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The maximal drawdown of the Brownian meander

by

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Summary. Motivated by evaluating the limiting distribution of randomly biased random walks on trees, we compute the exact value of a negative moment of the maximal drawdown of the standard Brownian meander.

Keywords. Brownian meander, Bessel process, maximal drawdown.

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1 Introduction

Let $(X(t), t \in [0, 1])$ be a random process. Its maximal drawdown on $[0, 1]$ is defined by

$$X^\#(1) := \sup_{s \in [0, 1]} [\overline{X}(s) - X(s)],$$

where $\overline{X}(s) := \sup_{u \in [0, s]} X(u)$. There has been some recent research interest on the study of drawdowns from probabilistic point of view ([7], [8]) as well as applications in insurance and finance ([1], [2], [3], [10], [12]).

We are interested in the maximal drawdown $m^\#(1)$ of the standard Brownian meander $(m(t), t \in [0, 1])$. Our motivation is the presence of the law of $m^\#(1)$ in the limiting distribution of randomly biased random walks on supercritical Galton–Watson trees ([4]); in particular, the value of $\mathbb{E}(\frac{1}{m^\#(1)})$ is the normalizing constant in the density function of

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this limiting distribution. The sole aim of the present note is to compute $E(\frac{1}{m^*(1)})$, which turns out to have a nice numerical value.

Let us first recall the definition of the Brownian meander. Let $W := (W(t), t \in [0, 1])$ be a standard Brownian motion, and let $g := \sup\{t \leq 1 : W(t) = 0\}$ be the last passage time at 0 before time 1. Since $g < 1$ a.s., we can define

$$m(s) := \frac{|W(g + s(1 - g))|}{(1 - g)^{1/2}}, \quad s \in [0, 1].$$

The law of $(m(s), s \in [0, 1])$ is called the law of the standard Brownian meander. For an account of general properties of the Brownian meander, see Yen and Yor [11].

**Theorem 1.1.** Let $(m(s), s \in [0, 1])$ be a standard Brownian meander. We have

$$E\left(\frac{1}{\sup_{s \in [0, 1]}[m(s) - m'(s)]}\right) = \left(\frac{\pi}{2}\right)^{1/2},$$

where $m(s) := \sup_{u \in [0, s]} m(u)$.

The theorem is proved in Section 2.

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N.B. from the first-named coauthors: This note originates from a question we asked our teacher, **Professor Marc Yor (1949–2014)**, who passed away in January 2014, during the preparation of this note. He provided us, in November 2012, with the essential of the material in Section 2.

### 2 Proof

Let $R := (R(t), t \geq 0)$ be a three-dimensional Bessel process with $R(0) = 0$, i.e., the Euclidean modulus of a standard three-dimensional Brownian motion. The proof of Theorem 1.1 relies on an absolute continuity relation between $(m(s), s \in [0, 1])$ and $(R(s), s \in [0, 1])$, recalled as follows.

**Fact 2.1.** (Imhof [5]) Let $(m(s), s \in [0, 1])$ be a standard Brownian meander. Let $(R(s), s \in [0, 1])$ be a three-dimensional Bessel process with $R(0) = 0$. For any measurable and non-negative functional $F$, we have

$$E\left[F(m(s), s \in [0, 1])\right] = \left(\frac{\pi}{2}\right)^{1/2} E\left[\frac{1}{R(1)} F(R(s), s \in [0, 1])\right].$$


We now proceed to the proof of Theorem 1.1. Let

\[ L := \mathbb{E}\left( \frac{1}{\sup_{s \in [0,1]} |\overline{\mathbf{m}}(s) - \mathbf{m}(s)|} \right). \]

