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J Aracena, Adrien Richard, L Salinas

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J. Aracena [a,*,1]

[a] *CI^2MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Av. Esteban Iturra s/n, Casilla 160-C, Concepción, Chile*

A. Richard [b,2]

[b] *Laboratoire I3S, UMR CNRS 7271 & Université de Nice-Sophia Antipolis, 2000 route des Lucioles, 06903 Sophia Antipolis, France*

L. Salinas [c,3]

[c] *Department of Computer Sciences and CI^2MA, University of Concepción, Edmundo Larenas 215, Piso 3, Concepción, Chile*

Abstract

We are interested in the number of fixed points in AND-OR-NOT networks, *i.e.* Boolean networks in which the update function of each component is either a conjunction or a disjunction of positive or negative literals. As main result, we prove that the maximum number of fixed points in a loop-less connected AND-OR-NOT network with n components is at most the maximum number of maximal independent sets in a loop-less connected graph with n vertices, a quantity already known.

Key words: Boolean network, AND-OR-NOT network, fixed point, maximal independent set.

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* Corresponding author.
Email addresses: `jaracena@dim.uchile.cl` (J. Aracena), `richard@unice.fr` (A. Richard), `lilisalinas@udec.cl` (L. Salinas).

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1 Introduction

A Boolean network is a system of n interacting Boolean variables, which evolve, in a discrete time, according to a predefined rule. The structure of such a network is often represented by a digraph, called interaction graph: vertices are network components, and there is an arc from one component to another when the evolution of the latter depends on the evolution of the former.

Boolean networks have applications in many areas, including circuit theory, computer science and social systems [4,8,25]. In particular, from the seminal works or Kauffman [13,14] and Thomas [23,24], they are extensively used as models of gene networks. In this context (as in many other applicative contexts) fixed points are of special interests: they correspond to the stable states of the systems and often have biological interpretations. Since experimental data often concern the structure of the network, it is interesting to try to extract, from this structure, information on fixed points (and in particular on the number of fixed points). Several works have been done in this direction, see [1,2,7,16,20,22,26] for example.

In this paper, we are interested in the number of fixed points in AND-OR-NOT networks (AND-OR-NOT-nets for short). These are Boolean networks in which the update function associated with each component is either a conjunction of literals or a disjunction of literals (*i.e.* each update function can be written as a Boolean formula which uses only the AND operator, or only the OR operator, and where the NOT operator can only precede a Boolean variable). Our interest for this class of Boolean networks is twofold. Firstly, every Boolean network can be represented, up to an increase of the number of components, under the form of an AND-OR-NOT-net. Secondly, an AND-OR-NOT-net can be represented, without loss of information, by a labelled digraph obtained from the interaction graph by labeling each arc by a sign (positive of negative) and each vertex by a type (AND or OR). This make easier the study of the relationships between structure and dynamics, in particular because graph theoretic tools and results can be used.

The main result of this paper, the following, illustrates this:

(1) *For every AND-OR-NOT-net N with n components and a loop-less connected interaction graph, there exists a connected graph G with at most n vertices such that the number of fixed points in N is at most the number of maximal independent sets in G.*

The maximum number $\mu(n)$ of maximal independent sets in a connected graph with n vertices is known [9]. According to (1), $\mu(n)$ is an upper-bound on the number of fixed points in an AND-OR-NOT-net with n components; and few additional arguments are needed to show that this upper bound is tight.

Two papers are particularly close to this work. Recently, Veliz-Cuba and Laubenbacher [26] study the number of fixed points in AND-NOT-nets (*i.e.* AND-OR-NOT-nets without OR-vertex) and in negative AND-NOT-nets, *i.e.* AND-NOT-nets in which each update function is a conjunction of negative literals. Mainly, they show two basic results that we independently obtain:

(2) *If a negative AND-NOT-net N has a loop-less symmetric interaction graph, then there is a one-to-one correspondence between the fixed points of N and the maximal independent sets of its interaction graph* [26].

(3) *The number of fixed points in a negative AND-NOT-net is at most the number of maximal independent sets in its interaction graph* [26].

As an immediate consequence, the authors pointed out that the number of fixed points in a negative AND-NOT-net with n components and a loop-less connected interaction graph is at most $\mu(n)$; this is a particular case the bound that we get with (1). Besides, the authors show that, given an AND-NOT-net with n components, there exists a negative AND-NOT-net with at most $2n$ components and the same number of fixed points. Consequently, an AND-NOT-net with n components and a loop-less connected interaction graph has at most $\mu(2n)$ fixed points. However, this is far from the right upper bound $\mu(n)$ given by (1), and $\mu(2n)$ does not really make sense because $2^n < \mu(2n)$ when $n \geq 9$.

The second close paper, by Aracena, Demongeot and Goles [2], is behind this work. It concerns AND-OR-nets (each update function is a conjunction or a disjunction of positive literals). The main result is the following:

(4) *For each AND-OR-net N with n components and a loop-less connected interaction graph, there exists a loop-less connected bipartite graph G with n vertices such that the number of fixed points in N is at most the number of maximal independent sets in G* [2].

The maximum number $\eta(n)$ of maximal independent sets in a loop-less connected bipartite graph with n vertices is known [15]. According to (4), $\eta(n)$ is an upper-bound on the number of fixed points in an AND-OR-net with n components; and again, few additional arguments are needed to show that this upper bound is tight. It is worth noting that even if AND-OR-nets are particular AND-OR-NOT-nets, our result is not a generalization of this one, because $\eta(n)$ is much smaller than $\mu(n)$.

In [2], the proof of the existence of the bipartite graph G with the property given in (4) is constructive: starting from N and using successively three graph transformations, G is obtained in polynomial time. Here, we use successively five polynomial graph transformations to obtain the graph G with the properties given in (1). Unfortunately none of the three graph transformations introduced in [2] is used; these transformations seem to be useless for the class of AND-OR-NOT-nets. Nevertheless, we think that some of the transformations introduced here could be of independent interest.

The paper is organized as follows. Definitions and notations are given in Section 2. The main results is formally stated and discuss in Section 3, and it is proved in Section 4. In Section 5 we characterize AND-OR-NOT-nets reaching the upper bound given in Section 3. Section 6 deals with the presence of loops in the interaction graph. A conclusion and some future research directions are given in Section 7.

2 Definitions and notations

A *digraph* (or *directed graph*) G consists in a finite set *vertices* (or *nodes*) $V(G)$ and a set of *arcs* (or *directed edges*) $E(G) \subseteq V(G) \times V(G)$; for convenience, we always assume that $V(G) = \{1, \dots, n\}$. Paths and cycles are always directed and seen as subdigraphs. An arc from a vertex to itself is a *loop*. A vertex of in-degree zero is a *source*. A digraph G is *symmetric* if $(u, v) \in E(G)$ for all $(v, u) \in E(G)$ with $u \neq v$, and G is *trivial* if it has a unique vertex and no arc. We see (undirected) *graph* as loop-less symmetric digraphs. The *underlying graph* of a digraph G is the (undirected) graph H defined as follows: $V(H) = V(G)$, and $(u, v) \in E(H)$ if and only if $u \neq v$ and $(u, v) \in E(G)$ or $(v, u) \in E(G)$. An *independent set* of G is a subset $I \subseteq V(G)$ such that $(u, v) \notin E(G)$ for all distinct $u, v \in I$. The set of independent sets of G is denoted $\mathrm{IS}(G)$, and the set of maximal independent sets of G (w.r.t. inclusion) is denoted $\mathrm{MIS}(G)$. Clearly, if H is the underlying graph of G then $\mathrm{IS}(H) = \mathrm{IS}(G)$ and $\mathrm{MIS}(H) = \mathrm{MIS}(G)$.

A *signed digraph* is a digraph G in which each arc is either *positive* or *negative*; the set of positive (resp. negative) arcs of G is denoted by $E^+(G)$ (resp. $E^-(G)$). If (u, v) is a positive (resp. negative) arc of G, we say that u is a *positive* (resp. *negative*) *predecessor* of v. The set of positive (resp. negative) predecessors of v in G is denoted $P^+_G(v)$ (resp. $P^-_G(v)$). The set of predecessor of v in G is $P_G(v) = P^+_G(v) \cup P^-_G(v)$. A cycle of G is *positive* (resp. *negative*) if it has an even (resp. odd) number of negative arcs.

An *AND-OR-NOT-net* is a signed digraph G in which each vertex is either an *AND-vertex* or a *OR-vertex*; the set of AND-vertices (resp. OR-vertices) is denoted by $V_{\mathrm{AND}}(G)$ (resp. $V_{\mathrm{OR}}(G)$). Vertices of G are often called *components*. Two vertices have the same *type* if they are both AND-vertices or both OR-vertices. An *AND-NOT-net* is an AND-OR-NOT-net with only AND-vertices (thus, AND-NOT-nets may be identified with signed digraphs). An AND-NOT-net is *negative* if every arc that is not a loop is negative (thus, loops-less negative AND-NOT-nets may be identified with digraphs).

