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On Extension and Torsion of a Compressible Elastic Circular Cylinder

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Dedicated to Millard F. Beatty

Abstract: In this paper we examine the combined extension and torsion of a compressible isotropic elastic cylinder of finite extent. The equilibrium equations are formulated in terms of the principal stretches and then applied to the special case of pure torsion superimposed on a uniform extension (an isochoric deformation). Explicit necessary and sufficient conditions on the strain-energy function for the material to support this deformation with vanishing traction on the lateral surfaces of the cylinder are obtained. Some strain-energy functions satisfying these conditions are considered, existing results are recovered as special cases and new results are obtained. We also point out how the strain-energy functions generated from the considered isochoric deformation considered (of a compressible material) can be used to generate energy functions and corresponding solutions for the incompressible theory.

1. INTRODUCTION

In the incompressible isotropic theory of non-linear elasticity the problem of finite torsion was first considered by Rivlin [1, 2, 3], while relevant experimental issues were discussed in [4] and [5]. Rivlin showed that finite torsion is a universal deformation, sustainable in all such materials in the absence of body forces by the application of surface tractions alone.

Finite torsion is not sustainable, however, in all compressible isotropic elastic materials, since it is not a homogeneous deformation, as required by Ericksen's theorem [6]. Examination of different classes of strain-energy functions for which torsion can be supported is therefore imperative. The torsion problem in the compressible case is discussed from both theoretical and experimental viewpoints in [7]. The same problem was also formulated in [8] and a formula derived for the couple required to maintain the deformation in respect of an arbitrary (isotropic) strain-energy function. Slight compressibility effects were investigated by Faulkner and Haddow [9] using the general theory of small deformation superimposed on large deformation and a Blatz–Ko (solid polyurethane rubber) material model [10]. Also, this same model together with the Levinson–Burgess polynomial material was used in [11] to study torsional deformations for slightly compressible materials.

Currie and Hayes [12] determined constitutive relations for which pure torsion is sustainable and proposed a general class of materials, which includes as a special case the Hadamard material. The Blatz–Ko material for foam polyurethane elastomers [10]
has been studied recently in respect of pure torsion by various authors (see, for example, [13], [14] and [15]). Loss of ellipticity for this material model during a pure torsional deformation was examined by Horgan and Polignone [16].

The work that is closest in character to the present article is that by Polignone and Horgan [17], in which the authors derive a necessary and sufficient condition on the strain energy for pure torsion to be sustainable, but without imposing zero-traction conditions on the lateral surface of the cylinder. They proposed and examined several material models expressed in terms of the principal invariants of the Cauchy–Green deformation tensors.

In the present article we first summarize, in section 2, the necessary kinematics. The required general form of constitutive law for a compressible elastic material is then given in section 3, where we also derive the equilibrium equations appropriate for the combination of torsion with a uniform extension of a circular cylinder. In section 4, we specialize the problem by requiring that the deformation is isochoric and that no change in radius accompanies the torsion (i.e., we consider pure torsion). New necessary and sufficient conditions on the strain-energy function for the material to sustain pure torsion with zero traction on the lateral surface of the cylinder are obtained in terms of the principal stretches. It is shown how these relate to the condition derived in [17], which did not impose the zero-traction requirement. We also show that, for any material satisfying these conditions a simple connection between the axial load on the cylinder and the torque required to achieve the torsion holds, a relation found previously for a special Blatz–Ko material by Beatty [13].

In section 5 we then focus on analysing the necessary and sufficient conditions when expressed in terms of the principal invariants $I_1, I_2, I_3$ of the Cauchy–Green deformation tensor. We note that a special form of the Blatz–Ko material satisfying these condition was shown by Polignone and Horgan [17] to meet the requirements considered. We also observe that a class of materials considered by Jiang and Ogden [18] in the context of axial shear of a circular cylindrical tube satisfies the same requirements. Several specific forms of strain-energy function are examined. We emphasize that for a strain-energy function that does not satisfy the necessary and sufficient conditions mentioned above but does satisfy the condition given in [17], a distribution of surface tractions is required on the lateral surface in order to maintain the pure torsional deformation. These tractions are calculated for some particular energy functions.

In section 6 we examine the modifications to the theory needed for consideration of energy functions expressed in terms of the principal invariants of the stretch tensors, and again some particular examples are studied.

In spirit this paper follows the work of Jiang and Ogden [18, 19] in that an isochoric form of deformation is used to place restrictions on the form of the strain-energy function by means of the radial equation of equilibrium. This general approach (which need not be restricted to consideration of isochoric deformations) has provided a systematic way of deriving new solutions to several boundary-value problems.

### 2. TORSION COMBINED WITH UNIFORM AXIAL EXTENSION

We consider a compressible non-linearly elastic circular cylinder whose cross-section in its natural (unstressed) configuration is defined by
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\[ 0 \leq R \leq A, \quad 0 \leq \Theta \leq 2\pi, \quad 0 \leq Z \leq L, \] (1)

where \( R, \Theta, Z \) are cylindrical polar coordinates.

In the deformed configuration the cylinder is subjected to a uniform axial stretch \( \lambda_z \), which is held fixed while a torsional deformation is superimposed. The plane face \( z = \lambda_z L \) is then rotated, while that at \( z = 0 \) is fixed, in such a way that plane sections of the cylinder normal to its axis remain plane and the radius turns through an angle \( \lambda_z \tau Z \), where \( \tau \geq 0 \) represents the angular twist per unit length of the deformed cylinder. The resulting deformation is then defined by

\[ r = r(R), \quad \theta = \Theta + \lambda_z \tau Z, \quad z = \lambda_z Z, \] (2)

where \( r, \theta, z \) are the cylindrical polar coordinates of a material point in the deformed configuration.