Write \( \overline{R}(t) := \sup_{u \in [0,t]} R(u) \) for \( t \geq 0 \). By Fact 2.1

\[ L = \left( \frac{\pi}{2} \right)^{1/2} \mathbb{E}\left[ \frac{1}{R(1)} \sup_{s \in [0,1]} \frac{1}{|\overline{R}(s) - R(s)|} \right] = \left( \frac{\pi}{2} \right)^{1/2} \int_{0}^{\infty} \mathbb{E}\left[ \frac{1}{R(1)} \mathbf{1}_{\{ \sup_{s \in [0,1]} |\overline{R}(s) - R(s)| < \frac{1}{a} \}} \right] da, \]

the last equality following from the Fubini–Tonelli theorem. By the scaling property,

\[ \mathbb{E}\left[ \frac{1}{R(1)} \mathbf{1}_{\{ \sup_{s \in [0,1]} |\overline{R}(s) - R(s)| < \frac{1}{a} \}} \right] = \mathbb{E}\left[ \frac{a}{R(a^2)} \mathbf{1}_{\{ \sup_{s \in [0,a^2]} |\overline{R}(u) - R(u)| < 1 \}} \right] \]

for all \( a > 0 \). So by means of a change of variables \( b = a^2 \), we obtain:

\[ L = \left( \frac{\pi}{8} \right)^{1/2} \int_{0}^{\infty} \mathbb{E}\left[ \frac{1}{R(b)} \mathbf{1}_{\{ \sup_{s \in [0,b]} |\overline{R}(u) - R(u)| < 1 \}} \right] db. \]

Define, for any random process \( X \),

\[ \tau_1^X := \inf\{ t \geq 0 : \overline{X}(t) - X(t) \geq 1 \}, \]

with \( \overline{X}(t) := \sup_{s \in [0,t]} X(s) \). For any \( b > 0 \), the event \( \{ \sup_{u \in [0,b]} |\overline{R}(u) - R(u)| < 1 \} \) means \( \{ \tau_1^R > b \} \), so

\[ L = \left( \frac{\pi}{8} \right)^{1/2} \int_{0}^{\infty} \mathbb{E}\left[ \frac{1}{R(b)} \mathbf{1}_{\{ \tau_1^R > b \}} \right] db = \left( \frac{\pi}{8} \right)^{1/2} \mathbb{E}\left( \int_{0}^{\tau_1^R} \frac{1}{R(b)} db \right), \]

the second identity following from the Fubini–Tonelli theorem. According to a relation between Bessel processes of dimensions three and four (Revuz and Yor [9], Proposition XI.1.11, applied to the parameters \( p = q = 2 \) and \( \nu = \frac{1}{2} \)),

\[ R(t) = U\left( \frac{1}{4} \int_{0}^{t} \frac{1}{R(b)} db \right), \quad t \geq 0, \]

where \( U := (U(s), s \geq 0) \) is a four-dimensional squared Bessel process with \( U(0) = 0 \); in other words, \( U \) is the square of the Euclidean modulus of a standard four-dimensional Brownian motion.

Let us introduce the increasing functional \( \sigma(t) := \frac{1}{4} \int_{0}^{t} \frac{1}{R(b)} db, t \geq 0 \). We have \( R = U \circ \sigma \), and

\[ \tau_1^R = \inf\{ t \geq 0 : \overline{R}(t) - R(t) \geq 1 \} = \inf\{ t \geq 0 : \overline{U}(\sigma(t)) - U(\sigma(t)) \geq 1 \} = \inf\{ \sigma^{-1}(s) : s \geq 0 \text{ and } \overline{U}(s) - U(s) \geq 1 \} \]