AND-OR-NOT-nets (which are simply labelled digraphs) take a sense in the light of the following definitions. Let G be an AND-OR-NOT-net. The set of possible *configurations* of G is the set of maps x from $V(G)$ to $\{0,1\}$; it is denoted $\{0,1\}^n$. The *update function* associated with a vertex v of G is the Boolean function f_v^G from $\{0,1\}^n$ to $\{0,1\}$ defined by:

(1) If $v \in V_{\mathrm{AND}}(G)$, then $f_v^G(x) = 0$ if and only if v has a positive predecessor u with $x(u) = 0$ or a negative predecessor u with $x(u) = 1$.

(2) If $v \in V_{\mathrm{OR}}(G)$, then $f_v^G(x) = 1$ if and only if v has a positive predecessor u with $x(u) = 1$ or a negative predecessor u with $x(u) = 0$.

In other words, if v is an AND-vertex then f_v^G is an AND-function (a conjunction of positive or negative literals corresponding to the positive or negative predecessors of v); and if v is an OR-vertex, then f_v^G is an OR-function (a disjunction of positive or negative literals).

The *global transition function* associated with G is the function f^G from $\{0,1\}^n$ to $\{0,1\}^n$ defined by $f^G(x)(v) = f_v^G(x)$ for all $x \in \{0,1\}^n$ and $v \in V(G)$. The (parallel) dynamics of an AND-OR-NOT-net is described by the successive iterations of f^G. The fixed points of f^G then correspond to the stable configurations of the network. In this paper, we are only interested in the number of fixed points of f^G. The set of fixed points of f^G is denoted $\mathrm{FP}(G)$. In the following, we abusively refer $\mathrm{FP}(G)$ as the set of fixed points of G.

3 Maximum number of fixed points

We are interested in the following question: what is the maximum number of fixed points that a loop-less connected AND-OR-NOT-nets with n vertices can have? Our starting point is the following easy but fundamental observation:

Proposition 1 *If H is a loop-less symmetric negative AND-NOT-net, then*

$$|\mathrm{FP}(H)| = |\mathrm{MIS}(H)|.$$

Thus, for loop-less negative AND-NOT-nets, our question is equivalent to the following question: what is the maximum number of maximal independent sets that a connected graph with n vertices can have? This question has been answered more than twenty years ago.

Theorem 1 [5,9] *The maximum number $\mu(n)$ of maximal independent sets in connected graph with n vertices is defined as follows: if $n < 6$ then $\mu(n) = n$, and otherwise*

$$\mu(n) = \begin{cases} 2 \cdot 3^{s-1} + 2^{s-1} & \text{if } n = 3s \\ 3^s + 2^{s-1} & \text{if } n = 3s+1 \\ 4 \cdot 3^{s-1} + 3 \cdot 2^{s-2} & \text{if } n = 3s+2 \end{cases}$$

The main result of this paper is the following. Together with Proposition 1 and Theorem 1, it answers our question.

Theorem 2 (Main result) *For every loop-less connected AND-OR-NOT-net G, there exists a loop-less connected symmetric negative AND-NOT-net H such that*

$$|V(H)| \leq |V(G)| \quad \text{and} \quad |\mathrm{FP}(H)| \geq |\mathrm{FP}(G)|.$$

Corollary 3 *The maximum number of fixed points in a loop-less connected AND-OR-NOT-net with n vertices is $\mu(n)$.*

In [9], it was showed that for every $n \geq 6$, there exists (up to isomorphism) a unique graph H_n with n vertices such that $|\mathrm{MIS}(H_n)| = \mu(n)$. So following Proposition 1, for all $n \geq 6$, the loop-less symmetric negative AND-NOT-net G_n with H_n as underlying graph is such that $|\mathrm{FP}(G_n)| = \mu(n)$. So the upper-bound given in Corollary 3 is the best possible.

Actually, the maximum number of maximal independent sets has been established in the general case and for several particular classes of graphs (see [18,19,5,9,17,11,15,12,10,3,6] for instance).

Theorem 4 [18] *The maximum number $\lambda(n)$ of maximal independent sets in a graph with n vertices is defined as follows: if $n = 1$ then $\lambda(n) = 1$, and otherwise*

$$\lambda(n) = \begin{cases} 3^s & \text{if } n = 3s \\ 4 \cdot 3^{s-1} & \text{if } n = 3s+1 \\ 2 \cdot 3^s & \text{if } n = 3s+2 \end{cases}$$

Clearly, Theorem 2 remains valid if "connected" is removed from the statement (in the condition and the conclusion). From this observation Theorem 4 and Proposition 1, we get:

Corollary 5 *The maximum number of fixed points in a loop-less AND-OR-NOT-net with n vertices is $\lambda(n)$.*

It was showed that there exists (up to isomorphism) a unique graph H_n with n vertices such that $|\mathrm{MIS}(H_n)| = \lambda(n)$ [18]. So, as above, we deduce that the upper-bound given in Corollary 5 is the best possible.

4 Proof of Theorem 2

The proof is constructive. It involves five AND-OR-NOT-net transformations, denoted from T_0 to T_4. The first transformation T_0 gives, from any loop-less connected AND-OR-NOT-net G with n vertices, a loop-less connected AND-NOT-net G^0 with n vertices and the same number of fixed points. The four other transformations are transformations on AND-NOT-nets. Each of them keeps the connectivity, never increases the number of vertices, and never decreases the number of fixed points. Moreover, the AND-net obtain from G^0 by applying successively T_1, T_2, T_3 and T_4 is always loop-less, symmetric and negative, and from this the theorem follows.

Before defining these transformations and their properties, we first state a lemma that will be used several times.

Lemma 6 *Let G be an AND-NOT-net, and let H be a strongly connected component of $G \setminus E^-(G)$. Then $x(u) = x(v)$ for every $x \in \mathrm{FP}(G)$ and $u, v \in V(H)$.*

PROOF. Let $x \in \mathrm{FP}(G)$. If $x(v) = 0$ for some $v \in V(H)$, then $x(u) = 0$ for every successor u of v in H, and since H is strong, we deduce that $x(v) = 0$ for all $v \in V(H)$. If $x(v) = 1$ for some $v \in V(H)$, then $x(u) = 1$ for every predecessor u of v in H, and since H is strong, we deduce that $x(v) = 1$ for all $v \in V(H)$. Thus $x(u) = x(v)$ for all $x \in \mathrm{FP}(G)$ and $u, v \in V(H)$. □

4.1 Transformation T_0 (making AND-NOT-nets from AND-OR-NOT-nets)

Transformation T_0 maps every AND-OR-NOT-net G to the AND-NOT-net G^0 obtained from G by changing the sign of each arc linking vertices of different type, and by changing the type of OR-vertices. Formally, denoting $E_0^+(G)$ (resp. $E_0^-(G)$) the set of positive (resp. negative) arcs (u, v) of G such that u

and v have not the same type, G^0 is defined by:

$$\begin{aligned} V(G^0) &= V_{\text{AND}}(G^0) = V(G) \\ E^+(G^0) &= (E^+(G) \setminus E_0^+(G)) \cup E_0^-(G) \\ E^-(G^0) &= (E^-(G) \setminus E_0^-(G)) \cup E_0^+(G) \end{aligned}$$

The following lemma is an easy exercise.

Lemma 7 *For every AND-OR-NOT-net G we have* $|\mathrm{FP}(G)| = |\mathrm{FP}(G^0)|$.

4.2 Transformation T_1 (removing constant vertices)

Let G be an AND-NOT-net. Transformation T_1 is a technical step allowing AND-NOT-nets to have some properties making possible the use of the other transformations. Roughly speaking, it consists in gluing together vertices with a constant level in fixed points.

Let $V_{\text{cst}}(G) = V_{\text{cst0}}(G) \cup V_{\text{cst1}}(G)$, where $V_{\text{cst0}}(G)$ and $V_{\text{cst1}}(G)$ are the subsets of $V(G)$ inductively defined in the following way:

(1) If there exists two strongly connected components H and H' in $G \setminus E^-(G)$ (not necessarily distinct) such that G has both a positive and a negative arc from $V(H')$ to $V(H)$, then $V(H) \subseteq V_{\text{cst0}}(G)$; and all the sources of G are in $V_{\text{cst1}}(G)$.

(2) For all $v \in V(G)$: if v has a positive predecessor in $V_{\text{cst0}}(G)$ or a negative predecessor in $V_{\text{cst1}}(G)$, then $v \in V_{\text{cst0}}(G)$; and if all the positive predecessors of v are in $V_{\text{cst1}}(G)$ and all the negative predecessors of v are in $V_{\text{cst0}}(G)$, then $v \in V_{\text{cst1}}(G)$.

See Figure 1 for an illustration.

