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The maximal drawdown of the Brownian meander

by

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Summary. Motivated by evaluating the limiting distribution of randomly biased random walks on trees, we compute the exact value of a negative moment of the maximal drawdown of the standard Brownian meander.

Keywords. Brownian meander, Bessel process, maximal drawdown.

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1 Introduction

Let \((X(t), t \in [0, 1])\) be a random process. Its maximal drawdown on \([0, 1]\) is defined by

\[ X^\#(1) := \sup_{s \in [0, 1]} [\overline{X}(s) - X(s)], \]

where \(\overline{X}(s) := \sup_{u \in [0, s]} X(u)\). There has been some recent research interest on the study of drawdowns from a probabilistic point of view ([7], [8]) as well as applications in insurance and finance ([1], [2], [3], [10], [12]).

We are interested in the maximal drawdown \(m^\#(1)\) of the standard Brownian meander \((m(t), t \in [0, 1])\). Our motivation is the presence of the law of \(m^\#(1)\) in the limiting distribution of randomly biased random walks on supercritical Galton–Watson trees ([4]); in particular, the value of \(\mathbb{E}(\frac{1}{m^\#(1)})\) is the normalizing constant in the density function of

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this limiting distribution. The sole aim of the present note is to compute \( \mathbb{E}\left(\frac{1}{m^\pi(1)}\right) \), which turns out to have a nice numerical value.

Let us first recall the definition of the Brownian meander. Let \( W := (W(t), t \in [0, 1]) \) be a standard Brownian motion, and let \( g := \sup\{t \leq 1 : W(t) = 0\} \) be the last passage time at 0 before time 1. Since \( g < 1 \) a.s., we can define
\[
m(s) := \frac{|W(g + s(1 - g))|}{(1 - g)^{1/2}}, \quad s \in [0, 1].
\]
The law of \( (m(s), s \in [0, 1]) \) is called the law of the standard Brownian meander. For an account of general properties of the Brownian meander, see Yen and Yor [11].

**Theorem 1.1.** Let \( (m(s), s \in [0, 1]) \) be a standard Brownian meander. We have
\[
(1.1) \quad \mathbb{E}\left(\frac{1}{\sup_{s \in [0, 1]} |m(s) - m(s)|}\right) = \left(\frac{\pi}{2}\right)^{1/2},
\]
where \( m(s) := \sup_{u \in [0, s]} m(u) \).

The theorem is proved in Section 2.

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N.B. from the first-named coauthors: This note originates from a question we asked our teacher, **Professor Marc Yor (1949–2014)**, who passed away in January 2014, during the preparation of this note. He provided us, in November 2012, with the essential of the material in Section 2.

## 2 Proof

Let \( R := (R(t), t \geq 0) \) be a three-dimensional Bessel process with \( R(0) = 0 \), i.e., the Euclidean modulus of a standard three-dimensional Brownian motion. The proof of Theorem 1.1 relies on an absolute continuity relation between \( (m(s), s \in [0, 1]) \) and \( (R(s), s \in [0, 1]) \), recalled as follows.

**Fact 2.1. (Imhof [5])** Let \( (m(s), s \in [0, 1]) \) be a standard Brownian meander. Let \( (R(s), s \in [0, 1]) \) be a three-dimensional Bessel process with \( R(0) = 0 \). For any measurable and non-negative functional \( F \), we have
\[
\mathbb{E}\left[F(m(s), s \in [0, 1])\right] = \left(\frac{\pi}{2}\right)^{1/2} \mathbb{E}\left[\frac{1}{R(1)} F(R(s), s \in [0, 1])\right].
\]
We now proceed to the proof of Theorem 1.1. Let

\[ L := \mathbb{E}\left( \frac{1}{\sup_{s \in [0,1]} |m(s) - m(s)|} \right). \]