The matrix representation, \( \mathbf{F} \) say, of the deformation gradient tensor \( \mathbf{F} \) with respect to the two sets of cylindrical polar coordinates has the form

\[
\mathbf{F} = \begin{pmatrix}
\dot{r} & 0 & 0 \\
0 & r/R & \lambda_z \tau R \\
0 & 0 & \lambda_z \\
\end{pmatrix},
\] (3)

where the dot signifies differentiation with respect to \( R \). The left Cauchy–Green strain tensor \( \mathbf{B} = \mathbf{F} \mathbf{F}^T \) has the matrix representation

\[
\mathbf{B} = \begin{pmatrix}
\dot{r}^2 & 0 & 0 \\
0 & (r/R)^2 + (\lambda_z^2 \tau^2 R^2) & \lambda_z^2 \tau R \\
0 & \lambda_z^2 \tau R & \lambda_z^2 \\
\end{pmatrix}.
\] (4)

Let \( \mathbf{v}^{(i)}, i = 1, 2, 3, \) be the unit Eulerian principal axes associated with this deformation—that is, the principal axes of \( \mathbf{B} \). We see that the radial unit vector \( \mathbf{e}_r \) is the Eulerian principal axis associated with the principal stretch

\[ \lambda_1 = \dot{r}. \] (5)

We may express the remaining two principal directions in terms of the cylindrical polar axes \( \mathbf{e}_\theta, \mathbf{e}_z \). Thus, we write

\[
\mathbf{v}^{(2)} = \cos \phi \mathbf{e}_\theta + \sin \phi \mathbf{e}_z, \quad \mathbf{v}^{(3)} = -\sin \phi \mathbf{e}_\theta + \cos \phi \mathbf{e}_z,
\] (6)

where \( \phi \) defines the orientation of the axes \( \mathbf{v}^{(2)}, \mathbf{v}^{(3)} \) relative to \( \mathbf{e}_\theta, \mathbf{e}_z \).

The polar decomposition theorem guarantees the existence and uniqueness of the (positive definite, symmetric) left stretch tensor \( \mathbf{V} \) through

\[
\mathbf{F} = \mathbf{VR},
\] (7)

where \( \mathbf{R} \) is a proper orthogonal tensor. The spectral decomposition of \( \mathbf{V} \) has the form
\[ \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \]  

which identifies the (positive) principal stretches \( \lambda_1, \lambda_2, \lambda_3 \) of the deformation.

By defining the rotation matrix

\[
\mathbf{P} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix}
\]

and comparing the entries of \( \mathbf{B} \) and \( \mathbf{V} \) (the matrix representation of \( \mathbf{V} \)) through \( \mathbf{B} = \mathbf{P} \mathbf{V} \mathbf{P}^T \),

we obtain the connections

\[ \lambda_2^2 \cos^2 \phi + \lambda_3^2 \sin^2 \phi = \frac{r^2}{R^2} + \lambda_2^2 \tau^2 R^2, \]  

\[ \lambda_2^2 \sin^2 \phi + \lambda_3^2 \cos^2 \phi = \lambda_2^2, \]  

\[ (\lambda_2^2 - \lambda_3^2) \sin \phi \cos \phi = \lambda_2^2 \tau R, \]

from which it may be deduced that

\[ \lambda_2^2 + \lambda_3^2 = \frac{r^2}{R^2} + \lambda_2^2 \tau^2 R^2 + \lambda_2^2, \]  

\[ (\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2) = \lambda_2^2 \tau^2 R^2, \]

and

\[ \lambda_2 \lambda_3 = \lambda_2^2 \frac{r}{R}. \]  

Further, we obtain the explicit expression for \( \phi \) in the form

\[ \cos 2\phi = \frac{\lambda_2^2 + \lambda_3^2 - 2\lambda_2^2}{\lambda_2^2 - \lambda_3^2}. \]

Counterparts of the above formulae for the Lagrangian principal axes can be found in ([20], section 5.2.5).

### 3. EQUILIBRIUM AND STRESS DEFORMATION EQUATIONS

For an isotropic elastic material, which we are considering here, the Cauchy stress tensor \( \mathbf{\sigma} \), is coaxial with the left Cauchy–Green strain tensor \( \mathbf{B} \). The non-zero components of \( \mathbf{\sigma} \) in the current configuration can therefore be expressed in terms of its principal stresses \( \sigma_1, \sigma_2, \sigma_3 \) through
\[ \sigma_{rr} = \sigma_1, \quad \sigma_{\theta \theta} = (\sigma_2 - \sigma_3) \cos \phi \sin \phi, \quad \sigma_{zz} = \sigma_2 \sin^2 \phi + \sigma_3 \cos^2 \phi. \]  

(17)

(18)

The connection

\[ \lambda_z^2 \tau \sigma_{\theta \theta} - \sigma_{zz} = (r^2/R^2 + \lambda_z^2 \tau^2 r^2 - \lambda_z^2) \sigma_{\theta \theta}. \]  

(19)

(see also Holzapfel et al. [21]) may be obtained from (17) and (18) on use of (16). Note that this is not a universal relation since \( r(R) \) depends on the solution of the equilibrium equation and hence, in general, on the form of the strain-energy function. This contrasts with the situation for incompressible materials, for which the counterpart of (19) is a universal relation. For a general discussion of universal relations we refer the reader to the recent review by Saccomandi [22].

Since we are considering compressible materials, the principal stresses are given by

\[ \lambda_1, \lambda_2, \lambda_3 \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i}, \quad i \in \{1, 2, 3\} \text{ (no summation over } i), \]  

(20)

where \( W = W(\lambda_1, \lambda_2, \lambda_3) \) is the strain energy per unit reference volume, which is a symmetric function of the principal stretches. For consistency with the classical theory \( W \) should satisfy

\[ W(1, 1, 1) = 0, \quad W_i(1, 1, 1) = 0, \quad i = 1, 2, 3, \]  

(21)

\[ W_{ij}(1, 1, 1) = \kappa - \frac{2}{3} \mu, : i \neq j, \quad W_{ii}(1, 1, 1) = \kappa + \frac{4}{3} \mu, \quad i = 1, 2, 3, \]  

(22)

where, in the latter, no summation is implied by the repetition of the index \( i \), and the notation \( W_i = \partial W/\partial \lambda_i, W_{ij} = \partial^2 W/\partial \lambda_i \partial \lambda_j \) is adopted. In (22), \( \mu(>0) \) denotes the shear modulus and \( \kappa(>0) \) the bulk modulus in the natural configuration.

