Local negative circuits and fixed points in Boolean networks

Adrien Richard

Local negative circuits and fixed points in Boolean networks

Adrien Richard

Laboratoire I3S, UMR 6070 CNRS & Université de Nice-Sophia Antipolis,
2000 route des Lucioles, 06903 Sophia Antipolis, France.

e-mail: richard@i3s.unice.fr
telephone: +33 4 92 94 27 51
fax: +33 4 92 94 28 98

Abstract: To each Boolean function $F : \{0,1\}^n \to \{0,1\}^n$ and each point $x \in \{0,1\}^n$, we associate the signed directed graph $G_F(x)$ of order n that contains a positive (resp. negative) arc from j to i if the discrete analogue of $(\partial f_i/\partial x_j)(x)$ is positive (resp. negative). We then focus on the following open problem: Is the absence of a negative circuit in $G_F(x)$ for all $x \in \{0,1\}^n$ a sufficient condition for F to have at least one fixed point? As main result, we settle this problem under the additional condition that, for all $x \in \{0,1\}^n$, the out-degree of each vertex of $G_F(x)$ is at most one.

Key words: Boolean network, Interaction graph, Discrete Jacobian matrix, Feedback circuit, Negative circuit, Fixed point.

Mathematics Subject Classification: 14R15, 37B99, 68R05, 92D99, 94C10.

1 Introduction

In the course of his analysis of discrete iterations, Robert introduced a discrete Jacobian matrix for Boolean maps and the notion of Boolean eigenvalue [2, 3, 4, 5]. This material allows Shih and Ho to state in 1999 a Boolean analogue of the Jacobian conjecture [7]: If a map from $\{0,1\}^n$ to itself is such that all the Boolean eigenvalues of the discrete Jacobian matrix of each element of $\{0,1\}^n$ are zero, then it has a unique fixed point. Thanks to the work of Shih and Dong [6], this conjecture is now a theorem.

Our starting point is an equivalent statement of the Shih-Dong theorem, the Theorem 1 below, in which the condition "all the Boolean eigenvalues of the discrete Jacobian matrix are zero" is expressed with the following few basic definitions and graph-theoretic notions.

Let n be a positive integer, and consider a Boolean map

$$F:\{0,1\}^n\to\{0,1\}^n,\qquad x=(x_1,\dots,x_n)\mapsto F(x)=(f_1(x),\dots,f_n(x)).$$

The *interaction graph of* F *evaluated at point* $x\in\{0,1\}^n$ is the directed graph on $\{1,\dots,n\}$ that contains an arc from a vertex j to a vertex i if the quantity

$$f_{ij}(x)=f_i(x_1,\dots,x_{j-1},1,x_{j+1},\dots,x_n)-f_i(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_n)$$

is not zero, *i.e.*, if the partial derivative of f_i with respect to x_j is not is not zero at point x. A *circuit* of length p in $G_F(x)$ is a sequence of p distinct vertices $i_1,i_2,\dots,i_p$ such that there is an arc from i_k to i_{k+1}, $1\le k<p$, and from i_p to i_1. An arc from a vertex to itself is thus a circuit of length one.

Theorem 1 (Shih and Dong, 2005)

If $G_F(x)$ has no circuit for all $x \in \{0,1\}^n$, then F has a unique fixed point.

Remy, Ruet and Thieffry [1] proved latter that F has at most one fixed point under a condition weaker than "$G_F(x)$ has no circuit for all $x \in \{0,1\}^n$". For that, they define the *sign of an arc* from j to i in $G_F(x)$ to be equals to $f_{ij}(x)$. And, as usual, they define the *sign of a circuit* to be the product of the signs of its edges.

Theorem 2 (Remy, Ruet and Thieffry, 2008)

If $G_F(x)$ has no positive circuit for all $x \in \{0,1\}^n$, then F has at most one fixed point.

This theorem positively answer a Boolean version of a conjecture of Thomas coming from theoretical biology (see [1] and the references therein).

