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To cite this version:
Lucio Boccardo, Gisella Croce, Chiara Tanteri. AN ELLIPTIC SYSTEM WITH DEGENERATE COERCIVITY. Rendiconti di Matematica, 2015, 36. hal-01302646

HAL Id: hal-01302646
https://hal.archives-ouvertes.fr/hal-01302646
Submitted on 14 Apr 2016

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AN ELLIPTIC SYSTEM WITH DEGENERATE COERCIVITY

LUCIO BOCCARDO, GISSELLA CROCE, CHIARA TANTERI

1. Introduction

1.1. Setting. In this paper we study the existence of solutions of the degenerate elliptic system

\[
\begin{align*}
-\text{div} \left( \frac{a(x)\nabla u}{(b(x) + |z|)^2} \right) + u &= f(x), \\
-\text{div} \left( \frac{A(x)\nabla z}{(B(x) + |u|)^2} \right) + z &= F(x),
\end{align*}
\]

(1.1)

where \( \Omega \) is a bounded, open subset of \( \mathbb{R}^N \), with \( N > 2 \), \( a(x) \) and \( A(x) \) are measurable matrices such that, for \( \alpha, \beta \in \mathbb{R}^+ \),

\[
\alpha |\xi|^2 \leq a(x)\xi \xi, \quad \alpha |\xi|^2 \leq A(x)\xi \xi; \quad |a(x)| \leq \beta, \quad |A(x)| \leq \beta.
\]

(1.2)

Moreover we assume

\[
0 < \lambda \leq b(x), \quad B(x) \leq \gamma,
\]

(1.3)

for some \( \lambda, \gamma \in \mathbb{R}^+ \) and

\[
f(x), \quad F(x) \in L^2(\Omega).
\]

(1.4)

Theorem 1.1. Under the assumptions (1.2), (1.3), (1.4), there exist \( u \in W^{1,1}_0(\Omega) \) and \( z \in W^{1,1}_0(\Omega) \), distributional solutions of the system (1.1).

1.2. Comments. First of all, we note that existence of solutions belonging to the nonreflexive space \( W^{1,1}_0(\Omega) \) is not so usual in the study of elliptic problems. Recently the existence of solutions in \( W^{1,1}_0(\Omega) \) was proved in [3], [4], [5], for elliptic scalar problems with degenerate coercivity (so that this paper is an extension to the systems of some of those results) and in some borderline cases of the Calderon-Zygmund theory of nonlinear Dirichlet problems in [9].

The main difficulty of the problem is that even if the differential operator is well defined between \( W^{1,2}_0(\Omega) \) and its dual, it is not coercive on \( W^{1,2}_0(\Omega) \): degenerate coercivity means that when \( |v| \) is “large”, \( \frac{1}{(a(x)+|v|)^2} \) goes to zero: for an explicit example see [18].
The study of problems involving degenerate equations begins with the paper [8] and it is developed in [1], [10], [11], [12], [3], [4], [5] (see also [2]).

2. Existence

2.1. A priori estimates. The first existence result is concerned with the case of a bounded data.

We recall the following definitions.

\[ T_k(s) = \begin{cases} 
  s, & \text{if } |s| \leq k; \\
  \frac{k}{|s|}, & \text{if } |s| > k; 
\end{cases} \]

\[ G_k(s) = s - T_k(s). \]

**Proposition 2.1.** Let \( \rho > 0, \sigma > 0 \) and \( g, G \in L^\infty(\Omega) \). Then there exist weak solutions \( w, W \) belonging to \( W^{1,2}_0(\Omega) \) of the system

\[
\begin{align*}
  w &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\div(a(x)\nabla w) + w = g(x), \\
  W &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\div(A(x)\nabla W) + W = G(x).
\end{align*}
\]

**Proof.** The existence is a consequence of the Leray-Lions theorem (or Schauder theorem) since the principal part is not degenerate, thanks to the presence of \( T_\rho \) and \( T_\sigma \). Moreover, if we take \( G_h(w) \) as test function in the first equation and \( G_k(W) \) as test function in the second equation, we have, dropping two positive terms,

\[
\begin{align*}
  \int_\Omega [||w|| - |g(x)||] |G_h(w)| &\leq 0, \\
  \int_\Omega [||W|| - |G(x)||] |G_k(w)| &\leq 0.
\end{align*}
\]

Then the choice \( h = \|g\|_{L^\infty(\Omega)} \), \( k = \|G\|_{L^\infty(\Omega)} \) implies

\[
\begin{align*}
  ||w|| &\leq \|g\|_{L^\infty(\Omega)}, \\
  ||W|| &\leq \|G\|_{L^\infty(\Omega)}.
\end{align*}
\]