Lemma 8 *Let G be an AND-NOT-net and $x \in \mathrm{FP}(G)$. If $v \in V_{\text{cst0}}(G)$ then $x(v) = 0$, and if $v \in V_{\text{cst1}}(G)$ then $x(v) = 1$.*

PROOF. Let $x \in \mathrm{FP}(G)$. We proceed by induction (following the inductive definition of $V_{\text{cst}}(G)$), and we only prove the base case, since the induction step is obvious. If v is a source, then by definition, f_v^G is a constant function equal to one thus $x(v) = 1$. Now, suppose that there exists two strongly connected components H and H' in $G \setminus E^-(G)$ such that G has a positive arc (u', u) and a negative arc (w', w) with $u', w' \in V(H')$ and $u, w \in V(H)$. Let $v \in V(H)$. If $x(u') = 0$ then $f_u^G(x) = 0 = x(u)$ and we deduce from

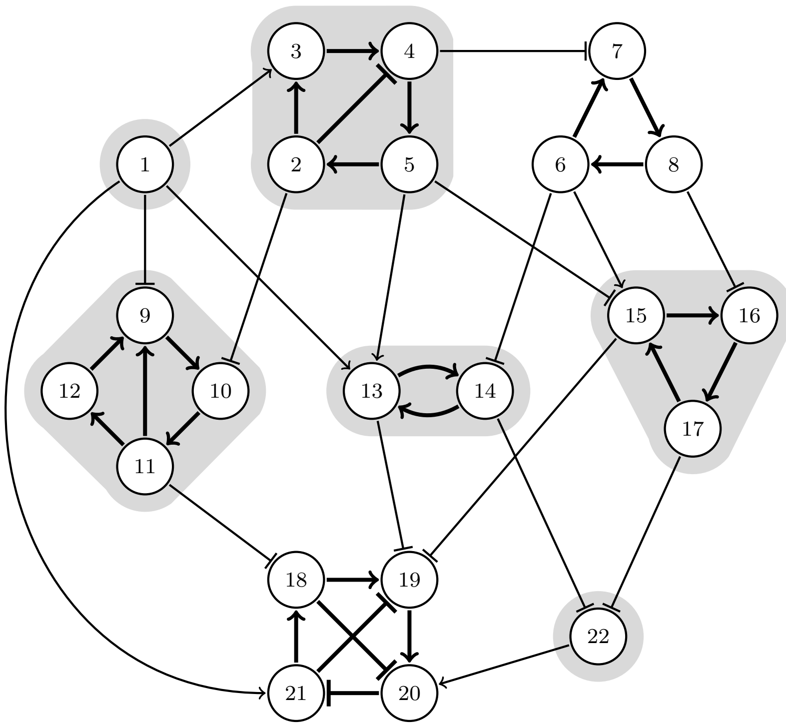

Fig. 1. An AND-NOT-net G. Negative arcs are represented by T–end arrows, and positive arcs by normal arrows (this graphical convention is used throughout the paper). Arcs that connect vertices belonging to the same strongly connected component are thick. Vertices in gray region are in $V_{\text{cst}}(G)$. More precisely, $V_{\text{cst1}}(G) = \{1, 22\}$ and $V_{\text{cst0}}(G) = \{2, 3, 4, 5\} \cup \{9, 10, 11, 12\} \cup \{13, 14\} \cup \{15, 16, 17\}$.

Lemma 6 that $x(v) = x(u) = 0$. Otherwise, $x(u') = 1$ thus by Lemma 6 we have $x(w') = 1$, so thus $f^G_w(x) = 0 = x(w)$ and we deduce from Lemma 6 that $x(v) = x(w) = 0$. $\square$

Remark 9 *Let $v \in V(G)$, and suppose that there exists a constant $c \in \{0, 1\}$ such that $x(v) = c$ for all $x \in \mathrm{FP}(G)$. Then v is not necessarily in $V_{\text{cst}}(G)$. For example, see Figure 2.*

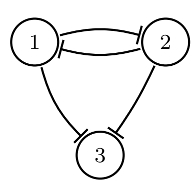

Fig. 2. Example of AND-NOT-net G where $x(3) = 0$ for all $x \in \mathrm{FP}(G)$ and $V_{\text{cst}}(G) = \emptyset$.

Transformation T_1 maps every AND-NOT-net G to the AND-NOT-net G^1

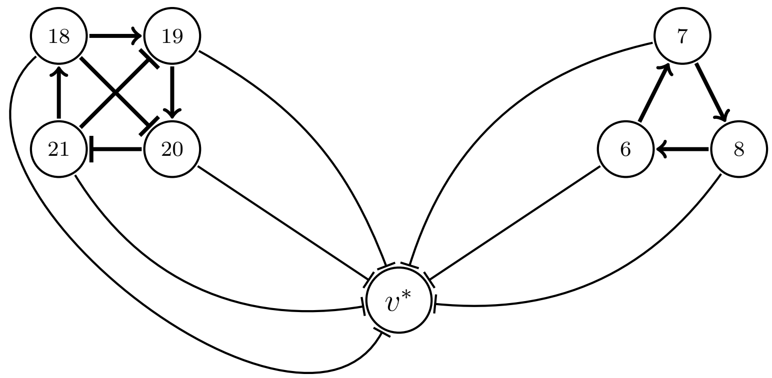

Fig. 3. The transformation G^1 by T_1 of the AND-NOT-net G of Figure 1.

defined in the following way. If $V_{\rm cst}(G)=\emptyset$, then $G^1=G$, and otherwise:

$$\begin{aligned}V(G^1)&=(V(G)\setminus V_{\rm cst}(G))\cup\{v^*\}\\E^+(G^1)&=E^+(G\setminus V_{\rm cst}(G))\\E^-(G^1)&=E^-(G\setminus V_{\rm cst}(G))\cup\{(u,v^*)\,|\,u\in V(G)\setminus V_{\rm cst}(G)\}\end{aligned}$$

Thus when $V_{\rm cst}(G)\neq\emptyset$, G^1 is obtained from the sub-AND-NOT-net $G\setminus V_{\rm cst}(G)$ by adding a new vertex v^* and a negative arc from u to v^* for each vertex u not in $V_{\rm cst}(G)$. Clearly, in all cases $|V(G^1)|\leq|V(G)|$. See Figure 3 for an illustration.

Lemma 10 *For every AND-NOT-net G we have* $|\mathrm{FP}(G)|\leq|\mathrm{FP}(G^1)|$.

PROOF. If $G=G^1$ there is nothing to prove, so assume that $G\neq G^1$. Consider the function that maps every configuration $x\in\mathrm{FP}(G)$ to the configuration $\tilde{x}$ of G^1 defined as follows: $\tilde{x}(v)=x(v)$ for all $v\neq v^*$, and

$$\tilde{x}(v^*)=\prod_{v\in V(G^1)\setminus v^*}(1-\tilde{x}(v)).$$

Clearly, $x\mapsto\tilde{x}$ is an injective function: if $x,y\in\mathrm{FP}(G)$ and $x\neq y$, then following Lemma 8, $x(v)\neq y(v)$ for some $v\in V(G)\setminus V_{\rm cst}(G)\subseteq V(G^1)$, and it follows that $\tilde{x}(v)\neq\tilde{y}(v)$. Thus, it is sufficient to prove that $\tilde{x}\in\mathrm{FP}(G^1)$ for all $x\in\mathrm{FP}(G)$. Let $x\in\mathrm{FP}(G)$ and $v\in V(G^1)\setminus v^*$.

(1) Suppose that $x(v)=1$. Then, there is no $u\in P^+_G(v)$ with $x(u)=0$ and no $u\in P^-_G(v)$ with $x(u)=1$. Since $G^1\setminus v^*=G\setminus V_{\rm cst}(G)$ and since $v^*\notin P_{G^1}(v)$, we deduce that $P^+_{G^1}(v)\subseteq P^+_G(v)$ and $P^-_{G^1}(v)\subseteq P^-_G(v)$. Since $\tilde{x}(u)=x(u)$ for all $u\neq v^*$ we deduce that $f^{G^1}_v(\tilde{x})=1=x(v)=\tilde{x}(v)$.

(2) Suppose that $x(v)=0$. Then either there exists $u\in P^+_G(v)$ with $x(u)=0$ or $u\in P^-_G(v)$ with $x(u)=1$. Suppose that there exists $u\in P^+_G(v)$

with $x(u) = 0$, the other case is similar. Since $v \notin V_{\rm cst}(G)$, we have $u \notin V_{\rm cst0}(G)$, and since $x(u) = 0$, by Lemma 8, we have $u \notin V_{\rm cst1}(G)$. Thus $u \notin V_{\rm cst}(G)$. Consequently $u \in P^+_{G^1}(v)$ and since $\tilde{x}(u) = x(u) = 0$, it follows that $f^{G^1}_v(\tilde{x}) = 0 = x(v) = \tilde{x}(v)$.

