.e.,* where i holds, to refute fallacious conclusion $\neg i$, one must "cancel" the proof that $\neg i$ is the case, *i.e.,* at the formal level of λ-calculus one must cancel inhabitant

$$(\mathrm{case}(z\ ;\ \lambda x.fx\ ;\ \lambda y.gy))t$$

of F obtained when reasoning in classical logic. Since z, f, g are its only free variables, and since f and g correspond to Which-way information theorem 3, one must "cancel" free variable z *i.e.,* one must give up exhaustive disjunction $z : a \vee b$, which says that after going through the first beam splitter, the photon is either on channel a or on channel b.

In the above context Γ formalizing Mach-Zehnder interferometer, one has by Theorem 4 that exhaustive disjunction $a \vee b$ is invalid *i.e.,* has no hard inhabitant. Therefore, one cannot reason by cases and apply the result of Scully *et al.* experiment—expressed here by Theorem 3—as in the fallacy, since that theorem requires hard inhabitants in its premiss. The above inhabitant of F thus vanishes.

Physically, exhaustive disjunction $a \vee b$ is refuted by experiment; it is not a phenomenon, only a phenomenon-to-be. The production of interference, which is what the quantum physical world does, therefore requires this physical non-existence of any hard, total information regarding $a \vee b$.

This solves the exhaustive disjunction fallacy.

Feynman [9] p. 144 concludes his discussion related to what is called here the "exhaustive disjunction fallacy" as follows:

> *To conclude that [the photon] goes either through [channel a] or [channel b] when you are not looking is to produce an error in prediction. That*

is the logical tight-rope on which we have to walk if we wish to interpret nature ... The question now is, how does it really work? What machinery is producing this thing?" Nobody knows any machinery ... The deep mystery is what I have described, and no one can go any deeper today.

It has been shown in this paper how Feynman's observation can be deduced in mathematics, from the logic formalism presented here, thus shedding some light on the nature of the "logical tight-rope" to be walked upon, and some of the machinery behind it.

To sum up, the wave particle paradox in quantum mechanics points out two fundamental properties of quantum mechanics: (i) the logical necessity of a distinction between phenomenon and phenomenon-to-be, and (ii) the physical non-validity of exhaustive disjunction $a \vee b$. Since by Grangier *et al.* experiment #1 a and b are mutually exclusive, if they are interpreted, as in quantum computing, as complementary Boolean values $true$, $false$, then the "exhaustive disjunction fallacy" part of the wave particle duality paradox points out the non-validity of the Law of the Excluded Middle Third $A \vee \neg A$ in quantum mechanics.

6 Conclusion

In this paper, we have presented a formalization, in terms of formal logic, of single photon self-interference in quantum mechanics, as well as a logic solution to the wave particle duality paradox. This solution uses a generalized Curry-Howard correspondence. It is based on the logic of partial information and on the sharp distinction between phenomenon and phenomenon-to-be pointed out in quantum physics, by N. Bohr and J.A. Wheeler.

We have shown that one way to reintegrate quantum mechanics reasoning and wave particle duality into logic, and eliminate its paradoxical character, is to extend classical logic into a logic of partial information, since this is what our experimental observations require.

References

1. M. A. Nait Abdallah. *The logic of partial information.* EATCS research monographs in Theoretical Computer Science. Spinger Verlag, 1995.
2. M.A. Nait Abdallah. On an application of Curry-Howard isomorphism to quantum mechanics. *in P. d'Aquino, Logic Colloquium '12: Manchester,UK July 2012, BSL*, pages 383–384, 2014.
3. D. Bohm. *Quantum theory.* Dover, 1989.
4. N. Bohr. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 48:696–702, 1935.
5. N. Bohr. *Atomic theory and the description of nature*, page 10. Cambridge University Press, 1961.
6. H. B. Curry and R. Feys. *Combinatory logic.* North Holland, 1974.
7. H. Everett. "Relative state" formulation of quantum mechanics. *Rev. Modern Physics*, 29(3):454–462, 1957.

8. R. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys*, 20:367–387, 1948.
9. R. Feynman. *The Character of Physical Law.* MIT Press, 1965.
10. R. Feynman, R. Leighton, and M. Sands. *The Feynman lectures on physics*, volume 3. Addison-Wesley, 1963.
11. P. Grangier, G. Roger, and A. Aspect. Experimental evidence for a photon anticorrelation effect on a beam splitter: a new light on single-photon interference. *Europhys. Lett.*, 1:173, 1986.
12. W. A. Howard. The formulae-as-types notion of construction. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. Academic Press, 1980.
13. I. Lakatos. Falsification and the methodology of scientific research programmes. In Worrall J. and G. Currie, editors, *Imre Lakatos, The Methodology of Scientific Research Programmes*, pages 8–101. Cambridge University Press, 1976.
14. Scully M.O., B.-G. Englert, and H. Walther. Quantum optical tests of complementarity. *Nature*, 351:111–116, 1991.
15. M.H.B. Sørensen and P. Urzyczyn. *Lectures on the Curry-Howard isomorphism.* Elsevier, 2006.
16. J.A. Wheeler. Law without law. In J.A. Wheeler and W.H. Zurek, editors, *Quantum Theory and Measurement*, pages 182–213. Princeton, 1984.
17. J.A. Wheeler. Information, physics, quantum: The search for links. In W. Zurek, editor, *Complexity, Entropy, and the Physics of Information*, pages 309–336. Addison-Wesley, 1990.