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which is $\sigma^{-1}(\tau_1^U)$. So $\tau_1^U = \sigma(\tau_1^R)$, i.e.,

$$\int_0^{\tau_1^R} \frac{1}{R(b)} \, db = 4\tau_1^U,$$

which implies that

$$L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U).$$

The Laplace transform of $\tau_1^U$ is determined by Lehoczky [6], from which, however, it does not seem obvious to deduce the value of $\mathbb{E}(\tau_1^U)$. Instead of using Lehoczky’s result directly, we rather apply his method to compute $\mathbb{E}(\tau_1^U)$. By Itô’s formula, $(U(t) - 4t, t \geq 0)$ is a continuous martingale, with quadratic variation $4 \int_0^t U(s) \, ds$; so applying the Dambis–Dubins–Schwarz theorem (Revuz and Yor [9], Theorem V.1.6) to $(U(t) - 4t, t \geq 0)$ yields the existence of a standard Brownian motion $B = (B(t), t \geq 0)$ such that

$$U(t) = 2B\left(\int_0^t U(s) \, ds\right) + 4t, \quad t \geq 0.$$

Taking $t := \tau_1^U$, we get

$$U(\tau_1^U) = 2B\left(\int_0^{\tau_1^U} U(s) \, ds\right) + 4\tau_1^U.$$

We claim that

$$\mathbb{E}\left[B\left(\int_0^{\tau_1^U} U(s) \, ds\right)\right] = 0.$$  

Then $\mathbb{E}(\tau_1^U) = \frac{1}{4} \mathbb{E}[U(\tau_1^U)]$; hence

$$L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U) = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)].$$

Let us admit (2.1) for the moment, and prove the theorem by computing $\mathbb{E}[U(\tau_1^U)]$ using Lehoczky [6]’s method; in fact, we determine the law of $U(\tau_1^U)$.

**Lemma 2.2.** The law of $U(\tau_1^U)$ is given by

$$\mathbb{P}\{U(\tau_1^U) > a\} = (a + 1)e^{-a}, \quad \forall a > 0.$$ 

In particular,

$$\mathbb{E}[U(\tau_1^U)] = \int_0^\infty (a + 1)e^{-a} \, da = 2.$$

Since $L = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)]$ (see (2.2)), this yields $L = \left(\frac{\pi}{2}\right)^{1/2}$ as stated in Theorem 1.1.
The rest of the note is devoted to the proof of Lemma 2.2 and (2.1).

**Proof of Lemma 2.2.** Fix $b > 1$. We compute the probability $\mathbb{P}\{\overline{U}(\tau(U)) > b\}$ which, due to the equality $\overline{U}(\tau(U)) = U(\tau(U)) + 1$, coincides with $\mathbb{P}\{U(\tau(U)) > b - 1\}$. By applying the strong Markov property at time $\sigma_0 := \inf\{t \geq 0 : U(t) = 1\}$, we see that the value of $\mathbb{P}\{\overline{U}(\tau(U)) > b\}$ does not change if the squared Bessel process $U$ starts at $U(0) = 1$. Indeed, observing that $\sigma_0 \leq \tau(U)$, $U(\sigma(U)) = 1$ and that $\overline{U}(\tau(U)) = \sup_{s \in [\sigma_0, \tau(U)]} U(s)$, we have

$$\mathbb{P}\{\overline{U}(\tau(U)) > b\} = \mathbb{P}\left\{ \sup_{s \in [\sigma_0, \tau(U)]} U(s) > b \right\} = \mathbb{P}_1\{\overline{U}(\tau(U)) > b\},$$

the subscript 1 in $\mathbb{P}_1$ indicating the initial value of $U$. More generally, for $x \geq 0$, we write $\mathbb{P}_x(\bullet) := \mathbb{P}(\bullet | U(0) = x)$, so $\mathbb{P} = \mathbb{P}_0$.