Thus $f^{G^1}_v(\tilde{x}) = \tilde{x}(v)$ for all $v \in V(G^1) \setminus v^*$. By the definition $f^{G^1}_{v^*}(\tilde{x}) = \tilde{x}(v^*)$, thus $\tilde{x} \in {\rm FP}(G^1)$. $\square$

Remark 11 *Actually,* $|{\rm FP}(G)| = |{\rm FP}(G^1)|$ *for every AND-NOT-net* G*, as showed in Appendix A. However,* $|{\rm FP}(G)| \leq |{\rm FP}(G^1)|$ *is sufficient for our propose.*

An AND-NOT-net G has the *property* P_1 if it is connected, has no loop, has no source, and satisfies the following *property* Q_1: for every strongly connected components H and H' of $G \setminus E^-(G)$ (not necessarily distinct), all the arcs of G from $V(H)$ to $V(H')$ are either positive or negative.

Lemma 12 *If* G *is a loop-less connected AND-NOT-net, then either* G^1 *is trivial or it has the property* P_1*.*

PROOF. Suppose that G^1 is not trivial, and let us prove that it has the property P_1. If $G^1 = G$, then $V_{\rm cst}(G) = \emptyset$ thus G has no source and the property Q_1; and since (by hypothesis) G is connected and has no loop, G^1 has the property P_1. So suppose that $G^1 \neq G$, that is, $V_{\rm cst}(G) \neq \emptyset$.

Clearly, G^1 is connected since $V(G^1) \setminus v^*$ is the set of predecessors of v^*. Also, there is no loop on v^*, and since G has no loop, $G^1 \setminus v^* = G \setminus V_{\rm cst}(G)$ has no loop. So G^1 has no loop. Suppose that G^1 has a source v. Since G^1 is not trivial v^* is not a source thus $v \in V(G)$. Since $v \notin V_{\rm cst}(G)$, v is not a source of G, and we deduce that, in G, all the predecessors of v are in $V_{\rm cst}(G)$. But then $v \in V_{\rm cst}(G)$, a contradiction. Thus G^1 has no source.

Suppose finally that G^1 has not the property Q_1. Let H and H' be strongly connected components of $G^1 \setminus E^-(G^1)$ (not necessarily distinct) such that G^1 has both a positive and a negative arc from $V(H)$ to $V(H')$. Then H and H' are distinct strongly connected components of $(G^1 \setminus E^-(G^1)) \setminus v^*$. Since $(G^1 \setminus E^-(G^1)) \setminus v^* = (G \setminus E^-(G)) \setminus V_{\rm cst}(V)$, there exists strongly connected components L and L' in $G \setminus E^-(G)$ with $V(H) \subseteq V(L)$ and $V(H') \subseteq V(L')$ (L and L' are not necessarily distinct). But then G has both a positive and a negative arc from $V(L)$ to $V(L')$, thus $V(H') \subseteq V(L') \subseteq V_{\rm cst0}(G) \subseteq V_{\rm cst}(G)$, a contradiction. So G^1 has the property Q_1, and we deduce that it has the property P_1. $\square$

4.3 Transformation T_2 (removing cycles with only positive arcs)

Transformation T_2 maps every AND-NOT-net G with the property Q_1 to the AND-NOT-net G^2 defined in the following way. Let $H_1, \dots, H_r$ be the strongly connected components of $G \setminus E^-(G)$. For every $1 \leq k \leq r$, let v_k be the smallest vertices in $V(H_k)$. Let $E_2^+(G)$ (resp. $E_2^-(G)$) be the set of couples (v_k, v_l) with $k \neq l$ such that G has at least one positive (resp. negative) arc from $V(H_k)$ to $V(H_l)$. Since G has the property Q_1, $E_2^+(G) \cap E_2^-(G) = \emptyset$. This allows us to define G^2 by

$$\begin{aligned}
&V(G^2) = V(G)\\
&E^+(G^2) = E_2^+(G)\\
&E^-(G^2) = E_2^-(G) \cup \{(v_k, u), (u, v_k) \,|\, 1 \leq k \leq r,\ u \in V(H_k) \setminus v_k\}
\end{aligned}$$

See Figure 4 for an illustration. Note that for every $u \in V(H_k)$, if $u \neq v_k$ then $f_u^{G^2}(x) = 1 - x(v_k)$.

An AND-NOT-net G has the *property* P_2 if it is connected, has no source, and has no cycle with only positive arcs (note that for every AND-NOT-net G with the property Q_1, G^2 has no cycle with only positive arcs).

Lemma 13 *If G is an AND-NOT-net with the property P_1, then G^2 has the property P_2 and* $|\mathrm{FP}(G)| \leq |\mathrm{FP}(G^2)|$.

PROOF. Since G has the property P_1, it is connected and has no source, and so G^2 has the property P_2. Let us prove that $|\mathrm{FP}(G)| \leq |\mathrm{FP}(G^2)|$. Let $H_1, \dots, H_r$ be the strongly connected components of $G \setminus E^-(G)$. For every $1 \leq k \leq r$, let v_k be the smallest vertices in $V(H_k)$. Let $U = V(G) \setminus \{v_1, v_2, \dots v_r\}$. Consider the permutation mapping each configuration x of G to the configuration $\tilde{x}$ of G defined by:

$$\tilde{x}(u) = \begin{cases} 1 - x(u) & \text{if } u \in U \\ x(u) & \text{ortherwise.} \end{cases}$$

We prove that $\tilde{x} \in \mathrm{FP}(G^2)$ for all $x \in \mathrm{FP}(G)$. Let $x \in \mathrm{FP}(G)$ and $1 \leq k \leq r$.

We first prove that $f_u^{G_2}(\tilde{x}) = \tilde{x}(u)$ given any $u \in V(H_k) \setminus v_k$. Indeed, if $\tilde{x}(u) = 0$ then $x(u) = 1$ thus, by Lemma 6, $\tilde{x}(v_k) = x(v_k) = 1$ and we deduce that $f_u^{G^2}(\tilde{x}) = 1 - \tilde{x}(v_k) = 0$. Similarly, if $\tilde{x}(u) = 1$ then $x(u) = 0$ thus, by Lemma 6, $\tilde{x}(v_k) = x(v_k) = 0$ and we deduce that $f_u^{G^2}(\tilde{x}) = 1 - \tilde{x}(v_k) = 1$. Thus $f_u^{G_2}(\tilde{x}) = \tilde{x}(u)$ in all cases.

We now prove that $f^{G^2}_{v_k}(\tilde{x}) = \tilde{x}(v_k)$. Suppose first that $\tilde{x}(v_k) = 0$. If H_k is not trivial then there exists $u \in P^-_{G^2}(v_k) \cap V(H_k)$ and since, by Lemma 6, $x(u) = x(v_k) = \tilde{x}(v_k) = 0$, we have $\tilde{x}(u) = 1$ and we deduce that $f^{G^2}_{v_k}(\tilde{x}) = 0$. Suppose that H_k is trivial. Since $x(v_k) = \tilde{x}(v_k) = 0$, one of the following two condition holds:

(1) There exists $u \in P^+_G(v_k)$ with $x(u) = 0$. Then $u \in V(H_l)$ for some $l \neq k$ so $v_l \in P^+_{G^2}(v_k)$. Since, by Lemma 6, we have $\tilde{x}(v_l) = x(v_l) = x(u) = 0$ we deduce that $f^{G^2}_{v_k}(\tilde{x}) = 0$.

(2) There exists $u \in P^-_G(v_k)$ with $x(u) = 1$. Then $u \in V(H_l)$ for some $l \neq k$ so $v_l \in P^-_{G_2}(v_k)$. Since, by Lemma 6, we have $\tilde{x}(v_l) = x(v_l) = x(u) = 1$ we deduce that $f^{G^2}_{v_k}(\tilde{x}) = 0$.

So in all cases, $f^G_{v_k}(\tilde{x}) = 0 = \tilde{x}(v_k)$. Suppose now that $\tilde{x}(v_k) = 1$, and suppose, for a contradiction, that $f^{G^2}_{v_k}(\tilde{x}) = 0$. Then one of the following two conditions holds:

(1) There exists $u \in P^+_{G_2}(v_k)$ with $\tilde{x}(u) = 0$. Then $u = v_l$ for some $l \neq k$, thus there exists an arc $(w, t) \in E^+(G)$ with $w \in V(H_l)$ and $t \in V(H_k)$. Since, by Lemma 6, $x(w) = x(v_l) = \tilde{x}(v_l) = \tilde{x}(u) = 0$, we have $f^G_t(x) = 0$. But, by Lemma 6, we have $x(t) = x(v_k) = \tilde{x}(v_k) = 1$, a contradiction.

(2) There exists $u \in P^-_{G^2}(v_k)$ with $\tilde{x}(u) = 1$. Suppose that $u \in V(H_k)$. Then $u \in U$ so $x(u) \neq \tilde{x}(u) = 1$, but by Lemma 6, $x(u) = x(v_k) = \tilde{x}(v_k) = 1$, a contradiction. So $u = v_l$ for some $l \neq k$, thus there exists an arc $(w, t) \in E^-(G)$ with $w \in V(H_l)$ and $t \in V(H_k)$. Since, by Lemma 6, $x(w) = x(v_l) = \tilde{x}(v_l) = \tilde{x}(u) = 1$, we have $f^G_t(x) = 0$. But, by Lemma 6, we have $x(t) = x(v_k) = \tilde{x}(v_k) = 1$, a contradiction.