The principal invariants \( I_1, I_2, I_3 \) of \( B \) are given in terms of the principal stretches by

\[ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2, \quad I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2, \]  

(23)

and we write \( \bar{W} \) to represent the energy function when expressed in terms of \( I_1, I_2, I_3 \). Similarly, we write \( \bar{W} \) for dependence on the principal invariants \( i_1, i_2, i_3 \) of \( V \), these being defined by

\[ i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2 \lambda_3 + \lambda_3 \lambda_1 + \lambda_1 \lambda_2, \quad i_3 = \lambda_1 \lambda_2 \lambda_3. \]  

(24)

We also note the connections

\[ I_1 = i_1^2 - 2i_2, \quad I_2 = i_2^2 - 2i_1i_3, \quad I_3 = i_3^2. \]  

(25)

Thus, we have
\[ W(\lambda_1, \lambda_2, \lambda_3) = \bar{W}(I_1, I_2, I_3) = \bar{W}(i_1, i_2, i_3). \]  

(26)

The analogues of (21) and (22) for \( \bar{W} \) and \( \bar{\ddot{W}} \), respectively, are

\[
\bar{W}(3, 3, 1) = 0, \quad \bar{W}_1 + \bar{W}_2 = -(\bar{W}_2 + \bar{W}_3) = \frac{\mu}{2}, \tag{27}
\]

\[
\bar{W}_{11} + 4\bar{W}_{12} + 4\bar{W}_{22} + 2\bar{W}_{13} + 4\bar{W}_{23} + \bar{W}_{33} = \frac{\kappa}{4} + \frac{\mu}{3}, \tag{28}
\]

where the derivatives are evaluated for \( I_1 = 3, I_2 = 3, I_3 = 1 \), and

\[
\bar{\ddot{W}}(3, 3, 1) = 0, \quad \bar{\ddot{W}}_1 + \bar{\ddot{W}}_2 = -(\bar{\ddot{W}}_2 + \bar{\ddot{W}}_3) = 2\mu, \tag{29}
\]

\[
\bar{\dddot{W}}_{11} + 4\bar{\dddot{W}}_{12} + 4\bar{\dddot{W}}_{22} + 2\bar{\dddot{W}}_{13} + 4\bar{\dddot{W}}_{23} + \bar{\dddot{W}}_{33} = \kappa + \frac{4\mu}{3}, \tag{30}
\]

with the derivatives evaluated for \( i_1 = 3, i_2 = 3, i_3 = 1 \). In each case the subscripts 1, 2, 3 indicate differentiation with respect to the relevant arguments, i.e., \( I_1, I_2, I_3 \) or \( i_1, i_2, i_3 \) respectively.

Since the deformation gradient depends only on the radial coordinate, the equilibrium equation \( \text{div} \ \sigma = 0 \) (in the absence of body forces) reduces to the radial equation

\[
\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0. \tag{31}
\]

In terms of the principal stretches equation (31) can be written in the form

\[
\frac{d}{dR} (RW_1) = \frac{\lambda_2}{\lambda_2 \lambda_3} \frac{\lambda_2 (\lambda_2^2 - \lambda_3^2) W_2 - \lambda_3 (\lambda_2^2 - \lambda_3^2) W_3}{\lambda_2^2 - \lambda_3^2}. \tag{32}
\]

This equation is identical to the one derived in [20] based on a Lagrangian formulation of the problem.

Note that for the problems considered here there is just one equilibrium equation. This contrasts with the situation for the (isochoric) azimuthal shear and axial shear problems for a circular cylindrical tube considered by Jiang and Ogden [19, 18] and others. In each of these problems there are two equilibrium equations to be satisfied for the (unknown) radially dependent deformation field. Compatibility of these two equations leads to restrictions on the form of strain-energy function. Appropriate shear and radial tractions are required on the lateral surfaces of the tube in order to maintain the deformation. In the current problem the equilibrium equation serves to determine \( r (R) \) for any given form of (compressible isotropic) strain-energy function. On the lateral surface of the cylinder there will in general be radial tractions, but we may also wish to consider the possibility that the lateral surface is traction free. In this case the boundary condition to be satisfied is
\[ \sigma_{rr} = \sigma_1 = 0 \quad \text{on } R = A, \] (33)

and the deformation is then maintained by applying an axial load and torque.

4. Isochoric Specialization

We now specialize the deformation so that it is isochoric, comprising an isochoric simple tension and an isochoric torsion. We therefore have \( r = \lambda_z^{-1/2} R \) and equations (5), (15) and (13) respectively reduce to

\[ \lambda_1 = \lambda_z^{-1/2}, \quad \lambda_2 \lambda_3 = \lambda_z^{1/2}, \quad \lambda_2^2 + \lambda_3^2 = \lambda_z^2 + \lambda_z^{-1} + \lambda_z \tau^2 R^2. \] (34)

Since the deformation is isochoric \( I_3 = 1 \) and the principal invariants \( I_1, I_2 \) are given by

\[ I_1 = \lambda_z^2 + 2\lambda_z^{-1} + \lambda_z \tau^2 R^2, \quad I_2 = 2\lambda_z + \lambda_z^{-2} + \tau^2 R^2. \] (35)

Corresponding expressions may be written down for \( i_1, i_2 \) but we do not need them here.

In terms of the stretches, the equations (17) and (18) for the stress components may now be given the explicit forms

\[ \sigma_{rr} = \lambda_z^{-1/2} W_1, \quad \sigma_{\theta \theta} = \sqrt{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)} \frac{\lambda_2 W_2 - \lambda_3 W_3}{\lambda_2^2 - \lambda_3^2}, \] (36)

\[ \sigma_{zz} = \frac{(\lambda_2^2 - \lambda_3^2)\lambda_2 W_2 - (\lambda_2^2 - \lambda_3^2)\lambda_3 W_3}{\lambda_2^2 - \lambda_3^2}, \] (37)

while the equilibrium equation (32) specializes accordingly.

On use of (34) we may write the equilibrium equation explicitly in terms of the stretches. After some manipulation this leads to

\[ (\lambda_2^2 + \lambda_3^2 - \lambda_z^2 - \lambda_z^{-1}) \frac{\lambda_2 W_{12} - \lambda_3 W_{13}}{\lambda_2^2 - \lambda_3^2} + W_1 = \lambda_z^{1/2} \frac{(\lambda_2^2 - \lambda_z^2)\lambda_2 W_2 - (\lambda_2^2 - \lambda_z^2)\lambda_3 W_3}{\lambda_2^2 - \lambda_3^2}, \] (39)

in which the derivatives of \( W \) are evaluated for (34). This provides a necessary and sufficient condition for the energy function to admit the deformation considered and generalizes the result given in [17] for \( \lambda_z = 1 \) to the case \( \lambda_z \neq 1 \).

When \( \lambda_z = 1 \) equation (39) reduces to

\[ \gamma (\dot{\lambda} W_{12} - \lambda^{-1} W_{13}) + (\dot{\lambda} + \lambda^{-1}) W_1 = \lambda^2 W_2 + \lambda^{-2} W_3, \] (40)
where \( \gamma \) is defined by

\[
\gamma = \lambda - \lambda^{-1} = \tau R, \tag{41}
\]

and we have set \( \lambda_2 = \lambda, \lambda_3 = \lambda^{-1} \). In this special case the deformation is locally a simple shear of magnitude \( \gamma \) in the \((e_\theta, e_z)\) plane and is referred to as pure torsion.