Let $b_0 = 1 < b_1 < \cdots < b_n := b$ be a subdivision of $[1, b]$ such that $\max_{1 \leq i \leq n}(b_i - b_{i-1}) \to 0$, $n \to \infty$. Consider the event $\{\overline{U}(\tau(U)) > b\}$: since $U(0) = 1$, this means $U$ hits position $b$ before time $\tau(U)$; for all $i \in [1, n - 1] \cap \mathbb{Z}$, starting from position $b_i$, $U$ must hit $b_{i+1}$ before hitting $b_i - 1$ (caution: not to be confused with $b_{i-1}$). More precisely, let $\sigma_i := \inf\{t \geq 0 : U(t) = b_i\}$ and let $U_i(s) := U(s + \sigma_i)$, $s \geq 0$; then

$$\{\overline{U}(\tau(U)) > b\} \subset \bigcap_{i=1}^{n-1}\{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$ 

By the strong Markov property, the events $\{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}$, $1 \leq i \leq n - 1$, are independent (caution: the processes $(U_i(s), s \geq 0)$, $1 \leq i \leq n - 1$, are not independent). Hence

$$\mathbb{P}_1\{\overline{U}(\tau(U)) > b\} \leq \prod_{i=1}^{n-1}\mathbb{P}_0\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$ 

Conversely, let $\varepsilon > 0$, and if $\max_{1 \leq i \leq n}(b_i - b_{i-1}) < \varepsilon$, then we also have

$$\mathbb{P}_1\{\overline{U}(\tau(U)) > b\} \geq \prod_{i=1}^{n-1}\mathbb{P}_0\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\},$$

with $\tau(U) := \inf\{t \geq 0 : \overline{U}(t) - U(t) \geq 1 + \varepsilon\}$. By scaling, $\overline{U}(\tau(U))$ has the same distribution as $(1 + \varepsilon)\overline{U}(\tau(U))$. So, as long as $\max_{1 \leq i \leq n}(b_i - b_{i-1}) < \varepsilon$, we have

$$\mathbb{P}_1\{\overline{U}(\tau(U)) > b\} \leq \prod_{i=1}^{n-1}\mathbb{P}_0\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} \leq \mathbb{P}_1\{\overline{U}(\tau(U)) > b \overline{1 + \varepsilon}}.$$
Since $\frac{1}{x}$ is a scale function for $U$, we have

$$
\mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1 \} = \frac{1}{b_{i+1} - b_i} - \frac{1}{b_i} = 1 - \frac{1}{b_{i+1} - b_i}.
$$

If $\lim_{n \to \infty} \max_{0 \leq i \leq n-1}(b_{i+1} - b_i) = 0$, then for $n \to \infty$,

$$
\sum_{i=1}^{n-1} \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{b_i - b_{i+1}} = \sum_{i=1}^{n-1} \frac{b_i - 1}{b_i} (b_{i+1} - b_i) + o(1)
$$

$$
\to \int_1^b \frac{r - 1}{r} \, dr = b - 1 - \log b.
$$

Therefore,

$$
\lim_{n \to \infty} \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1 \} = e^{-(b-1) - \log b} = b e^{-(b-1)}.
$$

Consequently,

$$
\mathbb{P}\{\overline{U}(\tau^U_1) > b\} = b e^{-(b-1)}, \quad \forall b > 1.
$$

We have already noted that $U(\tau^U_1) = \overline{U}(\tau^U_1) - 1$. This completes the proof of Lemma 2.2.

□

**Proof of (2.1).** The Brownian motion $B$ being the Dambis–Dubins–Schwarz Brownian motion associated with the continuous martingale $(U(t) - 4t, t \geq 0)$, it is a $(\mathcal{G}_r)_{r \geq 0}$-Brownian motion (Revuz and Yor [9], Theorem V.1.6), where, for $r \geq 0$,

$$
\mathcal{G}_r := \mathcal{F}_{C(r)}, \quad C(r) := A^{-1}(r), \quad A(t) := \int_0^t U(s) \, ds,
$$

and $A^{-1}$ denotes the inverse of $A$. [We mention that $\mathcal{F}_{C(r)}$ is well defined because $C(r)$ is an $(\mathcal{F}_t)_{t \geq 0}$-stopping time.] As such,