Seeing Theorems 1 and 2, it is natural to think about a proof by dichotomy of Theorem 1, and to study the following difficult question:

Question 1 *Is the absence of a negative circuit in $G_F(x)$ for all $x \in \{0,1\}^n$ a sufficient condition for F to have at least one fixed point?*

In this note, we partially answer this question by establishing the following theorem:

Theorem 3 *If $G_F(x)$ has no negative circuit for all $x \in \{0,1\}^n$, and if the out-degree of each vertex of $G_F(x)$ is at most one for all $x \in \{0,1\}^n$, then F has at least one fixed point.*

This partial answer is, in our knowledge, the first result about negative circuits in local interaction graphs associated with F. And it is not an obvious exercise. To see this, one can refer to the technical arguments used by Shih and Ho [7, pages 75-88] to prove that if

$G_F(x)$ has no circuit for all $x \in \{0,1\}^n$, and if the out-degree of each vertex of $G_F(x)$ is at most one for all $x \in \{0,1\}^n$, then F has at least one fixed point.

Finally, we also prove, using Theorem 2, the following theorem:

Theorem 4 *If $G_F(x)$ has no negative circuit for all $x \in \{0,1\}^n$, and if there exists a vertex $i \in \{1,\dots,n\}$ such that, for all $x \in \{0,1\}^n$, all the positive circuits of $G_F(x)$ contain i, then F has at least one fixed point.*

Note that Theorem 1 is an immediate consequence of Theorem 2 and Theorem 4.

The paper is organized as follows. After some preliminaries given in Section 2, Sections 3 and 4 are devoted to the proof of Theorems 3 and 4 respectively.

2 Preliminaries

As usual, we set $\overline{0}=1$ and $\overline{1}=0$. For all $x \in \{0,1\}$ and $I \subseteq \{1,\dots,n\}$, we denote by $\overline{x}^I$ the point y of $\{0,1\}^n$ defined by: $y_i = \overline{x_i}$ if $i \in I$, and $y_i = x_i$ otherwise ($i=1,\dots,n$). In order to simplify notations, we write $\overline{x}$ instead of $\overline{x}^{\{1,\dots,n\}}$, and $\overline{x}^i$ instead of $\overline{x}^{\{i\}}$.

Let F be a map from $\{0,1\}^n$ to itself. Using the previous notations, the partial derivative of f_i with respect to x_j can be defined by

$$f_{ij}(x) = \frac{f_i(\overline{x}^j) - f_i(x)}{\overline{x_j} - x_j}.$$

If $G_F(x)$ has an arc from j to i, we say that i (resp. j) is a *successor* (resp. *predecessor*) of j (resp. i), and we abusively write $j \to i \in G_F(x)$. The *out-degree* of a vertex is defined to be the number of successors of this vertex.

We are interested in maps F that have the following property $\mathfrak{P}$:

$$\forall x \in \{0,1\}^n, \textit{ the out-degree of each vertex of } G_F(x) \textit{ is at most one.} \qquad (\mathfrak{P})$$

Note that if F has the property $\mathfrak{P}$, then

$$j \to i \in G_F(x) \iff F(\overline{x}^i) = \overline{F(x)}^j.$$

The *Hamming distance* $d(x,y)$ between two points x,y of $\{0,1\}^n$ is the number of indices $i \in \{1,\dots,n\}$ such that $x_i \neq y_i$. So, for instance, $d(x,y)=n$ if and only if $y=\overline{x}$, and $d(x,y)=1$ if and only if there exists $i \in \{1,\dots,n\}$ such that $y=\overline{x}^i$. Note also that F has the property $\mathfrak{P}$ if and only if

$$\forall x,y \in \{0,1\}^n, \qquad d(x,y)=1 \;\Rightarrow\; d(F(x),F(y)) \leq 1.$$

We then deduce, by recurrence on $d(x,y)$, that F has the property $\mathfrak{P}$ if and only if

$$\forall x,y \in \{0,1\}^n, \qquad d(F(x),F(y)) \leq d(x,y).$$

We now associate with F two maps from $\{0,1\}^{n-1}$ to itself that will be used as inductive tools in the proof of Theorems 3 and 4. If $x \in \{0,1\}^{n-1}$ and $b \in \{0,1\}$, we denote by (x,b) the point $(x_1,\dots,x_{n-1},b)$ of $\{0,1\}^n$. Then, for $b \in \{0,1\}$, we define the map $F^{|b} = (f_1^{|b},\dots,f_n^{|b}) : \{0,1\}^{n-1} \to \{0,1\}^{n-1}$ by

$$f_i^{|b}(x) = f_i(x,b) \qquad (i=1,\dots,n-1).$$

We have then the following obvious property: for all $x \in \{0,1\}^{n-1}$ and $b \in \{0,1\}$,