Write \( \overline{R}(t) := \sup_{u \in [0, t]} R(u) \) for \( t \geq 0 \). By Fact 2.1

\[ L = \left( \frac{\pi}{2} \right)^{1/2} \mathbb{E} \left[ \frac{1}{\overline{R}(1)} \sup_{s \in [0,1]} \frac{1}{\overline{R}(s) - R(s)} \right] = \left( \frac{\pi}{2} \right)^{1/2} \int_0^\infty \mathbb{E} \left[ \frac{1}{\overline{R}(1)} \mathbf{1}_{\{ \sup_{s \in [0,1]} [\overline{R}(s) - R(s)] < \frac{1}{a} \}} \right] \, da , \]

the last equality following from the Fubini–Tonelli theorem. By the scaling property, \( \mathbb{E}[\frac{1}{\overline{R}(1)} \mathbf{1}_{\{ \sup_{s \in [0,1]} [\overline{R}(s) - R(s)] < \frac{1}{a} \}}] = \mathbb{E}[\frac{a}{\overline{R}(a^2)} \mathbf{1}_{\{ \sup_{s \in [0,a^2]} [\overline{R}(s) - R(s)] < 1 \}}] \) for all \( a > 0 \). So by means of a change of variables \( b = a^2 \), we obtain:

\[ L = \left( \frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[ \frac{1}{\overline{R}(b)} \mathbf{1}_{\{ \sup_{s \in [0,b]} [\overline{R}(u) - R(u)] < 1 \}} \right] \, db . \]

Define, for any random process \( X \),

\[ \tau^X_1 := \inf \{ t \geq 0 : \overline{X}(t) - X(t) \geq 1 \} , \]

with \( \overline{X}(t) := \sup_{s \in [0, t]} X(s) \). For any \( b > 0 \), the event \( \{ \sup_{u \in [0, b]} [\overline{R}(u) - R(u)] < 1 \} \) means \( \{ \tau^R_1 > b \} \), so

\[ L = \left( \frac{\pi}{8} \right)^{1/2} \int_0^\infty \mathbb{E} \left[ \frac{1}{\overline{R}(b)} \mathbf{1}_{\{ \tau^R_1 > b \}} \right] \, db = \left( \frac{\pi}{8} \right)^{1/2} \mathbb{E} \left( \int_0^{\tau^R_1} \frac{1}{\overline{R}(b)} \, db \right) , \]

the second identity following from the Fubini–Tonelli theorem. According to a relation between Bessel processes of dimensions three and four (Revuz and Yor [9], Proposition XI.1.11, applied to the parameters \( p = q = 2 \) and \( \nu = \frac{1}{2} \)),

\[ R(t) = U \left( \frac{1}{4} \int_0^t \frac{1}{\overline{R}(b)} \, db \right) , \quad t \geq 0 , \]

where \( U := (U(s), s \geq 0) \) is a four-dimensional squared Bessel process with \( U(0) = 0 \); in other words, \( U \) is the square of the Euclidean modulus of a standard four-dimensional Brownian motion.

Let us introduce the increasing functional \( \sigma(t) := \frac{1}{4} \int_0^t \frac{1}{\overline{R}(b)} \, db \), \( t \geq 0 \). We have \( R = U \circ \sigma \), and

\[ \tau^R_1 = \inf \{ t \geq 0 : \overline{R}(t) - R(t) \geq 1 \} \]
\[ = \inf \{ t \geq 0 : \overline{U}(\sigma(t)) - U(\sigma(t)) \geq 1 \} \]
\[ = \inf \{ \sigma^{-1}(s) : s \geq 0 \text{ and } \overline{U}(s) - U(s) \geq 1 \} \]
which is $\sigma^{-1}(\tau_1^U)$. So $\tau_1^U = \sigma(\tau_1^R)$, i.e.,

$$\int_0^{\tau_1^R} \frac{1}{R(b)} \, db = 4\tau_1^U,$$

which implies that

$$L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U).$$

The Laplace transform of $\tau_1^U$ is determined by Lehoczky [6], from which, however, it does not seem obvious to deduce the value of $\mathbb{E}(\tau_1^U)$. Instead of using Lehoczky’s result directly, we rather apply his method to compute $\mathbb{E}(\tau_1^U)$. By Itô’s formula, $(U(t) - 4t, t \geq 0)$ is a continuous martingale, with quadratic variation $4 \int_0^t U(s) \, ds$; so applying the Dambis–Dubins–Schwarz theorem (Revuz and Yor [9], Theorem V.1.6) to $(U(t) - 4t, t \geq 0)$ yields the existence of a standard Brownian motion $B = (B(t), t \geq 0)$ such that