Thus, if we set \( \rho = \|g\|_{L^\infty(\Omega)} \) and \( \sigma = \|G\|_{L^\infty(\Omega)} \), we can say that \( w \) and \( W \) are bounded weak solutions of the system

\[
\begin{align*}
  w &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\div(a(x)\nabla w) + w = g(x), \\
  W &\in W^{1,2}_0(\Omega) \cap L^\infty(\Omega) : -\div(A(x)\nabla W) + W = G(x).
\end{align*}
\]

\[ \square \]
Now we define
\[ f_n = \frac{f}{1 + \frac{1}{n}|f|}, \quad F_n = \frac{F}{1 + \frac{1}{n}|F|}, \]
so that
\[ (2.1) \quad \|f_n - f\|_{L^2(\Omega)} \to 0, \quad \|F_n - F\|_{L^2(\Omega)} \to 0. \]
Thanks to the Proposition 2.1, there exists a solution \((u_n, z_n)\) of the system
\[
\begin{cases}
  u_n \in W^{1,2}_0(\Omega) : -\text{div} \left( \frac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \right) + u_n = f_n(x), \\
  z_n \in W^{1,2}_0(\Omega) : -\text{div} \left( \frac{A(x)\nabla z_n}{(B(x) + |u_n|)^2} \right) + z_n = F_n(x),
\end{cases}
\]
Now we prove our first estimates.

**Lemma 2.2.** The sequences \(\{u_n\}\) and \(\{z_n\}\) are bounded in \(L^2(\Omega)\).

**Proof.** We take \(G_k(u_n)\) as a test function in the first equation and we have
\[
(2.3) \quad \alpha \int_{\Omega} \frac{|
abla G_k(u_n)|^2}{(b(x) + |z_n|)^2} + \int_{\Omega} |G_k(u_n)|^2 \leq \int_{\Omega} |f||G_k(u_n)|
\]
If we drop the first positive term and we use the Hölder inequality, then we have
\[
(2.4) \quad \left[ \int_{\Omega} |G_k(u_n)|^2 \right]^{\frac{1}{2}} \leq \left[ \int_{\{k \leq |u_n|\}} |f| \right]^{\frac{1}{2}}.
\]
Similar estimates hold true for \(z_n\). In particular, taking \(k = 0\), we have the boundedness of the sequences \(\{u_n\}\) and \(\{z_n\}\) in \(L^2(\Omega)\). So we have that there exist \(u, z\) such that, up to subsequences,
\[
(2.5) \quad u_n \rightharpoonup u, \quad z_n \rightharpoonup z \quad \text{weakly in} \quad L^2(\Omega).
\]
Then if we drop the second term in (2.3), we have
\[
(2.6) \quad \alpha \int_{\Omega} \frac{|
abla G_k(u_n)|^2}{(b(x) + |z_n|)^2} \leq \int_{\{k \leq |u_n|\}} |f|^2.
\]
A similar estimate for \(z_n\) comes from the second equation.

**Lemma 2.3.** The sequences \(\{u_n\}\) and \(\{z_n\}\) are bounded in \(W^{1,1}_0(\Omega)\).

**Proof.** A consequence of (2.6) and of the Hölder inequality is
\[
\int_{\Omega} |\nabla G_k(u_n)| = \int_{\Omega} \frac{|
abla G_k(u_n)|}{(b(x) + |z_n|)} (b(x) + |z_n|) \leq \left[ \int_{\{k \leq |u_n|\}} |f|^2 \frac{1}{\alpha} \right] \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).
\]
Similar estimates hold true for \( z_n \). In particular, with \( k = 0 \), we have

\[
\begin{align*}
\int_\Omega |\nabla u_n| & \leq \frac{\|f\|_{L^2(\Omega)} \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)}{\alpha^2}, \\
\int_\Omega |\nabla z_n| & \leq \frac{\|F\|_{L^2(\Omega)} \left( \|b\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right)}{\alpha^2}.
\end{align*}
\]

(2.7)

Now we improve the convergence (2.5).

**Lemma 2.4.** The sequences \( \{u_n\} \) and \( \{z_n\} \) are compact in \( L^2(\Omega) \).

**Proof.** The estimates (2.7) imply, thanks to the Rellich embedding Theorem, the \( L^1 \) compactenss and then the a.e. convergences

\[
(2.8) \quad u_n(x) \to u(x), \quad z_n(x) \to z(x).
\]

Now we use the Vitali Theorem: since we have the a.e. convergences (2.8), the compactness is achieved if we prove the equiintegrability.