$$f_{ij}^{|b}(x) = f_{ij}(x,b) \qquad (i,j = 1,\ldots,n-1).$$

Consequently, for all $x \in \{0,1\}^{n-1}$ and $b \in \{0,1\}$,

$$G_{F^{|b}}(x) \text{ is a } \textit{subgraph} \text{ of } G_F(x,b),$$

i.e., if $G_{F^{|b}}(x)$ has a positive (resp. negative) arc from j to i, then $G_F(x,b)$ has a positive (resp. negative) arc from j to i. It is then clear that if F has the property $\mathfrak{P}$ then $F^{|b}$ has the property $\mathfrak{P}$.

3 Proof of Theorem 3

Lemma 1 *If $d(x,F(x)) = 1$, then any circuit of $G_F(x)$ of length n is negative.*

Proof – Suppose that $d(x,F(x)) = 1$ and that $C = i_1,\ldots,i_n$ is a circuit of $G_F(x)$ of length n. Without loss of generality, we can suppose that $F(x) = \overline{x}^{i_1}$. Let $h(1) = 1$ and $h(0) = -1$. We have

$$f_{i_1 i_n}(x) = \frac{f_{i_1}(\overline{x}^{i_n}) - f_{i_1}(x)}{\overline{x_{i_n}} - x_{i_n}} = \frac{f_{i_1}(\overline{x}^{i_n}) - \overline{x_{i_1}}}{\overline{x_{i_n}} - x_{i_n}},$$

and since $f_{i_1 i_n}(x) \neq 0$ we obtain

$$f_{i_1 i_n}(x) = \frac{x_{i_1} - \overline{x_{i_1}}}{\overline{x_{i_n}} - x_{i_n}} = \frac{h(x_{i_1})}{h(\overline{x_{i_n}})}.$$

Furthermore, for $k = 1, \dots, n-1$, we have

$$f_{i_{k+1}i_k}(x) = \frac{f_{i_{k+1}}(\overline{x}^{i_k}) - f_{i_{k+1}}(x)}{\overline{x_{i_k}} - x_{i_k}} = \frac{f_{i_{k+1}}(\overline{x}^{i_k}) - x_{i_{k+1}}}{\overline{x_{i_k}} - x_{i_k}},$$

and since $f_{i_{k+1}i_k}(x) \neq 0$ we obtain

$$f_{i_{k+1}i_k}(x) = \frac{\overline{x_{i_{k+1}}} - x_{i_{k+1}}}{\overline{x_{i_k}} - x_{i_k}} = \frac{h(\overline{x_{i_{k+1}}})}{h(\overline{x_{i_k}})}.$$

Denoting by s the sign of C, we obtain

$$\begin{aligned} s &= f_{i_2i_1}(x) \cdot f_{i_3i_2}(x) \cdot f_{i_4i_3}(x) \cdots f_{i_ni_{n-1}}(x) \cdot f_{i_1i_n}(x) \\ &= \frac{h(\overline{x_{i_2}})}{h(\overline{x_{i_1}})} \cdot \frac{h(\overline{x_{i_3}})}{h(\overline{x_{i_2}})} \cdot \frac{h(\overline{x_{i_4}})}{h(\overline{x_{i_3}})} \cdots \frac{h(\overline{x_{i_n}})}{h(\overline{x_{i_{n-1}}})} \cdot \frac{h(x_{i_1})}{h(\overline{x_{i_n}})} = \frac{h(x_{i_1})}{h(\overline{x_{i_1}})} = -1. \end{aligned}$$

□

The rest of the proof is based on the following notion of opposition: given two points $x, y \in \{0,1\}^n$ and an index $i \in \{1, \dots, n\}$, we say that x and y are in *opposition* (*with respect to* i *in* F) if

$$F(x) = \overline{x}^i, \qquad F(y) = \overline{y}^i \qquad \text{and} \qquad x_i \neq y_i.$$

Lemma 2 *Let F be a map from $\{0,1\}^n$ to itself that has the property $\mathfrak{P}$. If F has two points in opposition, then there exists two distinct points x and y in $\{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit.*

Proof – We proceed by induction on n. The lemma being obvious for $n = 1$, we suppose that $n > 1$ and that the lemma holds for maps from $\{0,1\}^{n-1}$ to itself. We also suppose that F has at least two points in opposition.