Since there is a contradiction in both cases, $f^{G_2}_{v_k}(\tilde{x}) = 1 = \tilde{x}(v_k)$. $\square$

Remark 14 *Actually, we have* $|\mathrm{FP}(G)| = |\mathrm{FP}(G^2)|$, *as showed in Appendix B, but* $|\mathrm{FP}(G)| \leq |\mathrm{FP}(G^2)|$ *is enough for our propose.*

4.4 Transformation T_3 (removing positive arcs)

The transformation T_3 maps every AND-NOT-net G to the AND-NOT-net $G^3 = T_3(G)$ defined in the following way. Let $E^-_3(G)$ denotes the set of couples of vertices (u, v) such that for at least one vertex w, $(u, w) \in E^-(G)$ and G

has a path from w to v with only positive arcs. Then

$$V(G^3) = V(G)$$
$$E^+(G^3) = \emptyset$$
$$E^-(G^3) = E^-(G) \cup E_3^-(G)$$

See Figure 5 for an illustration.

An AND-NOT-net G has the *property* P_3 if it connected, has no source, and has no positive arc.

Lemma 15 *If G is an AND-NOT-net with the property P_2, then G^3 has the property P_3 and* $\mathrm{FP}(G) \subseteq \mathrm{FP}(G^3)$.

PROOF. We first prove that G^3 has no source, using the fact that G has no source and no cycle with only positive arcs. Let $v \in V(G)$, and let P be the longest path of G with only positive arcs and with v as terminal vertex. Let u be the initial vertex of P (if v has only negative predecessors, then the path is of length zero and $u = v$). Suppose that u has a positive predecessor w. If $w \notin V(P)$, then P is not of maximal length, and if $w \in V(P)$ then G has a cycle with only positive arcs, a contradiction. Thus u has only negative predecessors in G. Let w be one of them. Then $(w, v) \in E_3^-(G)$ so v is not a source of G^3.

We now prove that G^3 is connected. Suppose, for a contradiction, that G^3 is not connected. Since G is connected, G^3 has two connected components, say G_1^3 and G_2^3, such that G has at least one arc (v_1, v_2) with $v_1 \in V(G_1^3)$ and $v_2 \in V(G_2^3)$. Since (v_1, v_2) is not an arc of G^3, we have $(v_1, v_2) \in E^+(G)$. Let P be the longest path of G with only positive arcs and with (v_1, v_2) as final arc. Let u be the initial vertex of P. As above we show that u has only negative predecessors in G. Let w be one of them. Then (w, v_1) and (w, v_2) are negative arcs of G^3, thus G_1^3 and G_2^3 are connected, a contradiction.

Let $x \in \mathrm{FP}(G)$, and let us prove that $x \in \mathrm{FP}(G^3)$. Let $v \in V(G)$. If $x(v) = 0$, then one of the two following cases holds:

(1) There exists $u \in P_G^+(v)$ with $x(u) = 0$. Let P be the longest path of G with only positive arcs, with u as final vertex, and such that $x(w) = 0$ for all $w \in V(P)$. Let w be the initial vertex of P. If there exists $t \in P_G^+(w)$ with $x(t) = 0$, then $t \in V(P)$ (since P is of maximal length) and so G has a cycle with only positive arcs, a contradiction. Thus $x(t) = 1$ for all $t \in P_G^+(w)$. Since $x(w) = 0$, we deduce that there exists $t \in P_G^-(w)$ with $x(t) = 1$. Then $(t, v) \in E^-(G^3)$ thus $f_v^{G^3}(x) = 0 = x(v)$.

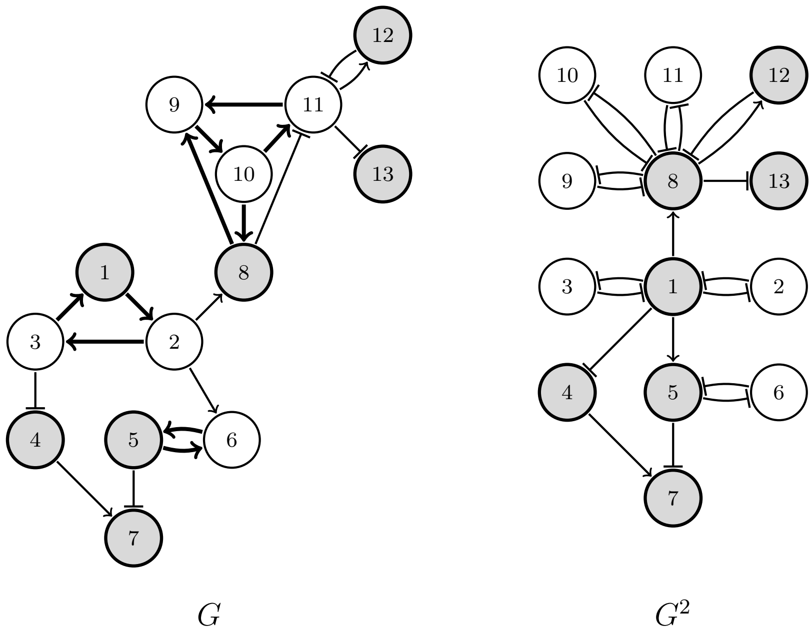

Fig. 4. An AND-NOT-net G (with the property Q_1) and its transformation G^2 by T_2. $G \setminus E^{-}(G)$ contains 7 strongly connected components. Arcs in strongly connected components are bolded, and the smallest vertex in each strongly connected component is in gray.

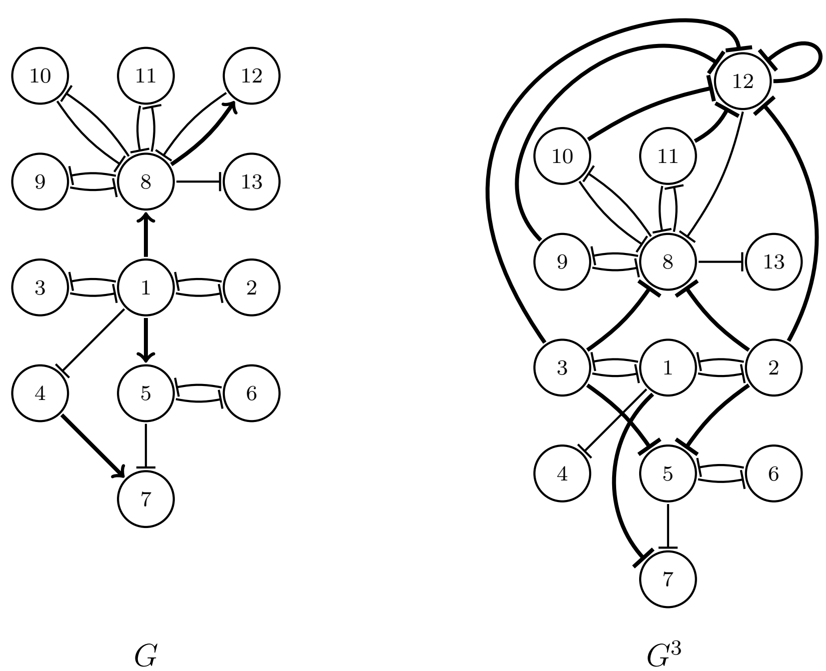

Fig. 5. An AND-NOT-net G and its transformation G^3 by T_3.

(2) There exists $u \in P^-_G(v)$ with $x(u) = 1$. Then $(u, v) \in E^-(G^3)$ thus $f^{G^3}_v(x) = 0 = x(v)$.

Suppose now that $x(v) = 1$, and suppose, for a contradiction, that $f^{G^3}_v(x) = 0$. Then there exists $u \in P^-_{G^3}(v)$ with $x(u) = 1$. If $(u, v) \in E^-(G)$ then $f^G_v(x) = 0$, a contradiction. Thus $(u, v) \in E^-_3(G)$, that is, G has a negative arc (u, w) and a path P from w to v with only positive arcs. Since $x(u) = 1$, we have $f^G_w(x) = 0 = x(w)$ and (following the path P) we deduce that $f^G_t(x) = 0 = x(t)$ for all $t \in V(P)$. In particular, $x(v) = 0$, a contradiction. Thus in all cases $f^{G^3}_v(x) = x(v)$ and so $x \in \mathrm{FP}(G^3)$. $\square$

Remark 16 *Actually, we have* $\mathrm{FP}(G) = \mathrm{FP}(G^3)$*, as showed in Appendix C, but* $\mathrm{FP}(G) \subseteq \mathrm{FP}(G^3)$ *is enough for our purpose.*

4.5 Transformation T_4 (symmetrization)

The transformation T_4 maps every signed AND-NOT-net G to the AND-NOT-net $T_4(G) = G^4$ defined by

$$V(G^4) = V(G)$$
$$E^+(G^4) = \emptyset$$
$$E^-(G^4) = (E^-(G) \cup \{(u, v) \mid (v, u) \in E^-(G)\}) \setminus \{(v, v) \mid v \in V(G)\}$$

Lemma 17 *If G is an AND-NOT-net with the property P_3, then G^4 is a loop-less connected symmetric AND-NOT-net with only negative arcs such that* $\mathrm{FP}(G) \subseteq \mathrm{FP}(G^4)$.