Equation (40) is equivalent to the result given in [17] except that a factor of \( \gamma \) has been removed from the latter. As pointed out in [17], satisfaction of (40) does not guarantee that the zero-traction boundary condition on the lateral surface of the cylinder is satisfied and therefore, in general, appropriate radial tractions need to be supplied in order to maintain the deformation.

We now obtain necessary and sufficient conditions on the energy function that ensure that the deformation is sustainable and that, in addition, the zero-traction boundary condition is met.

Consider the stress \( \sigma_{rr} \) given by (36). Since this is evaluated for the deformation given by (34) and \( \lambda_2 \) is constant, \( \sigma_{rr} \) depends on the deformation and the radial coordinate only through the combination \( \tau R \). We therefore write \( \sigma_{rr}(R) = \sigma(\tau R) \) to represent this dependence. On the lateral surface we then have \( \sigma_{rr}(A) = \sigma(\tau A) \). If we require the radial traction to vanish then, from (33),

\[
\sigma_{rr}(A) = \sigma(\tau A) = 0 \quad \text{for all } \tau \geq 0. \tag{42}
\]

Hence

\[
\frac{d}{d\tau} \sigma_{rr}(A) = \sigma'(\tau A)A = 0 \quad \text{for all } \tau > 0, \tag{43}
\]

and it then follows that, for any fixed \( \tau > 0 \),

\[
\frac{d}{dR} \sigma_{rr}(R) = \sigma'(\tau R)\tau = 0 \quad \text{for all } R > 0 \tag{44}
\]

(although we only require it to hold for \( 0 < R < A \)). Thus, \( \sigma_{rr}(R) \) is constant, and since it vanishes for \( R = A, \sigma_{rr} \equiv 0 \). From (31) we deduce that \( \sigma_{\theta \theta} \equiv 0 \) also.

The conditions on the strain-energy function that ensure these identities are, from (36) and (37),

\[
W_1 = 0, \quad \lambda_2(\lambda_2^2 - \lambda_3^2)W_2 - \lambda_3(\lambda_3^2 - \lambda_2^2)W_3 = 0, \tag{45}
\]

in which the terms are evaluated for the stretches given by (34). The conditions (45) are necessary and sufficient for the strain-energy function to admit the combined isochoric torsion and uniform extension with zero tractions on the lateral surface of the cylinder. It is worth noting that, since the tractions vanish, these conditions also ensure that the deformation is supported by a circular cylindrical tube. Note that derivation of the second equation in (45) requires the assumption that \( \lambda_2 \neq \lambda_3 \). This is justified since it is easy to show from (34) that \( \lambda_2 = \lambda_3 \) only in the trivial situation \( \tau = 0, \lambda_z = 1 \) with \( \lambda_2 = \lambda_3 = 1 \).

When \( \lambda_2 = 1 \) equations (45) reduce to
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\[ W_1 = 0, \quad \lambda^2 W_2 + \lambda^{-2} W_3 = 0, \quad (46) \]
evaluated for \( \lambda_2 = \lambda, \lambda_3 = \lambda^{-1}, \lambda - \lambda^{-1} = \tau R. \)

Beatty [13], in considering a special Blatz–Ko form of strain-energy function, showed that for pure torsion (\( \lambda_z = 1 \)) the resultant axial load \( \mathcal{N} \) on any cross-section of the cylinder and the resultant moment \( \mathcal{M} \) are related by

\[ \mathcal{N} = -\tau \mathcal{M}. \quad (47) \]

See also the discussion in [17]. We now show that this holds for any strain-energy function satisfying the necessary and sufficient conditions derived above.

For the more general situation with \( \lambda_z \neq 1 \) we have

\[ \mathcal{N} = 2\pi \lambda_z^{-1} \int_0^A \sigma_{zz} R \, dR, \quad \mathcal{M} = 2\pi \lambda_z^{-3/2} \int_0^A \sigma_{0z} R^2 \, dR, \quad (48) \]

where the integrals (which are independent of \( Z \)) are taken over any cross-section of the cylinder. On specializing the relation (19) to the deformation (34) and setting \( \sigma_{00} = 0 \), we obtain

\[ \mathcal{N} = -\tau \mathcal{M} + 2\pi \tau^{-1}(\lambda_z^{-1/2} - \lambda_z^{-7/2}) \int_0^A \sigma_{0z} \, dR, \quad (49) \]

so (47) follows when \( \lambda_z = 1 \). The formulae (48) and (49), with appropriate changes in the limits of the integrals, also apply for a tube.

When \( \lambda_z = 1 \) and pure torsion is considered equation (19) reduces to \( \sigma_{00} - \sigma_{zz} = \gamma \sigma_{0z} \). Thus, when \( \sigma_{00} = 0 \) we see that the connection \( \sigma_{zz} = \gamma \sigma_{0z} \), and hence (47), is a necessary condition for pure torsion to be admitted in the absence of surface tractions on the lateral surface. A similar statement can be made for the case \( \lambda_z \neq 1 \).

5. ENERGY FUNCTIONS IN TERMS OF \( I_1, I_2, I_3 \)

When expressed in terms of the strain energy \( \bar{W}(I_1, I_2, I_3) \), the Cauchy stress components (36)–(38) take the forms

\[
\begin{align*}
\sigma_{rr} &= 2\lambda_z^{-1}[\bar{W}_1 + 2\lambda_z^2 \bar{W}_2 + \lambda_z^2 \bar{W}_3], \\
\sigma_{0z} &= 2(\bar{W}_1 + \lambda_z^2 \bar{W}_2 - \lambda_z^2 \bar{W}_3), \\
\sigma_{00} &= 2\lambda_z^2 (\lambda_z^2 + \lambda_z^2 - \lambda_z^2 + 2\lambda_z^2 \bar{W}_2 + \lambda_z^2 \bar{W}_3), \\
\sigma_{zz} &= 2\lambda_z^2 \bar{W}_2 + 4\lambda_z^2 \lambda_z + 2\lambda_z^2 \bar{W}_3.
\end{align*}
\]

The combination of the identities \( \sigma_{rr} = \sigma_{00} \equiv 0 \) then leads to
\[ W_1 = 0, \quad \tilde{W}_2 (\lambda_2^2 + \lambda_3^2) + \tilde{W}_3 \lambda_z = 0, \]  

(54)
evaluated for (34). These are equivalent to (45). The corresponding equivalent of (40) is

\[ \tilde{W}_{11} + I \tilde{W}_{12} + (I - 1) \tilde{W}_{22} + \tilde{W}_{13} + \tilde{W}_{23} + \tilde{W}_2 - \frac{1}{2} \tilde{W}_1 = 0, \]  

(55)
as given by Polignone and Horgan [17], where \( I = I_1 = I_2 = 3 + \gamma^2, I_3 = 1, \gamma = \tau R \). In [17], \( I \) was left as \( 3 + \tau^2 R^2 \) and coefficients of terms involving \( \tau^2 R^2 \) were set to zero in order to find restrictions on the energy function that ensured satisfaction of (55). This approach is more restrictive than the one we adopt in section 5.1. For the case \( \lambda_z = 1 \), it is easy to show by differentiation of (54) and use of (35) that (55) follows from (54). For \( \lambda_z \neq 1 \) a similar deduction can be made.