First, suppose that α and β are two points in opposition with respect to i in F such that $\alpha \neq \overline{\beta}$. Then there exists $j \neq i$ such that $\alpha_j = \beta_j$, and without loss of generality we can suppose that $\alpha_n = \beta_n = b$. Set $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})$ and $\tilde{\beta} = (\beta_1, \ldots, \beta_{n-1})$ so that $\alpha = (\tilde{\alpha}, b)$ and $\beta = (\tilde{\beta}, b)$. Then, $\tilde{\alpha}_i = \alpha_i \neq \beta_i = \tilde{\beta}_i$, and since $F(\alpha) = \overline{\alpha}^i$, we have

$$F^{|b}(\tilde{\alpha}) = (f_1(\alpha), \ldots, f_i(\alpha), \ldots, f_{n-1}(\alpha)) = (\alpha_1, \ldots, \overline{\alpha_i}, \ldots, \alpha_{n-1}) = \overline{\tilde{\alpha}}^i,$$

and we show similarly that $F^{|b}(\tilde{\beta}) = \overline{\tilde{\beta}}^i$. Consequently, $\tilde{\alpha}$ and $\tilde{\beta}$ are in opposition with respect to i in $F^{|b}$. Since F has the property $\mathfrak{P}$, $F^{|b}$ has the property $\mathfrak{P}$, and so, by induction hypothesis, there exists two distinct points $x, y \in \{0,1\}^{n-1}$ such that $G_{F^{|b}}(x)$ and $G_{F^{|b}}(y)$ have a common negative circuit. Since $G_{F^{|b}}(x)$ and $G_{F^{|b}}(y)$ are subgraphs of $G_F(x, b)$ and $G_F(y, b)$ respectively, we deduce that $G_F(x, b)$ and $G_F(y, b)$ have a common negative circuit and the lemma holds.

So in the following, we assume the following hypothesis $\mathfrak{H}$:

$$\textit{If } F \textit{ has two points } \alpha \textit{ and } \beta \textit{ in opposition, then } \alpha = \overline{\beta}. \qquad (\mathfrak{H})$$

We need the following four claims to complet the proof.

Claim 1 *F has no fixed point.*

Proof – Let α and β be two points in opposition with respect to i in F. Suppose, by contradiction, that x is a fixed point of F. If $x_i = \alpha_i$, then $d(F(x), F(\alpha)) = d(x, \overline{a}^i) > d(x, \alpha)$ and this contradicts the fact that F has the property $\mathfrak{P}$. Otherwise, $x_i = \beta_i$, thus

$d(F(x), F(\beta)) = d(x, \overline{\beta}^i) > d(x, \beta)$ and we arrive to the same contradiction. $\square$

Notation: *In the following, for all $x \in \{0,1\}^n$, we set*

$$x^1 = x \qquad \textit{and} \qquad x^{k+1} = F(x^k) \qquad (k = 1, 2, 3, \dots).$$

Claim 2 *If α and β are in opposition in F, then there exists a permutation $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$ such that α^k and β^k are in opposition with respect to i_k in F $(k = 1, \dots, n)$.*

Proof – Suppose that $\alpha = \alpha^1$ and $\beta = \beta^1$ are in opposition with respect to i in F. For $p = 1, \dots, n$, we denote by S_p the set of sequences $(i_1, i_2, \dots, i_p)$ of p distinct indices of $\{1, \dots, n\}$ such that $\alpha^{k+1} = \overline{\alpha^k}^{i_k}$ for $k = 1, \dots, p$. S_1 is not empty since, by definition, $(i) \in S_1$. So in order to prove that S_n is not empty, it is sufficient to prove that

$$S_p \neq \emptyset \;\Rightarrow\; S_{p+1} \neq \emptyset \qquad (p = 1, \dots, n-1).$$

Suppose that $(i_1, \dots, i_p) \in S_p$ $(1 \leq p < n)$. Then $\alpha^{p+1} = \overline{a^p}^{i_p}$, so $d(\alpha^{p+1}, \alpha^p) = 1$ and since F has the property $\mathfrak{P}$, we deduce that

$$d(F(\alpha^{p+1}), \alpha^{p+1}) = d(F(\alpha^{p+1}), F(\alpha^p)) \leq d(\alpha^{p+1}, \alpha^p) = 1.$$