PROOF. It is obvious that G^4 is symmetric, has no loop, and no positive arcs. Then, since G is connected and has no positive arc, G^4 is connected too. It remains to prove that $\mathrm{FP}(G) \subseteq \mathrm{FP}(G^4)$. Let $x \in \mathrm{FP}(G)$ and $v \in V(G)$. If $x(v) = 0$ then there exists $u \in P^-_G(v)$ with $x(u) = 1$. Thus $u \neq v$, so $u \in P^-_{G^4}(v)$ and we deduce that $f^{G^4}_v(x) = 0$. Suppose now that $x(v) = 1$, and suppose, for a contradiction, that $f^{G^4}_v(x) = 0$. Then there exists $u \in P^-_{G^4}(v)$ with $x(u) = 1$. If $u \in P^-_G(v)$ then $f^G_v(x) = 0 \neq x(v)$, a contradiction. Thus $v \in P^-_G(u)$, and since $x(v) = 1$, we have $f^G_u(x) = 0 \neq x(u)$, a contradiction. Thus $f^{G^4}_v(x) = 1 = x(v)$. $\square$

Remark 18 *The inclusion in Lemma 17 is sometimes strict. For instance, if C_n is a directed cycle of length n with only negative arcs then* $|\mathrm{FP}(C_n)| \leq 2$ *(since C_n has no fixed point if n is odd and two fixed points otherwise) while the number of fixed points in $C^4_n = T_4(C_n)$ growths exponentially with n:* $|\mathrm{FP}(C^4_n)| \sim p^n$ *where $p > 1.3$ is the plastic number* [5].

4.6 Proof of Theorem 2

Let G be a loop-less connected AND-OR-NOT-net. Let $G^0 = T_0(G)$. If $G^1 = T_1(G^0)$ is trivial, then the theorem is obvious. So suppose that G^1 is not trivial. Following Lemmas 7, 10, 12, 13, 15 and 17: AND-NOT-nets $G^2 = T_2(G^1)$, $G^3 = T_3(G^2)$ and $G^4 = T_4(G^3)$ are well defined; G^4 is loop-less, connected, symmetric and negative. Furthermore

$$|\mathrm{FP}(G)| = |\mathrm{FP}(G^0)| \leq |\mathrm{FP}(G^1)| \leq |\mathrm{FP}(G^2)| \leq |\mathrm{FP}(G^3)| \leq |\mathrm{FP}(G^4)|.$$

Since it is clear that

$$|V(G)| = |V(G^0)| \geq |V(G^1)| = |V(G^2)| = |V(G^3)| = |V(G^4)|$$

the theorem is proved. Note that according to Remarks 11, 14 and 16:

$$|\mathrm{FP}(G^0)| = |\mathrm{FP}(G^1)| = |\mathrm{FP}(G^2)| = |\mathrm{FP}(G^3)|.$$

5 Extremal AND-NOT-nets

In this section, we characterize AND-OR-NOT-nets reaching the upper bound given in Corollaries 3 and 5, that is, we characterize loop-less connected AND-OR-NOT-nets with $\mu(n)$ fixed points, and loop-less AND-OR-NOT-nets with $\lambda(n)$ fixed points.

Let $n = 3s + r$ with $0 \leq r \leq 2 \leq s$. Let H_n be the graph described in Figure 6. It has been proved in [9] that H_n is the unique connected graph with n vertices and $\mu(n)$ maximal independent sets. Let G_n be the loop-less symmetric negative AND-NOT-net with H_n as underlying graph. According to Proposition 1, G_n is the unique loop-less symmetric connected negative AND-NOT-net with n vertices and $\mu(n)$ fixed points. Let $\mathcal{G}_n$ be the family of AND-NOT-nets containing G_n and all the connected AND-NOT-nets that we can obtain from G_n by removing some arcs that does not belong to a triangle (cycle of length three); since there are $2(s-1)$ such arcs, $|\mathcal{G}_n| = 3^{s-1}$.

Theorem 19 *Let G be a loop-less connected AND-NOT-net with n vertices. If $n \geq 6$, then G has $\mu(n)$ fixed points if and only if $G \in \mathcal{G}_n$. If $n \leq 5$, then G has $\mu(n)$ fixed points if and only if G is isomorphic to one of the AND-NOT-nets given in Figure 7.*

PROOF. Graphs with at most five vertices that maximize the number of maximal independent sets are given [9], and from this it is easy to check the

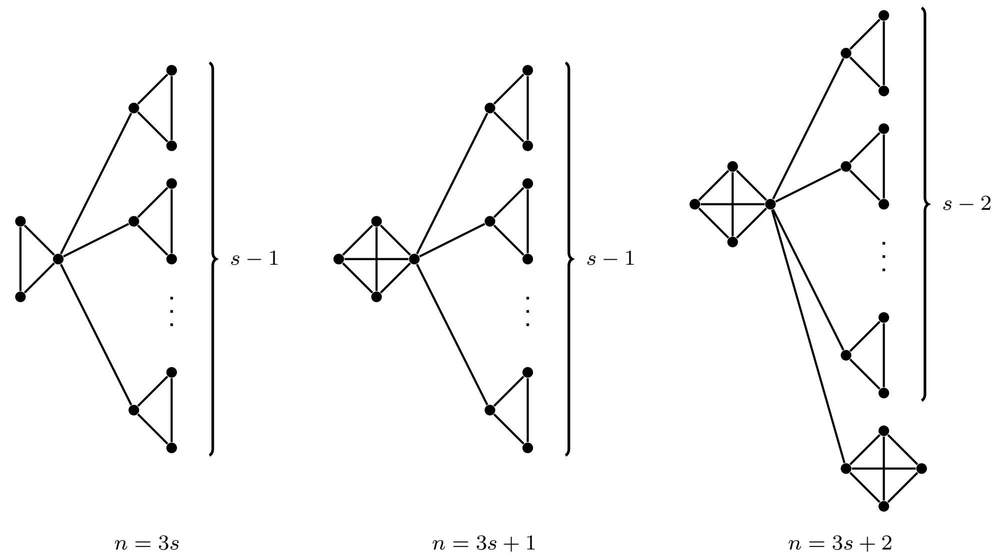

Fig. 6. Graphs H_n (a line between two nodes u and v means that both (v, u) and (u, v) are arcs of the graph). Note that in every case, there are $2(s-1)$ arcs that does belong to no triangles.

case $n \leq 5$. Suppose that $n \geq 6$. It is also easy to check that $|\mathrm{FP}(G)| = \mu(n)$ if $G \in \mathcal{G}_n$. So suppose that $|\mathrm{FP}(G)| = \mu(n)$, and let us prove that $G \in \mathcal{G}_n$. Let $G^1 = T_1(G)$. It is clear that G^1 is not trivial, thus $G^2 = T_2(G^1)$, $G^3 = T_3(G^2)$ and $G^4 = T_4(G^3)$ are well defined. Following Corollary 3 and Lemmas 10, 12, 13, 15 and 17, we have

$$|\mathrm{FP}(G)| = |\mathrm{FP}(G^1)| = |\mathrm{FP}(G^2)|, \qquad \mathrm{FP}(G^2) = \mathrm{FP}(G^3) = \mathrm{FP}(G^4).$$

Since G^4 is loop-less, connected, symmetric and negative, G^4 is isomorphic to G_n. Without loss of generality, assume that $G^4 = G_n$.

Let us prove that $G^3 \in \mathcal{G}_n$. It is easy to check that, for all vertex v, there exists $x, y \in \mathrm{FP}(G_n)$ such that $x(v) \neq y(v)$. We deduce that G^3 has no negative loops. Since G^3 is negative, it follows that G^3 is a sub-AND-NOT-net of G_n, which is connected since G^4 is. Let (u, v) be an arc that belong to at least one triangle of G_n. It is also easy to check that every sub-AND-NOT-net of G_n that does not contain (u, v) cannot have the same set of fixed points than G_n. Consequently, $G^3 \in \mathcal{G}_n$.