When equations (54) hold it follows from (51) and (53) that

\[ \sigma_{\theta z} = 2\lambda_z^{-1} \tilde{W}_2 \sqrt{(\lambda_2^2 - \lambda_z^2)(\lambda_3^2 - \lambda_z^2)} \]  

(56)
and

\[ \sigma_{zz} = -2\lambda_z^{-1} \tilde{W}_2 (\lambda_2^2 + \lambda_3^2 - 2\lambda_z^2), \]  

(57)
and when \( \lambda_z = 1 \) these reduce to

\[ \sigma_{\theta z} = 2\gamma \tilde{W}_2, \quad \sigma_{zz} = -2\gamma^2 \tilde{W}_2. \]  

(58)

Since \( I_3 = 1 \) the first equation in (54) shows that \( W(I_1, I_2, 1) \) is independent of \( I_1 \) provided that \( I_1 \) and \( I_2 \) are independent. Note, however, that in the second equation in (54) the term \( \tilde{W}_3 (I_1, I_2, 1) \) may depend on \( I_1 \). In this case

\[ \lambda_2^2 + \lambda_3^2 = I_1 - \lambda_z^{-1} = \lambda_z I_2 - \lambda_z^2, \]  

(59)
can be used (in principle) to determine \( \lambda_z \) in terms of \( I_1 \) and \( I_2 \) and the second equation in (54) can then be written as an ordinary differential equation with \( I_z \) as the independent variable. In general this is impractical if not impossible. To illustrate the procedure, therefore, we shall restrict attention to \( \lambda_z = 1 \) in what follows.

5.1. Pure torsion

If \( \lambda_z = 1 \) we have \( I_2 = I_1 \), and (54) does not then imply that \( W \) is independent of \( I_1 \) since it applies only for the restricted manifold \( I_2 = I_1, I_3 = 1 \) in \( (I_1, I_2, I_3) \)-space. If, however, \( W \) is independent of \( I_1 \), then (54) is automatically satisfied.

5.1.1. JO2 materials

An example of a class of strain-energy functions independent of \( I_1 \) is that introduced by Jiang and Ogden [18]. We write this as \( W = W_{J02}(I_2, I_3) \), where
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\[ W_{JO2}(I_2, I_3) = g(I_2)h_1(I_3) + h_2(I_3), \]  

(60)

\[ g, h_1, h_2 \] are functions to be determined and in the subscript JO2 the 2 is used to distinguish (60) from a class of energy functions introduced by Jiang and Ogden [19], for which the subscript JO1 will be used later.

Substitution of (60) into (54)_2 leads to an expression for the function \( g \), and hence (60) may be given the explicit form

\[ \tilde{W}_{JO2}(I_2, I_3) = \frac{\mu}{2k} (I_2 - 1)^k h_1(I_3) + h_2(I_3), \]  

(61)

where \( k \neq 0 \) is a disposable parameter and, for consistency with (27) and (28), \( h_1, h_2 \) must satisfy

\[ h_1(1) = 1, \quad h_1'(1) = -k, \quad h_2(1) = -\frac{\mu}{k}, \quad h_2'(1) = 0, \]  

(62)

and

\[ h_2''(1) + \frac{\mu}{k} h_2'''(1) = \frac{\kappa}{4} + \frac{\mu}{3} + (k + 1)\mu. \]  

(63)

In the case \( k = 0 \) equation (61) is replaced by

\[ \tilde{W}_{JO2}(I_2, I_3) = \mu \log (I_2 - 1)h_1(I_3) + h_2(I_3), \]  

(64)

and (62) and (63) by

\[ h_1(1) = 1, \quad h_1'(1) = 0, \quad h_2(1) = -\mu \log 2, \quad h_2'(1) = -\mu, \]  

(65)

and

\[ h_2''(1) + \mu (\log 2) h_2'''(1) = \frac{\kappa}{4} + \frac{4\mu}{3}. \]  

(66)

Using (58) the shear stress \( \sigma_{\theta z} \) is now calculated from (61) and (64) as

\[ \sigma_{\theta z} = \frac{\mu (2 + \gamma^2)^k - 1}{2^{k-1}} \gamma, \]  

(67)

which applies for all \( k \).

Differentiation of (67) with respect to \( \gamma \) gives

\[ \frac{d\sigma_{\theta z}}{d\gamma} = \frac{\mu}{2^{k-1}}(2 + \gamma^2)^{k-2} [2 + (2k - 1)\gamma^2]. \]  

(68)

Three separate cases arise:

(i) \( k > \frac{1}{2} \): \( \sigma_{\theta z} \) increases monotonically and \( \sigma_{\theta z} \to \infty \) as \( \gamma \to \infty \);

(ii) \( k = \frac{1}{2} \): \( \sigma_{\theta z} \) increases monotonically and \( \sigma_{\theta z} \to \sqrt{2}\mu \) as \( \gamma \to \infty \);

(iii) \( k < \frac{1}{2} \): \( \sigma_{\theta z} \) has a maximum and \( \sigma_{\theta z} \to 0 \) as \( \gamma \to \infty \).
Of the three cases (i)–(iii), only (i) can be regarded as yielding a stable solution for all values of the applied twist \( \tau > 0 \). But even in this case stability may be lost at some critical value of \( \tau \) if there is an energetic preference for departure from the pure torsion. Analysis of stability is beyond the scope of the present paper, but will be examined elsewhere.

Finally in this section we calculate the resulting torque \( \mathcal{M} \) using (48)\(_2\) with \( \lambda_z = 1 \) and (67). This gives

\[
\mathcal{M} = \frac{\pi \mu A^3}{2^{k-1} k(k+1) \gamma_4^2} \left[ \frac{(2 + \gamma_4^2)^k (k \gamma_4^2 - 2) + 2^{k+1}}{k \neq 0, -1, (69)} \right]
\]

where \( \gamma_4 = \tau A \). Corresponding results may be obtained for \( k = 0 \) and \( k = -1 \) but we omit them here.