Let \( E \) be a measurable subset of \( \Omega \). Since \( u_n = T_k(u_n) + G_k(u_n) \), we have (we use (2.4))

\[
\int_E |u_n|^2 \leq 2 \int_E |T_k(u_n)|^2 + 2 \int_E |G_k(u_n)|^2 \leq 2 k^2 |E| + \int_\Omega |G_k(u_n)|^2 \leq 2 k^2 |E| + 2 \int_{\{k \leq |u_n|\}} |f|^2,
\]

where \( |E| \) denotes the measure of \( E \). Now we recall that a consequence of Lemma 2.3 is that the sequence \( \{u_n\} \) is bounded in \( L^1(\Omega) \), so that if we fix \( \epsilon > 0 \), there exists \( k_\epsilon \) such that (uniformly with respect to \( n \))

\[
\int_{\{k \leq |u_n|\}} |f|^2 \leq \epsilon, \quad k \geq k_\epsilon.
\]

Then

\[
\int_E |u_n|^2 \leq 2 k^2 |E| + 2 \epsilon
\]

implies

\[
\lim_{|E| \to 0} \int_E |u_n|^2 \leq 2 \epsilon, \text{ uniformly with respect to } n.
\]

Similar inequality holds true for \( z_n \).

**Lemma 2.5.** The sequences \( \{u_n\} \) and \( \{z_n\} \) are weakly compact in \( W_0^{1,1}(\Omega) \).
Proof. Here we follow [4], [5]. Let again \( E \) be a measurable subset of \( \Omega \), and let \( i \) be in \( \{1, \ldots, N\} \). Then
\[
\int_E |\partial_i u_n| \leq \int_E |\nabla u_n| = \int_E \frac{|\nabla u_n|}{b(x) + |z_n|} (b(x) + |z_n|)
\]
\[
\leq \left[ \int_\Omega \frac{|\nabla u_n|^2}{(b(x) + |z_n|)^2} \right]^{\frac{1}{2}} \left[ \int_E (b(x) + |z_n|)^2 \right]^{\frac{1}{2}}
\]
\[
\leq \left[ \frac{1}{\alpha} \int_\Omega |f|^2 \right]^{\frac{1}{2}} \left\{ \left[ \int_\Omega b(x) \right] \frac{1}{2} + \left[ \int_E |z_n|^2 \right] \frac{1}{2} \right\},
\]
where we have used the inequality (2.6) in the last passage. Since the sequence \( \{u_n\} \) is compact in \( L^2(\Omega) \), we have that the sequence \( \{\partial_i u_n\} \) is equiintegrable. Thus, by Dunford-Pettis theorem, and up to subsequences, there exists \( Y_i \) in \( L^1(\Omega) \) such that \( \partial_i u_n \) weakly converges to \( Y_i \) in \( L^1(\Omega) \). Since \( \partial_i u_n \) is the distributional derivative of \( u_n \), we have, for every \( n \) in \( \mathbb{N} \),
\[
\int_\Omega \partial_i u_n \phi = -\int_\Omega u_n \partial_i \phi, \quad \forall \phi \in C_0^\infty(\Omega).
\]
We now pass to the limit in the above identities, using that \( \partial_i u_n \) weakly converges to \( Y_i \) in \( L^1(\Omega) \), and that \( u_n \) strongly converges to \( u \) in \( L^2(\Omega) \); we obtain
\[
\int_\Omega Y_i \phi = -\int_\Omega u \partial_i \phi, \quad \forall \phi \in C_0^\infty(\Omega),
\]
which implies that \( Y_i = \partial_i u \), and this result is true for every \( i \). Since \( Y_i \) belongs to \( L^1(\Omega) \) for every \( i \), \( u \) belongs to \( W^{1,1}_0(\Omega) \). A similar result holds true for \( z_n \).

Thus, thanks to Lemma 2.4 and Lemma 2.5, we can improve the convergence (2.5):
\[
(2.9) \quad \begin{cases} u_n \text{ converges weakly in } W^{1,1}_0(\Omega) \text{ and strongly in } L^2(\Omega) \text{ to } u, \\ z_n \text{ converges weakly in } W^{1,1}_0(\Omega) \text{ and strongly in } L^2(\Omega) \text{ to } z. \end{cases}
\]

2.2. Proof of Theorem 1.1 - First of all, we use the equiintegrability proved in Lemma 2.5: fix \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that, for every measurable subset \( E \) with \( |E| \leq \delta(\varepsilon) \), we have
\[
\int_E |\nabla u_n| \leq \varepsilon.
\]
Taking into account the absolute continuity of the Lebesgue integral, we have, for some \( \tilde{\delta}(\varepsilon) > 0 \),
\[
\int_E |\nabla u_n| \leq \varepsilon, \quad \int_E |\nabla u| \leq \varepsilon,
\]
for every measurable subset \( E \) with \( |E| \leq \tilde{\delta}(\varepsilon) \).
On the other hand, since $|\Omega|$ is finite and the sequence