Since, by Claim 1, we have $F(\alpha^{p+1}) \neq \alpha^{p+1}$, we deduce that $d(F(\alpha^{p+1}), \alpha^{p+1}) = 1$. In other words, there exists $j \in \{1, \dots, n\}$ such that

$$F(\alpha^{p+1}) = \overline{\alpha^{p+1}}^j.$$

Suppose that there exists $k \in \{1, \dots, p\}$ such that $j = i_k$. Then,

$$F(\alpha^k) = \overline{\alpha^k}^j$$

and since

$$\alpha^{p+1} = \overline{\alpha^p}^{\{i_p\}} = \overline{\alpha^{p-1}}^{\{i_{p-1}, i_p\}} = \cdots = \overline{\alpha^k}^{\{i_k, \dots, i_{p-1}, i_p\}},$$

we have

$$\alpha^k_j = \alpha^k_{i_k} \neq \alpha^{p+1}_{i_k} = \alpha^{p+1}_j.$$

Thus α^k and α^{p+1} are in opposition with respect to i in F. But since $\{i_k, \dots, i_{p-1}, i_p\}$ is strictly included in $\{1, \dots, n\}$, we have $\alpha^{p+1} \neq \overline{\alpha^k}$ and this contradicts the hypothesis $\mathfrak{H}$. Thus $j \notin \{i_1, \dots, i_p\}$ and we deduce that $(i_1, \dots, i_p, j)$ belongs to S_{p+1}. Thus S_{p+1} is not empty and it follows that S_n is not empty. Thus, there exists a permutation $\{i_1, \dots, i_n\}$ of $\{1, \dots, n\}$ such that $\alpha^{p+1} = \overline{a^p}^{i_p}$ for $p = 1, \dots, n$, and we show similarly that there exists a permutation $\{j_1, \dots, j_n\}$ of $\{1, \dots, n\}$ such that $\beta^{p+1} = \overline{\beta^p}^{j_p}$ for $p = 1, \dots, n$. Observe that, following the hypothesis $\mathfrak{H}$, we have $\alpha = \overline{\beta}$ and thus

$$\alpha^{n+1} = \overline{\alpha}^{\{i_1, \dots, i_n\}} = \overline{\alpha} = \beta \quad \text{and} \quad \beta^{n+1} = \overline{\beta}^{\{j_1, \dots, j_n\}} = \overline{\beta} = \alpha. \tag{1}$$

We are now in possition to prove, by recurrence on k decreasing from n to 1, that α^k and β^k are in opposition with respect to i_k in F. Since F has the property $\mathfrak{P}$, and from (1), we have

$$d(\alpha^n, \beta^n) \geq d(F(\alpha^n), F(\beta^n)) = d(\alpha^{n+1}, \beta^{n+1}) = d(\beta, \alpha) = d(\beta, \overline{\beta}) = n.$$

thus

$$d(\alpha^n, \beta^n) = n = d(\alpha^{n+1}, \beta^{n+1}) = d(\overline{\alpha^n}^{i_n}, \overline{\beta^n}^{j_n})$$

We deduce that $i_n = j_n$ and $\alpha^n_{i_n} \neq \beta^n_{i_n}$. It is then clear that α^n and β^n are in opposition with respect to i_n in F. Now, suppose that α^k and β^k are in opposition with respect to i_k in F $(2 \leq k \leq n)$. Then, following the hypothesis $\mathfrak{H}$, $\alpha^k = \overline{\beta^k}$, and since F has the property $\mathfrak{P}$, we deduce that

$$d(\alpha^{k-1}, \beta^{k-1}) \geq d(F(\alpha^{k-1}), F(\beta^{k-1})) = d(\alpha^k, \beta^k) = d(\overline{\beta^k}, \beta^k) = n$$

Thus

$$d(\alpha^{k-1}, \beta^{k-1}) = n = d(\alpha^k, \beta^k) = d(\overline{\alpha^{k-1}}^{i_{k-1}}, \overline{\beta^{k-1}}^{j_{k-1}}).$$

We deduce that $i_{k-1} = j_{k-1}$ and $\alpha^{k-1}_{i_{k-1}} \neq \alpha^{k-1}_{i_{k-1}}$ and thus that α^{k-1} and β^{k-1} are in opposition with respect to i_{k-1} in F. □

Claim 3 *If α and β are in opposition with respect to i in F, then i has at most one predecessor in $G_F(\alpha)$.*