We give results for two specific values of \( k \). For \( k = 1 \) equation (69) gives the standard result

\[
\mathcal{M} = \frac{1}{2} \pi \mu \tau A^4
\]

obtainable for the neo-Hookean strain energy in the incompressible theory and also for the special Blatz–Ko material considered in [17]. For \( k = 2 \) the corresponding specialization of (69) is

\[
\mathcal{M} = \frac{1}{2} \pi \mu \tau A^4 \left( 1 + \frac{1}{3} \tau^2 A^2 \right).
\]

Since \( \mathcal{M} \) is positive for all \( \tau > 0 \) in these examples it follows from (47) that \( \mathcal{N} < 0 \). Thus, a compressive axial load is required to maintain \( \lambda_z = 1 \), suggesting that in the absence of axial load the cylinder would lengthen. This conclusion also applies for all other \( k \) since it is easy to show from (69) that \( \mathcal{M} > 0 \) for all \( \tau > 0 \).

5.1.2. Blatz–Ko considerations

We now consider a modification of theJO2 material class (60) of the form

\[
\tilde{W} = g(I_2/I_3)h_1(I_3) + h_2(I_3).
\]

This is motivated by reference to the special Blatz–Ko material of the form

\[
\tilde{W} = \frac{\mu}{2} \left[ I_2 - 1 - \frac{1}{v} + \frac{1 - 2v}{v} I_3^{\nu/(1-2\nu)} \right],
\]

where \( v \) is Poisson’s ratio, given by

\[
v = \frac{3\kappa - 2\mu}{2(3\kappa + \mu)}
\]

in terms of the bulk modulus \( \kappa \) and shear modulus \( \mu \).

The procedure used to determine the function \( g \) in respect of theJO2 materials leads, in this case, to \( \tilde{W} \) having the form...
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\[ \bar{W} = \frac{\mu}{2k} \exp[k(I_2/I_3 - 3)]h_1(I_3) + h_2(I_3), \] (75)

provided that \( k \neq 0 \), where

\[ h_1(1) = 1, \quad h'_1(1) = k, \quad h_2(1) = -\frac{\mu}{2k}, \quad h'_2(1) = 0 \] (76)

and

\[ h''_2(1) + \frac{\mu}{2k} h''_1(1) = \frac{\kappa}{4} + \frac{k\mu}{2} - \frac{2\mu}{3}. \] (77)

These should be contrasted with the corresponding expressions (61)–(63) for JO2 materials.

If \( k = 0 \) equations (75)–(77) are replaced by

\[ \bar{W} = \frac{\mu I_3}{2 I_3} h_1(I_3) + h_2(I_3), \] (78)

\[ h_1(1) = 1, \quad h'_1(1) = 0, \quad h_2(1) = -\frac{3\mu}{2}, \quad h''_2(1) = \mu \frac{2}{3} \] (79)

and

\[ h''_2(1) + \frac{3\mu}{2} h'_1(1) = \frac{\kappa}{4} - \frac{2\mu}{3}, \] (80)

respectively.

For the special case in which \( h_1(I_3) \equiv 1 \), equations (78)–(80) reduce to

\[ \bar{W} = \frac{\mu I_3}{2 I_3} + h_2(I_3), \] (81)

\[ h_2(1) = -\frac{3\mu}{2}, \quad h''_2(1) = \mu \frac{2}{3}, \quad h''_2(1) = 0, \] (82)

which provides a slight generalization of the Blatz–Ko model (73) in that \( h_2(I_3) \) need not be of the specific form required by (73).

In respect of (75) and (78) the shear stress is given by \( \sigma_{\theta z} = \mu \gamma \exp(k\gamma^2) \), and this is a monotonically increasing function of \( \gamma \) for all \( \gamma \) if and only if \( k \geq 0 \). The resultant torque on the ends of the cylinder is calculated as

\[ \mathcal{M} = \frac{n \mu A^3}{k^2 \gamma^2} [1 - \exp(k\gamma^2) + k\gamma A \exp(k\gamma^2)], \quad k \neq 0, \] (83)

while for \( k = 0 \) it is again given by (70). In each case \( \mathcal{M} > 0 \) and \( \mathcal{N} < 0 \), as for the JO2 materials.
5.1.3. Generation of other energy functions

One possible approach to finding energy functions that can support pure torsion is to start with a relatively simple energy function and then extend it by the addition of a term which can then be determined by application of the necessary and sufficient conditions. We illustrate this method with just one example. Consider the energy function containing a term linear in \( I_2 \) and given by

\[
W = \frac{\mu a}{2} [I_2 h_1(I_3) + g(I_2)h_2(I_3) + h_3(I_3)],
\]  

(84)

where \( a \) is a constant and the function \( g \) is to be determined, while the functions \( h_1, h_2, h_3 \) are to be consistent with the requirements (27) and (28). Two separate possibilities arise when (84) is substituted into (54)\(_2\) for \( \lambda_2 = 1 \).

In the first case \( g \) is quadratic in \( I_2 \) and (84) can be written as

\[
W = \frac{\mu a}{2} [I_2 h_1(I_3) + \frac{1}{2}(k + 1)I_2^2 h_2(I_3) + h_3(I_3)],
\]  

(85)

where

\[
a = \frac{1}{(3k + 4)}, \quad h_1(1) = h_2(1) = 1, \quad h_1'(1) = k, \quad h_2'(1) = -2, \quad h_3'(1) = 1,
\]  

(86)

\[
h_3(1) = \frac{3}{2}(3k + 5), \quad h_3''(1) = \frac{9}{2}(k + 1)h_2''(1) + 3h_1''(1) = (3k + 4)\frac{\kappa}{2\mu} + 18k + \frac{68}{3},
\]  

(87)

(88)

and \( k \) is a disposable constant.

In the second case (84) becomes

\[
W = \frac{\mu a}{2} [I_2 h_1(I_3) - \frac{(a - 1)}{ma2^m-1}(I_2 - 1)^m h_2(I_3) + h_3(I_3)],
\]  

(90)

where

\[
h_1(1) = h_2(1) = 1, \quad h_1'(1) = -1, \quad h_2'(1) = -m,
\]  

(89)

\[
h_3(1) = -\frac{1}{ma}[2 + (3m - 2)a], \quad h_3'(1) = 1,
\]  

(90)

and

\[
h_3''(1) - \frac{2}{ma}(a - 1)h_2''(1) + 3h_1''(1) = \frac{\kappa}{2\mu a} + 4 + \frac{2}{3a} - 2(m + 1)\frac{(a - 1)}{a}.
\]  

(91)
Here, both $a$ and $m$ are disposable constants, but in the special case $m = 2$, equation (85) is not recovered since $k = -1$ and the energy is then linear in $I_2$. Thus, equation (85) is not a special case of (90).