We now prove that $G = G^1 = G^2 = G^3$. Suppose, for a contradiction, that $G^2 \neq G^3$. Then G^2 has at least one positive arc. Since G^2 has no cycles with only positive arcs, G^2 has at least one positive arc (u, v) such that u has no positive predecessors. Then, it is clear that $P^-_{G^3}(u) = P^-_{G^2}(u)$. Since $G^3 \in \mathcal{G}_n$ and since each vertex of each AND-NOT-net in $\mathcal{G}_n$ has in-degree at

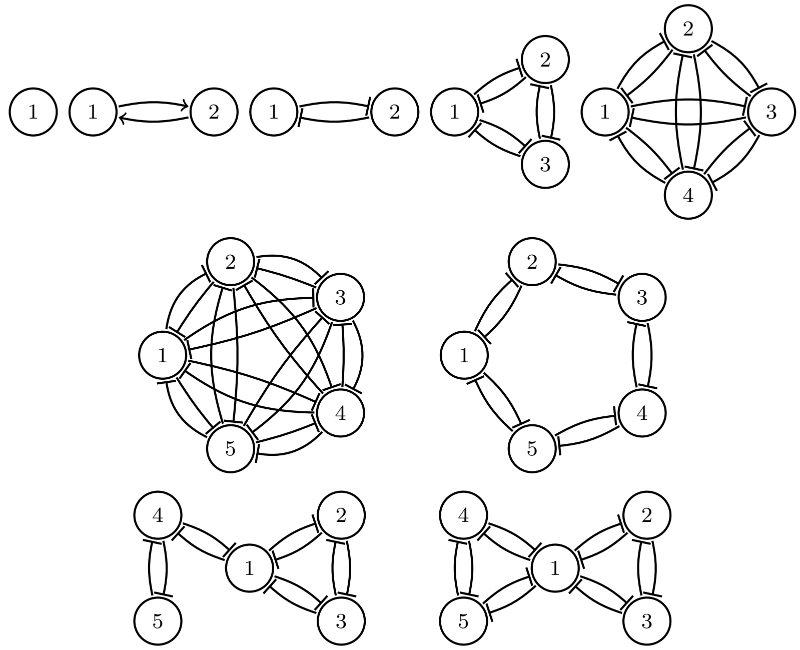

Fig. 7. Extremal loop-less connected AND-NOT-nets with at most five vertices.

least two, we deduce that $P^-_{G^2}(u)$ contains at least two vertices, say w_1 and w_2. Consequently, (w_1,u), (w_2,u), (w_1,v) and (w_2,v) are arcs of G^3, and since $G^3 \in \mathcal{G}_n$, it follows that $(v,u) \in E^-(G^3)$ (because squares in graphs of $\mathcal{G}_n$ contain diagonals). Since $P^-_{G^2}(u) = P^-_{G^3}(u)$, we have $(v,u) \in E^-(G^2)$, and since $(u,v) \in E^+(G^2)$, we deduce that G^3 has a loop on v, a contradiction. This proves that $G^2 = G^3 \in \mathcal{G}_n$. If $G^1 \neq G^2$, then $G^1 \setminus E^-(G^1)$ contains at least one non-trivial strongly connected component H. But, then $|V(H)|-1 > 0$ vertices of H has in-degree one in G^2, a contradiction with the fact that $G^2 = G^3 \in \mathcal{G}_n$. Thus $G^1 = G^2 \in \mathcal{G}_n$. Since $|V(G^1)| = |V(G^4)| = n$, if $G \neq G^1$ then G^1 contains a vertex of in-degree $n-1$, a contradiction with the fact that $G^1 \in \mathcal{G}_n$. Thus $G = G^1 \in \mathcal{G}_n$. $\square$

From this characterization and Lemma 7, we deduce that a loop-less connected AND-OR-NOT-net G with $n \geq 6$ vertices has $\mu(n)$ fixed points if and only if it has the following property: its underlying graph H is isomorphic to H_n; every arc of H that is not in a triangle is an arc of G; an arc of G is negative if and only if it connects vertices with the same type. We can also derived easily from Figure 7 and Lemma 7, the extremal loop-less connected AND-OR-NOT-nets with at most five vertices.

To characterize extremal loop-less AND-OR-NOT-nets, additional definitions are needed. Given two graphs H and H', let $H+H'$ denotes the disjoint union of H and H', and let nH denotes the disjointed union of n copies of H. As usual, K_n is the complete graph with n vertices. For $n \geq 2$, let $\mathcal{H}_n$ be the set of graphs defined as follows: if $n = 3s$ then $\mathcal{H}_n$ only contains $3K_n$; if $n = 3s+1$

then $\mathcal{H}_n$ contains $K_4+(s-1)K_3$ and $2K_2+(s-1)K_3$; and if $n=3s+2$ then $\mathcal{H}_n$ only contains $K_2+(s-1)K_3$. In [18] the following is proved: a graph H with n vertices has $\eta(n)$ maximal independent sets if and only if H is isomorphic to a graph in $\mathcal{H}_n$. Using this characterization and arguments similar to the ones used in the proof of Theorem 19, we can prove the following.

Theorem 20 *If G is a loop-less AND-NOT-net with $n \geq 2$ vertices, then G has $\eta(n)$ fixed points if and only if G has the following properties: (i) the underlying graph H of G is isomorphic to a graph in $\mathcal{H}_n$; (ii) G is symmetric (so H is actually the underlying digraph of G); (iii) copies of K_3 and K_4 have only negative arcs; and copies of K_2 have either two negative arcs or two positive arcs.*

From this characterization and Lemma 7, we deduce that a loop-less connected AND-OR-NOT-net G with $n \geq 2$ vertices has $\eta(n)$ fixed points if and only if it has the properties *(i)* and *(iii)* given above and the following property: an arc in a copy of K_3 and K_4 is negative if and only if it connects vertices with the same type.

6 Allowing loops

In this section, we establish the maximal number of fixed points in a connected AND-OR-NOT-nets when the presence of loops is allowed.

Lemma 21 *Let G be an AND-NOT-net. If H is an AND-NOT-net obtained from G by removing a positive loop, then* $\mathrm{FP}(H) \subseteq \mathrm{FP}(G)$*; and if H is an AND-NOT-net obtained from G by removing a negative loop, then* $\mathrm{FP}(G) \subseteq \mathrm{FP}(H)$*.*

PROOF. Suppose that G has a positive loop on v, and let H be the AND-NOT-net obtained from G by removing this loop. Let $x \in \mathrm{FP}(H)$. If $x(v)=0$ then $f_v^G(x)=0$ because of the presence of the positive loop, and it follows that $x \in \mathrm{FP}(G)$. If $x(v)=1$, then, in H, $x(u)=1$ for all positive predecessors of v, and $x(u)=0$ for all negative predecessors of v. Since this situation remains true in G, we have $f_v^G(x)=1$, so $x \in \mathrm{FP}(G)$.

Suppose that G has a negative loop on v, and let H be the AND-NOT-net obtained from G by removing this loop. Let $x \in \mathrm{FP}(G)$. Clearly, since G has a negative loop on v, we have $x(v)=0$. Thus, in G, v has a positive predecessor u with $x(u)=0$ or a negative predecessor u with $x(u)=1$. So $u \neq v$ and we deduce that $f_v^H(x)=0$. Thus $x \in \mathrm{FP}(H)$. $\square$

Remark 22 *These inclusions are sometimes strict. Indeed: the AND-NOT-net with one vertex and a positive loops has two fixed points; the AND-NOT-net with one vertex and a negative loop has no fixed point; and the AND-NOT-net with one vertex and no arc has a unique fixed point.*

Lemma 23 *Let G be an AND-NOT-net with a positive loop on each vertex, let v be a vertex of G, and let H be the AND-NOT-net obtained from G by making negative each arc starting from v. Then $|\mathrm{FP}(G)| \leq |\mathrm{FP}(H)|$.*

PROOF. Let U be the set of vertices $u \in V(G) \setminus v$ such that $(v,u) \in E^+(G)$. For each $x \in \mathrm{FP}(G)$, let $\tilde{x}$ be the configuration defined as follows: $\tilde{x}(u) = x(u)$ for all $u \neq v$, and

$$\tilde{x}(v) = \begin{cases} 0 & \text{if } \exists u \in U \text{ with } x(u) = 1 \\ x(v) & \text{ortherwise.} \end{cases}$$

Let us prove that the map $x \mapsto \tilde{x}$ is an injection. Let $x, y \in \mathrm{FP}(G)$ with $x \neq y$, and suppose, for a contradiction, that $\tilde{x} = \tilde{y}$. Then $x(u) = y(u)$ for all $u \neq v$, thus $x(v) \neq y(v)$. If $x(u) = y(u) = 0$ for all $u \in U$, then $\tilde{x}(v) = x(v) \neq y(v) = \tilde{y}(v)$, a contradiction. Thus $x(u) = y(u) = 1$ for some $u \in U$. Since G has a positive arc (v,u), we deduce that $x(v) = y(v) = 1$, a contradiction. This prove that $x \mapsto \tilde{x}$ is an injection.