$$D_n = \frac{a(x)}{(b(x) + |z_n|)^2}$$

converges almost everywhere (recall (2.9)), the Egorov theorem says that for every $q > 0$, there exists a measurable subset $F$ of $\Omega$ such that $|F| < q$, and $D_n$ converges to $D$ uniformly on $\Omega \setminus F$. We choose $q = \delta$ so that we have, for every $\varphi \in \text{Lip}(\Omega)$,

$$\left| \int_{\Omega} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| \leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + \left| \int_F [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| \leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + \frac{\beta}{\lambda^2} \| \nabla \varphi \|_{L^\infty(\Omega)} \left( \int_F |\nabla u_n| + \int_F |\nabla u| \right) \leq \left| \int_{\Omega \setminus F} [D_n \nabla u_n \nabla \varphi - D \nabla u \nabla \varphi] \right| + 2 \varepsilon \frac{\beta}{\lambda^2} \| \nabla \varphi \|_{L^\infty(\Omega)},$$

which proves that

$$\int_{\Omega} \frac{a(x)}{(b(x) + |z_n|)^2} \to \int_{\Omega} \frac{a(x)}{(b(x) + |z|)^2}. \tag{2.10}$$

Thus, thanks to the above limit, (2.1) and Lemma 2.4, it is possible to pass to the limit in the weak formulation of (2.2), for every $\varphi, \psi \in \text{Lip}(\Omega)$,

$$\begin{cases} \int_{\Omega} \frac{a(x)}{(b(x) + |z_n|)^2} + \int_{\Omega} u_n \varphi = \int_{\Omega} f_n(x) \varphi, \\ \int_{\Omega} \frac{A(x)}{(B(x) + |u_n|)^2} \nabla z_n \nabla \psi = \int_{\Omega} z_n \psi = \int_{\Omega} F_n(x); \end{cases} \tag{2.11}$$

and we prove that $u$ and $z$ are solutions of our system, in the following distributional sense

$$\begin{cases} \int \frac{a(x)}{(b(x) + |z|)^2} + \int_u \varphi = \int f(x) \varphi, \quad \forall \varphi \in \text{Lip}(\Omega); \\ \int \frac{A(x)}{(B(x) + |u|)^2} \nabla z \nabla \psi = \int z \psi = \int F(x) \psi, \quad \forall \psi \in \text{Lip}(\Omega). \end{cases} \tag{2.12}$$

Now we show that, in the above definition of solution, it is possible to use less regular test functions: it possible to use functions only belonging to $W^{1,2}_0(\Omega)$.
Proposition 2.6. The above functions $u$ and $z$ are solutions of our system, in the following sense:

\begin{equation}
\begin{cases}
\int_{\Omega} \frac{a(x)\nabla u \nabla w}{(b(x) + |z|)^2} + \int_{\Omega} u v = \int_{\Omega} f(x) v, \quad \forall v \in W^{1,2}_0(\Omega); \\
\int_{\Omega} \frac{A(x)\nabla z \nabla w}{(B(x) + |u|)^2} + \int_{\Omega} z w = \int_{\Omega} F(x) w, \quad \forall w \in W^{1,2}_0(\Omega).
\end{cases}
\end{equation}

Proof. In order to avoid technicalities, here we also assume that

\begin{equation}
a(x) \quad \text{and} \quad A(x) \quad \text{are scalar functions.}
\end{equation}

We start with the following inequalities (we use (2.6) with $k = 0$)

\[ \int_{\Omega} \left| \frac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \right|^2 \leq \frac{\alpha^2}{\lambda^2} \int_{\Omega} \frac{|\nabla u_n|^2}{(b(x) + |z_n|)^2} \leq \frac{\alpha^2}{\lambda^2} \int_{\Omega} |f|^2. \]

Thus, up to subsequences, we can say that, for some $\Psi \in (L^2(\Omega))^N$,

\begin{equation}
\int_{\Omega} \frac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \Phi \to \int_{\Omega} \Psi \Phi,
\end{equation}

for every $\Phi \in (L^2(\Omega))^N$. Now we compare the limit (2.10) with the limit (2.15), taking $\Phi = \nabla \varphi$, and we deduce that

\[ \int_{\Omega} \left[ \frac{a(x)\nabla u}{(b(x) + |z|)^2} - \Psi \right] \Phi = 0. \]

Thus we proved that

\[ \frac{a(x)\nabla u_n}{(b(x) + |z_n|)^2} \quad \text{weakly converges in} \quad (L^2(\Omega))^N \text{ to} \quad \frac{a(x)\nabla u}{(b(x) + |z|)^2}, \]

which allows us to pass to the limit in (2.11) only assuming $\varphi, \psi \in W^{1,2}_0(\Omega)$.

Acknowledgments

This paper contains the unpublished part of the results presented by the first author in a talk at the conference “Calculus of Variations and Differential Equations - Conférence en l'honneur du 60ème anniversaire de Bernard Dacorogna” (Lausanne, 10-12 juin 2013).

References

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