Restrictions on the values of $k, a, m$ may be determined by considering the monotonicity of $\sigma_{\theta_2}$ as in sections 5.1.1 and 5.1.2, but we omit the details here.

Further forms of energy function can be generated by replacing the linear term in $I_2$ in (84) by other specific functions of $I_2$ and in this way an extensive catalogue of energy functions that can support pure torsion with zero lateral traction can be built up.

5.1.4. Energy functions with $\sigma_{\theta_2} \neq 0$

If it is not required to satisfy the zero-traction condition on $R = A$, then in order for an energy function to support pure torsion it is only necessary that equation (55) be satisfied. For this purpose we examine the class of strain-energy functions $\tilde{W} = W_{3O1}(I_1, I_3)$ introduced by Jiang and Ogden [19]. These have the form

$$\tilde{W}_{3O1}(I_1, I_3) = f(I_1)h_1(I_3) + h_2(I_3).$$  \hspace{1cm} (94)

Substitution of (94) into (55) with $I_3 = 1$ leads to

$$\tilde{W}_{3O1}(I_1, I_3) = \frac{\mu}{2k} \exp[k(I_1 - 3)]h_1(I_3) + h_2(I_3),$$  \hspace{1cm} (95)

where $k \neq 0$ is a disposable constant and

$$h_1(1) = 1, \quad h'_1(1) = \frac{1}{2} - k, \quad h_2(1) = -\frac{\mu}{2k}, \quad h'_2(1) = -\frac{\mu}{4k}, \quad h''_2(1) = \frac{k}{4} + \frac{\mu}{6}(3k - 1).$$  \hspace{1cm} (96)

The resulting shear stress is $\sigma_{\theta_2} = \mu \gamma \exp(k \gamma^2)$, which is the same as that discussed in respect of (81). From (50) the radial stress is calculated as

$$\sigma_r = \frac{\mu}{2k} \exp(k \gamma^2) - 1.$$  \hspace{1cm} (98)

It follows that $\sigma_{\gamma} > 0$ for $\gamma \neq 0$ for all $k$. This is consistent with the discussion in section 5.1.1 where we noted that an axial compression was required to prevent the cylinder extending during torsion. In this case, however, it follows from (50) and (53) that $\sigma_{\gamma} = \sigma_r$ and hence that the axial load is positive, in contrast to the situation in section 5.1.1. Thus, the tendency for the axial load to extend the cylinder is counterbalanced by the tensile radial traction (which tends to shorten the cylinder).

In a similar way to that in section 5.1.3 we may generate classes of strain-energy functions, except that now this is done using (55) rather than (54). We consider the energy function
\[ W(I_1, I_2, I_3) = f(I_1)h_1(I_3) + g(I_2)h_2(I_3) + h_3(I_3), \]  
(99)

where \( f \) and \( g \) are to be determined and \( h_1, h_2, h_3 \) play the same roles as previously. This can be regarded as a generalization of the Blatz–Ko material

\[
W(I_1, I_2, I_3) = \frac{\mu}{2} a \left( I_1 - 1 - \frac{1}{v} + \frac{1 - 2v}{v} I_3^{v/(1-2v)} \right) \\
+ \frac{\mu}{2} (1 - a) \left( I_2^2 - 1 - \frac{1}{v} + \frac{1 - 2v}{v} I_3^{v/(1-2v)} \right),
\]  
(100)

where \( a \) is a constant and \( v \) is Poisson’s ratio, as discussed earlier. For purposes of illustration we set \( f(I_1) = aI_1 \), where \( a \) is a constant.

Equation (55) then leads to an equation for \( g(I) \), the solution of which reveals two separate cases. Firstly, we obtain

\[
W = aI_1 h_1(I_3) + \left[ b \frac{(I_2 - 1)^m}{2^m} + a \frac{2k - 1}{2m - 1} I_2 \right] h_2(I_3) + h_3(I_3),
\]  
(101)

for \( m \neq 1 \), where

\[
h_1(1) = 1, \quad h_2(1) = 1, \quad h'_1(1) = k, \quad h'_2(1) = -m.
\]  
(102)

Three of the constants \( a, b, k, m \) are independent. They are related by

\[
2m + 2k - 3 \quad a \quad \frac{2m + 2k - 3}{m - 1} + bm = \mu,
\]  
(103)

while \( h_3(1) \) or \( h'_3(1) \) are given in terms of the constants through (27). Equation (28) provides a connection between \( h'_1(1), h'_2(1), h'_3(1), a, b, k, m \) and \( \mu \) and \( \kappa \). We omit the details.

The second case corresponds to \( m = 1 \) and gives

\[
W = aI_1 h_1(I_3) + \left[ bI_2 - \frac{a}{2} (2k - 1)(I_2 - 1) \log(I_2 - 1) \right] h_2(I_3) + h_3(I_3),
\]  
(104)

where (102) hold with \( m = 1 \) and (103) is replaced by

\[
3a - 2ka - (2k - 1)a \log 2 + 2b = \mu.
\]  
(105)

In each case the stresses \( \sigma_{\theta z}, \sigma_{rr}, \sigma_{zz} \) may be calculated but no new features arise so we omit the details. By taking specific values of the constants several of the particular examples discussed previously can be recovered. A further class of energy functions can be generated by replacing \( g(I_2) \) by \( g(I_2/I_3) \) in (99) in a similar way to that in section 5.1.2.
6. ENERGY FUNCTIONS IN TERMS OF $i_1, i_2, i_3$

Here we consider an alternative formulation of the pure torsion problem based on the principal invariants $i_1, i_2, i_3$ defined in (24). The strain energy is written as $W = \tilde{W}(i_1, i_2, i_3)$, as in (26). The Cauchy stress components (50)–(53) become

$$\sigma_{rr} = \tilde{W}_1 + (i - 1)\tilde{W}_2 + \tilde{W}_3,$$  \hspace{1cm} (106)
$$\sigma_{\theta \theta} = \frac{1}{i - 1} (\tilde{W}_1 \gamma^2 + 2) + (i - 2)(i + 1)\tilde{W}_2 + (i - 1)\tilde{W}_3,$$ \hspace{1cm} (107)
$$\sigma_{\theta z} = \frac{\gamma}{i - 1} (\tilde{W}_1 + \tilde{W}_2),$$ \hspace{1cm} (108)
$$\sigma_{zz} = \frac{1}{i - 1} (2\tilde{W}_1 + (i + 1)\tilde{W}_2 + (i - 1)\tilde{W}_3)$$ \hspace{1cm} (109)

when evaluated for

$$i = i_1 = i_2 = \lambda + \lambda^{-1} + 1, \hspace{1cm} \gamma = \lambda - \lambda^{-1}, \hspace{1cm} i_3 = 1,$$  \hspace{1cm} (110)

while the equilibrium equation (55) becomes

$$\left(\tilde{W}_{11} + i\tilde{W}_{12} + (i - 1)\tilde{W}_{22} + \tilde{W}_{31} + \tilde{W}_{32}\right)(i + 1) + i(\tilde{W}_2 - \tilde{W}_1) = 0.$$ \hspace{1cm} (111)

We recall that the subscripts on $\tilde{W}$ denote derivatives with respect to $i_1, i_2, i_3$.