$$U(t) = 2B\left(\int_0^t U(s) \, ds\right) + 4t, \quad t \geq 0.$$

Taking $t := \tau_1^U$, we get

$$U(\tau_1^U) = 2B\left(\int_0^{\tau_1^U} U(s) \, ds\right) + 4\tau_1^U.$$

We claim that

$$(2.1) \quad \mathbb{E}\left[B\left(\int_0^{\tau_1^U} U(s) \, ds\right)\right] = 0.$$

Then $\mathbb{E}(\tau_1^U) = \frac{1}{4} \mathbb{E}[U(\tau_1^U)]$; hence

$$(2.2) \quad L = (2\pi)^{1/2} \mathbb{E}(\tau_1^U) = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)].$$

Let us admit (2.1) for the moment, and prove the theorem by computing $\mathbb{E}[U(\tau_1^U)]$ using Lehoczky [6]’s method; in fact, we determine the law of $U(\tau_1^U)$.

**Lemma 2.2.** The law of $U(\tau_1^U)$ is given by

$$\mathbb{P}\{U(\tau_1^U) > a\} = (a + 1)e^{-a}, \quad \forall a > 0.$$

In particular,

$$\mathbb{E}[U(\tau_1^U)] = \int_0^\infty (a + 1)e^{-a} \, da = 2.$$

Since $L = \left(\frac{\pi}{8}\right)^{1/2} \mathbb{E}[U(\tau_1^U)]$ (see (2.2)), this yields $L = \left(\frac{\pi}{2}\right)^{1/2}$ as stated in Theorem 1.1.
The rest of the note is devoted to the proof of Lemma 2.2 and (2.1).

**Proof of Lemma 2.2.** Fix $b > 1$. We compute the probability $\mathbb{P}\{\overline{U}(\tau_U^1) > b\}$ which, due to the equality $\overline{U}(\tau_U^1) = U(\tau_U^1) + 1$, coincides with $\mathbb{P}\{U(\tau_U^1) > b - 1\}$. By applying the strong Markov property at time $\sigma_0^U := \inf\{t \geq 0 : U(t) = 1\}$, we see that the value of $\mathbb{P}\{\overline{U}(\tau_U^1) > b\}$ does not change if the squared Bessel process $U$ starts at $U(0) = 1$. Indeed, observing that $\sigma_0^U \leq \tau_U^1$, $U(\sigma_0^U) = 1$ and that $\overline{U}(\tau_U^1) = \sup_{s \in [\sigma_0^U, \tau_U^1]} U(s)$, we have

$$\mathbb{P}\{\overline{U}(\tau_U^1) > b\} = \mathbb{P}\left\{ \sup_{s \in [\sigma_0^U, \tau_U^1]} U(s) > b \right\} = \mathbb{P}_1\{\overline{U}(\tau_U^1) > b\},$$

the subscript 1 in $\mathbb{P}_1$ indicating the initial value of $U$. More generally, for $x \geq 0$, we write $\mathbb{P}_x(\cdot) := \mathbb{P}(\cdot | U(0) = x)$; so $\mathbb{P} = \mathbb{P}_0$.

Let $b_0 = 1 < b_1 < \cdots < b_n := b$ be a subdivision of $[1, b]$ such that $\max_{1 \leq i \leq n}(b_i - b_{i-1}) \to 0$, $n \to \infty$. Consider the event $\{\overline{U}(\tau_U^1) > b\}$: since $U(0) = 1$, this means $U$ hits position $b$ before time $\tau_U^1$; for all $i \in [1, n - 1] \cap \mathbb{Z}$, starting from position $b_i$, $U$ must hit $b_{i+1}$ before hitting $b_i - 1$ (caution: not to be confused with $b_{i-1}$). More precisely, let $\sigma_i^U := \inf\{t \geq 0 : U(t) = b_i\}$ and let $U_i(s) := U(s + \sigma_i^U)$, $s \geq 0$; then

$$\{\overline{U}(\tau_U^1) > b\} \subset \bigcap_{i=1}^{n-1} \{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$ 