$$
\int_0^{\tau^U_1} U(s) \, ds = A(\tau^U_1).
$$

For all $r \geq 0$, $\{A(\tau^U_1) > r\} = \{\tau^U_1 > C(r)\} \in \mathcal{F}_{C(r)} = \mathcal{G}_r$ (observing that $\tau^U_1$ is an $(\mathcal{F}_t)_{t \geq 0}$-stopping time), which means that $A(\tau^U_1)$ is a $(\mathcal{G}_r)_{r \geq 0}$-stopping time. If $A(\tau^U_1) = \int_0^{\tau^U_1} U(s) \, ds$ has a finite expectation, then we are entitled to apply the (first) Wald identity to see that $\mathbb{E}[B(A(\tau^U_1))] = 0$ as claimed in (2.1).
It remains to prove that $\mathbb{E}[A(\tau_1^U)] < \infty$.

Recall that $U$ is the square of the Euclidean modulus of an $\mathbb{R}^4$-valued Brownian motion. By considering only the first coordinate of this Brownian motion, say $\beta$, we have

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) < a^{1 - \varepsilon} \right\} \leq \mathbb{P}\left\{ \sup_{s \in [0, a]} |\beta(s)| < a^{(1 - \varepsilon)/2} \right\} = \mathbb{P}\left\{ \sup_{s \in [0, 1]} |\beta(s)| < a^{-\varepsilon/2} \right\};$$

so by the small ball probability for Brownian motion, we obtain:

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) < a^{1 - \varepsilon} \right\} \leq \exp(-c_1 a^\varepsilon),$$

for all $a \geq 1$ and all $\varepsilon \in (0, 1)$, with some constant $c_1 = c_1(\varepsilon) > 0$. On the event $\{\sup_{s \in [0, a]} U(s) \geq a^{1 - \varepsilon}\}$, if $\tau_1^U > a$, then for all $i \in [1, a^{1 - \varepsilon} - 1] \cap \mathbb{Z}$, the squared Bessel process $U$, starting from $i$, must first hit position $i + 1$ before hitting $i - 1$ (which, for each $i$, can be realized with probability $\leq 1 - c_2$, where $c_2 \in (0, 1)$ is a constant that does not depend on $i$, nor on $a$). Accordingly,

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) \geq a^{1 - \varepsilon}, \tau_1^U > a \right\} \leq (1 - c_2)^{|a^{1 - \varepsilon} - 1|} \leq \exp(-c_3 a^{1 - \varepsilon}),$$

with some constant $c_3 > 0$, uniformly in $a \geq 2$. We have thus proved that for all $a \geq 2$ and all $\varepsilon \in (0, 1)$,

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_3 a^{1 - \varepsilon}) + \exp(-c_1 a^\varepsilon).$$

Taking $\varepsilon := \frac{1}{2}$, we see that there exists a constant $c_4 > 0$ such that

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_4 a^{1/2}), \quad \forall a \geq 2.$$

On the other hand, $U$ being a squared Bessel process, we have, for all $a > 0$ and all $b \geq a^2$,

$$\mathbb{P}\{A(a) \geq b\} = \mathbb{P}\{A(1) \geq \frac{b}{a^2}\} \leq \mathbb{P}\{ \sup_{s \in [0, 1]} U(s) \geq \frac{b}{a^2} \} \leq e^{-c_5 b/a^2},$$

for some constant $c_5 > 0$. Hence, for $b \geq a^2$ and $a \geq 2$,

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \mathbb{P}\{\tau_1^U > a\} + \mathbb{P}\{A(a) \geq b\} \leq \exp(-c_4 a^{1/2}) + e^{-c_5 b/a^2}.$$

Taking $a := b^{2/5}$ gives that

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \exp(-c_6 b^{1/5}),$$

for some constant $c_6 > 0$ and all $b \geq 4$. In particular, $\mathbb{E}[A(\tau_1^U)] < \infty$ as desired. \[\square\]

\footnote{This is the special case $b_1 := i$ of the argument we have used to obtain (2.3).}
References


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