Proof – Let $\{i_1, \ldots, i_n\}$ be a permutation of $\{1, \ldots, n\}$ with the property of Claim 2. Then $\overline{\alpha}^{i_1} = F(\alpha) = \overline{\alpha}^i$ thus $i = i_1$. Suppose, by contradiction, that i_1 has at least two predecessors in $G_F(\alpha)$. Then i_1 has a predecessor $i_k \neq i_n$ in $G_F(\alpha)$. Using the property $\mathfrak{P}$, we deduce that

$$F(\overline{\alpha}^{i_k}) = \overline{F(\alpha)}^{i_1} = \overline{\overline{\alpha}^{i_1}}^{i_1} = \alpha = \overline{\overline{\alpha}^{i_k}}^{i_k} \quad \text{and} \quad F(\alpha^k) = \overline{\alpha^k}^{i_k}.$$

If $k = 1$, then $\alpha^k = \alpha$ and so

$$(\alpha^k)_{i_k} = \alpha_{i_k} \neq (\overline{\alpha}^{i_k})_{i_k} \quad \text{and} \quad \alpha^k_{i_n} = (\overline{\alpha}^{i_k})_{i_n}. \tag{2}$$

Otherwise, $\alpha^k = \overline{\alpha}^{\{i_1,\dots,i_{k-1}\}}$ and so (2) holds again. Consequently, in both cases, α^k and $\overline{\alpha}^{i_k}$ are in opposition with respect to i_k in F and $\alpha^k \neq \overline{\overline{\alpha}^{i_k}}$. This contradicts the hypothesis $\mathfrak{H}$. $\square$

Claim 4 *If α et β are in opposition in F, then $G_F(\alpha^n)$ has a circuit of length n.*

Proof – Let $\{i_1,\dots,i_n\}$ be a permutation of $\{1,\dots,n\}$ with the property of Claim 2. We will show that $i_1,\dots,i_n$ is a circuit of $G_F(\alpha^n)$. We have

$$F\left(\overline{\alpha^k}^{i_{k-1}}\right) = F\left(\overline{\overline{\alpha^{k-1}}^{i_{k-1}}}^{i_{k-1}}\right) = F(\alpha^{k-1}) = \alpha^k = \overline{\overline{a^k}^{i_k}}^{i_k} = \overline{F(\alpha^k)}^{i_k} \qquad (k = 2,\dots,n)$$

and thus

$$i_{k-1} \to i_k \in G_F(\alpha^k) \qquad (k = 2,\dots,n). \tag{3}$$

In addition,

$$F\left(\overline{\alpha^k}^{i_k}\right) = F(\alpha^{k+1}) = \overline{a^{k+1}}^{i_{k+1}} = \overline{F(\alpha^k)}^{i_{k+1}} \qquad (k = 1,\dots,n-1)$$

and thus

$$i_k \to i_{k+1} \in G_F(\alpha^k) \qquad (k = 1,\dots,n-1).$$

Let k be any index of $\{1, \dots, n-1\}$, and suppose, by contradiction, that

$$i_k \to i_{k+1} \notin G_F(\alpha^n).$$

Since $i_k \to i_{k+1} \in G_F(\alpha^k)$, there exists $p \in \{k+1, \dots, n\}$ such that

$$i_k \to i_{k+1} \in G_F(\alpha^{p-1}) \qquad \text{and} \qquad i_k \to i_{k+1} \notin G_F(\alpha^p).$$

Following (3), we have $i_p \neq i_{k+1}$. Furthermore, from $i_k \to i_{k+1} \in G_F(\alpha^{p-1})$ we deduce that

$$f_{i_{k+1}}(\alpha^{p-1}) \neq f_{i_{k+1}}(\overline{\alpha^{p-1}}^{i_k}), \tag{4}$$

and from both $i_k \to i_{k+1} \notin G_F(\alpha^p)$ and $\alpha^p = \overline{\alpha^{p-1}}^{i_{p-1}}$ we deduce that

$$f_{i_{k+1}}(\overline{\alpha^{p-1}}^{i_{p-1}}) = f_{i_{k+1}}\left(\overline{\overline{\alpha^{p-1}}^{i_{p-1}}}^{i_k}\right) = f_{i_{k+1}}\left(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\right). \tag{5}$$