By the strong Markov property, the events $\{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}$, $1 \leq i \leq n - 1$, are independent (caution: the processes $(U_i(s), s \geq 0)$, $1 \leq i \leq n - 1$, are not independent). Hence

$$(2.3) \quad \mathbb{P}_1\{\overline{U}(\tau_U^1) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}.$$ 

Conversely, let $\varepsilon > 0$, and if $\max_{1 \leq i \leq n}(b_i - b_{i-1}) < \varepsilon$, then we also have

$$\mathbb{P}_1\{\overline{U}(\tau_U^{1+\varepsilon}) > b\} \geq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\},$$

with $\tau_U^{1+\varepsilon} := \inf\{t \geq 0 : \overline{U}(t) - U(t) \geq 1 + \varepsilon\}$. By scaling, $\overline{U}(\tau_U^{1+\varepsilon})$ has the same distribution as $(1 + \varepsilon)\overline{U}(\tau_U^1)$. So, as long as $\max_{1 \leq i \leq n}(b_i - b_{i-1}) < \varepsilon$, we have

$$\mathbb{P}_1\{\overline{U}(\tau_U^1) > b\} \leq \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} \leq \mathbb{P}_1\{\overline{U}(\tau_U^1) > \frac{b}{1 + \varepsilon}\}.$$
Since \( \frac{1}{x} \) is a scale function for \( U \), we have
\[
\mathbb{P}_{b_i} \{ U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1 \} = \frac{1}{b_{i+1} - b_i} - \frac{1}{b_i} = 1 - \frac{1}{b_{i+1} - b_i}.
\]
If \( \lim_{n \to \infty} \max_{0 \leq i \leq n-1} (b_{i+1} - b_i) = 0 \), then for \( n \to \infty \),
\[
\sum_{i=1}^{n-1} \frac{b_i - 1}{b_i - b_{i+1}} = \sum_{i=1}^{n-1} \frac{b_i - 1}{b_i} (b_{i+1} - b_i) + o(1)
\]
\[
\to \int_1^b \frac{r - 1}{r} \, dr = b - 1 - \log b.
\]
Therefore,
\[
\lim_{n \to \infty} \prod_{i=1}^{n-1} \mathbb{P}_{b_i} \{ U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1 \} = e^{-(b-1) \log b} = b e^{-(b-1)}.
\]
Consequently,
\[
\mathbb{P} \{ U(\tau_U) > b \} = b e^{-(b-1)}, \quad \forall b > 1.
\]
We have already noted that \( U(\tau_U) = \bar{U}(\tau_U) - 1 \). This completes the proof of Lemma 2.2.
\( \square \)

**Proof of (2.1).** The Brownian motion \( B \) being the Dambis–Dubins–Schwarz Brownian motion associated with the continuous martingale \( (U(t) - 4t, t \geq 0) \), it is a \((\mathcal{G}_r)_{r \geq 0}\)-Brownian motion (Revuz and Yor [9], Theorem V.1.6), where, for \( r \geq 0 \),
\[
\mathcal{G}_r := \mathcal{F}_{C(r)}, \quad C(r) := A^{-1}(r), \quad A(t) := \int_0^t U(s) \, ds,
\]
and \( A^{-1} \) denotes the inverse of \( A \). [We mention that \( \mathcal{F}_{C(r)} \) is well defined because \( C(r) \) is an \((\mathcal{F}_t)_{t \geq 0}\)-stopping time.] As such,
\[
\int_0^{\tau_U} U(s) \, ds = A(\tau_U).
\]
For all \( r \geq 0 \), \( \{ A(\tau_U) > r \} = \{ \tau_U > C(r) \} \in \mathcal{F}_{C(r)} = \mathcal{G}_r \) (observing that \( \tau_U \) is an \((\mathcal{F}_t)_{t \geq 0}\)-stopping time), which means that \( A(\tau_U) \) is a \((\mathcal{G}_r)_{r \geq 0}\)-stopping time. If \( A(\tau_U) = \int_0^{\tau_U} U(s) \, ds \) has a finite expectation, then we are entitled to apply the (first) Wald identity to see that \( \mathbb{E}[B(A(\tau_U))] = 0 \) as claimed in (2.1).
It remains to prove that $\mathbb{E}[A(\tau_1^U)] < \infty$.