When $\sigma_{rr} = \sigma_{\theta \theta} = 0$ we obtain, after some rearrangement,

$$i\tilde{W}_1 + \tilde{W}_2 = 0, \hspace{1cm} (i^2 - i - 1)\tilde{W}_1 - \tilde{W}_3 = 0,$$ \hspace{1cm} (112)

again evaluated for (110). These are the counterparts and equivalents of (54). In general, application of (112) leads to different classes of energy function from those generated on the basis of (54).

To be specific we consider energy functions of the form

$$\tilde{W}(i_1, i_2, i_3) = f(i_1)h_1(i_3) + g(i_2)h_2(i_3) + h_3(i_3),$$ \hspace{1cm} (113)

which includes many well-known energy functions as special cases: for example, the class I, class II and class III materials considered by Carroll [15], which are, respectively, linear in $(i_2, i_3)$, $(i_1, i_3)$ and $(i_1, i_2)$; see also Carroll and Horgan [23] and the recent review by Horgan [24]. For materials of class III the torsion problem was studied in [17], and it was found that pure torsion can be sustained only in the presence of a (uniformly distributed) tensile loading on the lateral surface of the cylinder. Clearly, class III materials cannot satisfy (112), except trivially. It was also shown in [17] that class II materials cannot sustain pure torsion and that (in general) neither can class I materials.
On substitution of (113) into (112) we obtain

\[ if'(i) + g'(i) = 0, \quad (i^2 - i - 1)f'(i) - kf(i) - mg(i) - h'_3(1) = 0, \quad (114) \]

where, on application of (29), we have

\[ h_3(1) = h_2(1) = 1, \quad h'_1(1) = k, \quad h'_2(1) = m, \quad (115) \]
\[ h_3(1) = -f(3) - g(3), \quad f'(3) + g'(3) = 2\mu, \quad (116) \]
\[ g'(3) + kf(3) + mg(3) + h'_3(1) = -2\mu. \quad (117) \]

We have not written down the specialization of (30) explicitly.

Equations (114) may be solved to give \( f \) and \( g \). In general the solution involves logarithms and we omit the details. There is one special case in which logarithms do not arise, and we therefore focus on this for purposes of illustration. This corresponds to \( k = -1, m = -2 \) in (115) and leads to

\[ \tilde{W} = -\mu i_3 h_1(i_3) + \frac{\mu}{2} i_3^2 h_2(i_3) + h_3(i_3), \quad (118) \]

with

\[ h_3(1) = -\frac{3\mu}{2}, \quad h'_3(1) = \mu, \quad h''_3(1) + \frac{9\mu}{2} h''_2(1) - 3\mu h''_1(1) = \kappa + \frac{5\mu}{3}. \quad (119) \]

From the specialization of (25) for \( i_3 = 1 \) we note that \( I_1 = i_1^2 - 2i_2, I_2 = i_2^2 - 2i_1 \), and hence it is clear that (118) is not independent of \( I_1 \) when regarded as a function of \( I_1 \) and \( I_2 \).

Note that for the material (118) we obtain \( \sigma_{xx} = \mu y, \sigma_{zz} = -\mu y^2 \), which are the same results as are obtained for (78) with \( k = 0 \). This means, in particular, that it is not possible to distinguish between the two different strain-energy functions on the basis of pure torsion alone.

Finally in this section, we consider the compressible Varga material defined by

\[ \tilde{W} = a(i_1 - 3) + b(i_2 - 3) + h(i_3), \quad (120) \]

where \( a \) and \( b \) are constants. This corresponds to the class III materials introduced by Carroll [15] (see also Haughton [25]). As already noted, these materials can support pure torsion, but require a radial traction on the lateral surface of the cylinder in order to do so. For (120) to support pure torsion it is necessary and sufficient that \( b = a \). The energy function (120) is interesting because it can also support pure azimuthal shear [26] and pure axial shear [18]. The present authors have also shown that the same is true for helical shear [27] and have obtained a closed-form solution to the helical shear problem in this case (see Beatty and Jiang [28] for a discussion of helical shear).
It is evident that the procedure discussed in the above sections can be used to derive further classes of energy function and to derive material models that best fit experimental data.

7. INCOMPRESSIBLE MATERIALS

In [19] and [18] we discussed the procedure for generating strain-energy functions and corresponding solutions for incompressible materials in respect of pure azimuthal shear and pure axial shear of a circular cylindrical tube. A similar procedure can be adopted in respect of the pure torsion considered here.

For this purpose, we define the strain-energy functions

\[ \bar{w}(I_1, I_2) = \bar{W}(I_1, I_2, 1), \quad \bar{w}(i_1, i_2) = \bar{W}(i_1, i_2, 1), \]  

(121)

where \( \bar{W}(I_1, I_2, i_3) \) or \( \bar{W}(i_1, i_2, i_3) \), corresponds to one of the models derived in the foregoing sections. The Cauchy stress tensor \( \sigma \) for an incompressible material is written as

\[ \sigma = 2\bar{w}_1 \mathbf{B} - 2\bar{w}_2 \mathbf{B}^{-1} - p \mathbf{I}, \]  

(122)

where \( p \) is an arbitrary hydrostatic pressure to be determined by the equilibrium equations and the boundary conditions. Equivalently, it may be written as

\[ \sigma = \bar{w}_1 \mathbf{V} - \bar{w}_2 \mathbf{V}^{-1} - p \mathbf{I}, \]  

(123)

where \( \mathbf{V} \) is the left stretch tensor and, in general, \( p \) in (123) differs from that in (122).

Thus, results obtained for pure torsion for the considered compressible materials apply also to incompressible materials. Instead of imposing restrictions on the strain-energy function, the equilibrium equation serves to determine \( p \). The procedure of obtaining results for an incompressible problem using corresponding results from the isochoric compressible case was illustrated in [19] and [18].

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