If

$$f_{i_{k+1}}(\alpha^{p-1}) \neq f_{i_{k+1}}(\overline{\alpha^{p-1}}^{i_{p-1}})$$

then i_{k+1} and i_p are distinct successors of i_{p-1} in $G_F(\alpha^{p-1})$, and this contradicts the fact that F has the property $\mathfrak{P}$. Thus

$$f_{i_{k+1}}(\alpha^{p-1}) = f_{i_{k+1}}(\overline{\alpha^{p-1}}^{i_{p-1}})$$

and from (4) and (5) we deduce that

$$f_{i_{k+1}}(\overline{\alpha^{p-1}}^{i_k}) \neq f_{i_{k+1}}\left(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\right).$$

Thus $i_{p-1} \to i_{k+1} \in G_F\big(\overline{\alpha^{p-1}}^{i_k}\big)$ and since F has the property $\mathfrak{P}$, we have

$$F\big(\overline{\alpha^p}^{i_k}\big) = F\left(\overline{\overline{\alpha^{p-1}}^{i_{p-1}}}^{i_k}\right) = F\left(\overline{\overline{\alpha^{p-1}}^{i_k}}^{i_{p-1}}\right) = \overline{F\big(\overline{\alpha^{p-1}}^{i_k}\big)}^{i_{k+1}}$$

Since $i_k \to i_{k+1} \in G_F(\alpha^{p-1})$, we have $F\big(\overline{\alpha^{p-1}}^{i_k}\big) = \overline{F(\alpha^{p-1})}^{i_{k+1}}$ and using the property $\mathfrak{P}$ we obtain

$$F\big(\overline{\alpha^p}^{i_k}\big) = \overline{\overline{F(\alpha^{p-1})}^{i_{k+1}}}^{i_{k+1}} = F(\alpha^{p-1}) = \alpha^p = \overline{\overline{\alpha^p}^{i_p}}^{i_p} = \overline{F(\alpha^p)}^{i_p}$$

So i_k and i_{p-1} are predecessors of i_p in $G_F(\alpha^p)$, and $i_k \neq i_{p-1}$ since $i_p \neq i_{k+1}$. We have now a contradiction: following Claim 2, α^p and β^p are in opposition with respect to i_p in F, and so, following Claim 3, i_p has at most one predecessor in $G_F(\alpha^p)$. We have thus prove that

$$i_k \to i_{k+1} \in G_F(\alpha^n) \qquad (k = 1, \ldots, n-1)$$

To prove the claim, it is thus sufficient to prove that $i_n \to i_1 \in G_F(\alpha^n)$, and this is obvious. Indeed, following the hypothesis $\mathfrak{H}$, we have $\overline{\alpha} = \beta$, thus

$$F(\alpha^n) = \alpha^{n+1} = \overline{\alpha}^{\{i_1,\ldots,i_n\}} = \overline{\alpha} = \beta$$

and so

$$F\left(\overline{\alpha^n}^{i_n}\right) = F(\alpha^{n+1}) = F(\beta) = \overline{\beta}^{i_1} = \overline{F(\alpha^n)}^{i_1}.$$

$\square$

We are now in position to prove the lemma. Let α and β be two points in opposition in F. Following Claim 2 and Claim 4, α^n and β^n are two points in opposition, and thus distinct, such that $G_F(\alpha^n)$ and $G_F(\beta^n)$ have a common circuit of length n, and according to Lemma 1, this circuit is negative, both in $G_F(\alpha^n)$ and $G_F(\beta^n)$. $\square$

Lemma 3 *Let F be a map from $\{0,1\}^n$ to itself that has the property $\mathfrak{P}$. If there is no distinct points $x, y \in \{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit, then F has at least one fixed point.*

Proof – We proceed by induction on n. The lemma being obvious for $n = 1$, we suppose that $n > 1$ and that the lemma holds for maps from $\{0,1\}^{n-1}$ to itself. Let F be as in the statement, and let $b \in \{0,1\}$. Since $G_{F^{|b}}(x)$ is a subgraph of $G_F(x,b)$ for all $x \in \{0,1\}^{n-1}$, $F^{|b}$ has the property $\mathfrak{P}$ and there is no distinct points $x, y \in \{0,1\}^n$ such that $G_{F^{|b}}(x)$ and $G_{F^{|b}}(y)$ have a common negative circuit. So, by induction hypothesis, $F^{|b}$ has at least one fixed point that we denote by ξ^b. Now, we prove that $(\xi^0, 0)$ or $(\xi^1, 1)$ is a fixed point of