Recall that $U$ is the square of the Euclidean modulus of an $\mathbb{R}^4$-valued Brownian motion. By considering only the first coordinate of this Brownian motion, say $\beta$, we have

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) < a^{1-\varepsilon} \right\} \leq \mathbb{P}\left\{ \sup_{s \in [0, a]} |\beta(s)| < a^{(1-\varepsilon)/2} \right\} = \mathbb{P}\left\{ \sup_{s \in [0, 1]} |\beta(s)| < a^{-\varepsilon/2} \right\};$$

so by the small ball probability for Brownian motion, we obtain:

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) < a^{1-\varepsilon} \right\} \leq \exp(-c_1 a^{\varepsilon}),$$

for all $a \geq 1$ and all $\varepsilon \in (0, 1)$, with some constant $c_1 = c_1(\varepsilon) > 0$. On the event \{sup\,s\in[0,a]|U(s)| \geq a^{1-\varepsilon}\}, if $\tau_1^U > a$, then for all $i \in [1, a^{1-\varepsilon} - 1] \cap \mathbb{Z}$, the squared Bessel process $U$, starting from $i$, must first hit position $i + 1$ before hitting $i - 1$ (which, for each $i$, can be realized with probability $\leq 1 - c_2$, where $c_2 \in (0, 1)$ is a constant that does not depend on $i$, nor on $a$). Accordingly,

$$\mathbb{P}\left\{ \sup_{s \in [0, a]} U(s) \geq a^{1-\varepsilon}, \tau_1^U > a \right\} \leq (1 - c_2) a^{1-\varepsilon} \leq \exp(-c_3 a^{1-\varepsilon}),$$

with some constant $c_3 > 0$, uniformly in $a \geq 2$. We have thus proved that for all $a \geq 2$ and all $\varepsilon \in (0, 1)$,

$$\mathbb{P}\{ \tau_1^U > a \} \leq \exp(-c_3 a^{1-\varepsilon}) + \exp(-c_1 a^{\varepsilon}).$$

Taking $\varepsilon := \frac{1}{2}$, we see that there exists a constant $c_4 > 0$ such that

$$\mathbb{P}\{ \tau_1^U > a \} \leq \exp(-c_4 a^{1/2}), \quad \forall a \geq 2.$$

On the other hand, $U$ being a squared Bessel process, we have, for all $a > 0$ and all $b \geq a^2$,

$$\mathbb{P}\{ A(a) \geq b \} = \mathbb{P}\{ A(1) \geq \frac{b}{a^2} \} \leq \mathbb{P}\left\{ \sup_{s \in [0, 1]} U(s) \geq \frac{b}{a^2} \right\} \leq e^{-c_5 b/a^2},$$

for some constant $c_5 > 0$. Hence, for $b \geq a^2$ and $a \geq 2$,

$$\mathbb{P}\{ A(\tau_1^U) \geq b \} \leq \mathbb{P}\{ \tau_1^U > a \} + \mathbb{P}\{ A(a) \geq b \} \leq \exp(-c_4 a^{1/2}) + e^{-c_5 b/a^2}.$$

Taking $a := b^{2/5}$ gives that

$$\mathbb{P}\{ A(\tau_1^U) \geq b \} \leq \exp(-c_6 b^{1/5}),$$

for some constant $c_6 > 0$ and all $b \geq 4$. In particular, $\mathbb{E}[A(\tau_1^U)] < \infty$ as desired. \[\square\]

\[1\] This is the special case $b_i := i$ of the argument we have used to obtain (2.3).
References