F. If not, then for $b \in \{0,1\}$,

$$\begin{aligned} F(\xi^b, b) &= (f_1(\xi^b, b), \dots, f_{n-1}(\xi^b, b), f_n(\xi^b, b)) \\ &= (f_1^{|b}(\xi^b), \dots, f_{n-1}^{|b}(\xi^b), f_n(\xi^b, b)) \\ &= (\xi_1^b, \dots, \xi_{n-1}^b, f_n(\xi^b, b)) \\ &= (\xi^b, f_n(\xi^b, b)) \\ &= (\xi^b, \overline{b}) \\ &= \overline{(\xi^b, b)}^n. \end{aligned}$$

We deduce that $(\xi^0, 0)$ and $(\xi^1, 1)$ are in opposition with respect to n in F, and so, by Lemma 2, there exists two distinct points $x, y \in \{0,1\}^n$ such that $G_F(x)$ and $G_F(y)$ have a common negative circuit, a contradiction. □

Theorem 1 is an obvious consequence of Lemma 3.

4 Proof of Theorem 4

We proceed by induction on n. The case $n = 1$ being obvious, we suppose that $n > 1$ and that the theorem holds for maps from $\{0,1\}^{n-1}$ to itself. Let F be a map from $\{0,1\}^n$ to itself, and without loss of generality, suppose that, for all $x \in \{0,1\}^n$, all the positive circuits of $G_F(x)$ contain the vertex n.

For $b \in \{0,1\}$ and $x \in \{0,1\}^{n-1}$, it is clear that $G_{F^{|b}}(x)$ has no circuit since $G_{F^{|b}}(x)$ is a subgraph of $G_F(x, b)$ that does not contains the vertex n. So $F^{|b}$ trivilally satisfies the conditions of the theorem. So, by induction hypothesis, $F^{|b}$ has at least one fixed point that we denote by ξ^b.

We will show that $\alpha = (\xi^0, 0)$ or $\beta = (\xi^b, 1)$ is a fixed point of F. Suppose, by contradiction, that neither α nor β is a fixed point of F. Then, as in Lemma 3, we prove that $F(\alpha) = \overline{\alpha}^n$ and that $F(\beta) = \overline{\beta}^n$.

Consider the map $\bar{F}$ from $\{0,1\}^n$ to $\{0,1\}^n$ defined by

$$\bar{F}(x) = \overline{F(x)}^n.$$

It is clear that α and β are distinct fixed points of $\bar{F}$. So, by Theorem 2, there exists $x \in \{0,1\}^n$ such that $G_{\bar{F}}(x)$ has a positive circuit C. If n does not belong to C, then since

$$\bar{f}_{ij} = f_{ij} \quad \text{for } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n, \tag{6}$$

we deduce that C is a positive circuit of $G_F(x)$ that does not contains n, a contradiction. Otherwise, n belongs to C, and we then deduce from (6) and the fact that

$$\bar{f}_{nj} = -f_{nj} \quad \text{for } j = 1, \dots, n$$

that C is a negative circuit of $G_F(x)$, a contradiction.

References

[1] E. Remy, P. Ruet, D. Thieffry, Graphics requirement for multistability and attractive cycles in a Boolean dynamical framework, *Adv. Appl. Math.*, 41 (2008) 335-350.

[2] F. Robert, Théorèmes de Perron-Frobenius et Stein-Rosenberg Booléens, Linear Algebra Appl. 19 (1978) 237-250.

[3] F. Robert, Dérivée Discrète et Comportement Local d'Une Itération Discrète, Linear Algebra Appl. 52/53 (1983) 547-589.

[4] F. Robert, Discrete iterations: a metric study, in: Series in Computational Mathematics, Vol. 6, Springer-Verlag, Berlin-Heidelber-New York, 1986.

[5] F. Robert, Les systèmes dynamiques discrets, in: Mathématiques et Applications, Vol. 19, Springer-Verlag, Berlin-Heidelber-New York, 1995.

[6] M.-H. Shih, J.-L. Dong, A combinatorial analogue of the Jacobian problem in automata networks, Adv. in Appl. Math. 34 (2005) 30-46.

[7] M.-H. Shih, J.-L. Ho, Solution of the Boolean Markus-Yamabe Problem, Adv. in Appl. Math. 22 (1999) 60-102.