Let $x \in \mathrm{FP}(G)$ and let us prove that $\tilde{x} \in \mathrm{FP}(H)$. Let $u \in U$. If $\tilde{x}(u) = 0$ then $f^H_u(\tilde{x}) = 0$ since, in H, u has a positive loop. If $\tilde{x}(u) = 1$ then $f^G_u(x) = x(u) = \tilde{x}(u) = 1$ and $\tilde{x}(v) = 0$. So, if, in H, u has a positive predecessor w with $\tilde{x}(w) = 0$ or a negative predecessor w with $\tilde{x}(w) = 1$, then $w \neq v$, thus $x(w) = \tilde{x}(w)$ and we deduce that $f^G_v(x) = 0$, a contradiction. Thus, in H, u has no positive predecessor w with $\tilde{x}(w) = 0$ and no negative predecessor w with $\tilde{x}(w) = 1$. Thus $f^H_u(\tilde{x}) = 1 = \tilde{x}(u)$. So for all $u \in U$, we have $f^H_u(\tilde{x}) = \tilde{x}(u)$. It remains to prove that $f^H_v(\tilde{x}) = \tilde{x}(v)$. If $\tilde{x}(v) = 0$ then $f^H_v(\tilde{x}) = 0$ since H has a positive loop on v. If $\tilde{x}(v) = 1$ then $\tilde{x} = x$ and since $f^H_v = f^G_v$ we have $f^H_v(\tilde{x}) = f^G_v(\tilde{x}) = \tilde{x}(v)$. Thus $f^H_v(\tilde{x}) = \tilde{x}(v)$ in all cases, and we deduce that $\tilde{x} \in \mathrm{FP}(H)$. $\square$

Remark 24 *The inequality is sometimes strict. See an example in Figure 8.*

The following observation is straightforward.

Lemma 25 *If G is a negative AND-NOT-net with a positive loop on each vertex, then $|\mathrm{FP}(G)| = |\mathrm{IS}(G)|$.*

Remark 26 *If G is a loop-less negative AND-NOT-net, then applying transformation T_4 consists in a symmetrization of G that may increase the number*

Fig. 8. Let G and H be two AND-NOT-nets, where G satisfies the conditions of the statement of Lemma 23, H is obtained from G by making negative each arc starting from vertex 1, and $|\mathrm{FP}(G)| = 4 < |\mathrm{FP}(H)| = 5$.

of fixed points (cf. Remark 18). However, if G is as in the previous lemma, a symmetrization does not change fixed points. To see this, let G be a negative AND-NOT-net with a positive loop on each vertex, and consider the symmetric version G^s of G, obtained from G by adding a negative arc (u, v) for each $(v, u) \in E^-(G)$ (such that $(u, v) \notin E^-(G)$). Clearly, the symmetrization has no influence on independent sets: $|\mathrm{IS}(G^s)| = |\mathrm{IS}(G)|$. Hence, according to Lemma 25, $|\mathrm{FP}(G^s)| = |\mathrm{FP}(G)|$.

A *star* is a graph G that contains a vertex v, called *center*, such there is an arc (u, w) if and only if $v = u \neq w$ or $u \neq w = v$. Thus, a star with $n + 1$ vertices is isomorphic to $K_{1,n}$ (the complete bipartite graph with a part of size 1 and a part of size n), and a star has a unique center when $n \neq 2$.

Clearly, if G is a connected graph and T a spanning tree of G, then $|\mathrm{IS}(T)| \geq |\mathrm{IS}(G)|$. Besides, in [19] it was proved that among all trees on n vertices, $K_{1,n-1}$ is the one that maximizes the number of independent sets. As a consequence, we have the following property:

Lemma 27 *If G is a connected graph with n vertices, then $|\mathrm{IS}(G)| \leq 2^{n-1}+1$, and the bound is reached if and only if G is a star.*

For each $n > 1$, let S_n be the AND-NOT-net defined as follows: there are n vertices, denoted from 1 to n, a positive loop on each vertex, and a positive arc $(1, k)$ for each $1 < k \leq n$. Note that the underlying graph of S_n is a star. We are now in position to state the main result of this section.

Theorem 28 *If G is a connected AND-NOT-net with n vertices, then*

$$|\mathrm{FP}(G)| \leq 2^{n-1} + 1.$$

The bound is reached if and only if G is isomorphic to S_n or has the following properties: (i) a star as underlying graph; (ii) a positive loop on each vertex; (iii) no negative cycles; (iv) no positive arc leaving the center of this star.

PROOF. Let G be a connected AND-NOT-net with n vertices. Let G' be the AND-NOT-net obtained from G by: (1) removing all the negative loops; (2) adding a positive loop on each vertex (if it does not already exist); and (3) making negative each arc that is not a loop. Clearly, G and G' have the same underlying graph; we denote it by H. According to Lemmas 21, 23, 25 and 27,

$$|\mathrm{FP}(G)| \leq |\mathrm{FP}(G')| = |\mathrm{IS}(G')| = |\mathrm{IS}(H)| \leq 2^{n-1}+1.$$

This prove the first assertion. Let us prove the second. Suppose that H is a star, and let v be the center of H. If G is isomorphic to S_n or has the properties *(i)*–*(iv)*, then it is easy to check that G has 2^{n-1} fixed points x such that $x(v)=0$, and a (unique) fixed point x such that $x(v)=1$. So finally, suppose that $|\mathrm{FP}(G)| = 2^{n-1}+1$, suppose that G is not isomorphic to S_n, and let us prove that this implies that G has the properties *(i)*–*(iv)*.

(i) We have already seen that $|\mathrm{FP}(G)| \leq |\mathrm{IS}(H)|$, and it follows from Lemma 27 that H is a star (thus G satisfies *(i)*). Let v be the center of this star.

(ii) Let v be a vertex of G, and for every $x \in \mathrm{FP}(G)$, let $\tilde{x}$ be the configuration such that for all $u \neq v$, $\tilde{x}(v) = 1 - x(v)$ and $\tilde{x}(u) = x(u)$. If G has no positive loop on v, then $x \in \mathrm{FP}(G) \Rightarrow \tilde{x} \notin \mathrm{FP}(G)$, and we deduce that $|\mathrm{FP}(G)| \leq 2^{n-1}$, a contradiction. Thus G has the property *(ii)*.

(iii) Suppose that G has a negative cycle. It follows from *(i)* and *(iii)* that there exists $u \neq v$ such that (u,v) and (v,u) are arcs of G with opposite signs. Let $x \in \mathrm{FP}(G)$. Suppose that (v,u) is positive (and (u,v) negative); the other case being similar. Then, $x(v)=0$ implies $x(u)=0$, and $x(v)=1$ implies $x(u)=0$. Thus $x(u)=0$ for all $x \in \mathrm{FP}(G)$, so $|\mathrm{FP}(G)| \leq 2^{n-1}$, a contradiction. Thus G has the property *(iii)*.

(iv) We first need to prove the following property: for all $u \neq v$,

$$(v,u) \in E^+(G) \;\Rightarrow\; (u,v) \notin E(G). \qquad (*)$$

Suppose that $(v,u) \in E^+(G)$ with $u \neq v$. By *(ii)* $(u,v) \notin E^-(G)$, and if $(u,v) \in E^+(G)$, then it is easy to see that $x(v) = x(u)$ for all $x \in \mathrm{FP}(G)$, so that $|\mathrm{FP}(G)| \leq 2^{n-1}$, a contradiction. This prove $(*)$. Now suppose, for a contradiction, that G has a positive arc (v,u) leaving the center $(u \neq v)$. Suppose first that there exists $w \neq u,v$ such that $(v,w) \in E^-(G)$. Let $x \in \mathrm{FP}(G)$. Then $x(v)=0$ implies $x(u)=0$ and $x(v)=1$ implies $x(w)=0$. Thus there are at most 2^{n-2} fixed points x such that $x(v)=0$ and at most 2^{n-2} fixed points x such that $x(v)=1$. Thus $|\mathrm{FP}(G)| \leq 2^{n-1}$, a contradiction. We deduce that all the arcs leaving v are positive. From $(*)$ and the fact that G is not isomorphic to S_n, we deduce that there exists $w \neq u,v$ such that $(w,v) \in E(G)$. Let $c=1$ if this arc is positive, and $c=0$ otherwise. Then, $x(v)=0$ implies $x(u)=0$, and $x(v)=1$ implies $x(w)=c$. Hence, as above, we deduce that $|\mathrm{FP}(G)| \leq 2^{n-2}+2^{n-2} = 2^{n-1}$, a contradiction. Thus there is no positive arc leaving the center v. □

7 Conclusion and perspectives

We have proved that the number of fixed points in a connected AND-OR-NOT-net G with n vertices is bounded above by the maximal number of maximal independent sets in a connected graph with n vertices if G has no loops, and by the maximal number of independent sets in a connected graph with n vertices otherwise. In this way, using results on independent sets, we obtain tight upper-bounds on the number of fixed points in AND-OR-NOT-nets, and we characterize AND-OR-NOT-nets reaching these bounds.

Considering AND-OR-NOT-nets reaching the bounds is interesting. For example, in the loop-less case, AND-NOT-nets reaching the bounds are symmetric, contains only negative arcs, and a lot of "triangles" that is cycles of length 3. Thus, in the loop-less case, to reach the bound, a lot of negative cycles are necessary, and this is not very intuitive since negative cycles are mostly known to be unfavorable to fixed points. Now, when loops are allowed, AND-NOT-nets reaching the bound have no negative cycles. This shows that the influence of negative cycles on the number of fixed points is subtile, not yet well understood while the influence of positive cycle is rather well understand: the number of fixed points is at most 2^{τ_p} , where τ_p is the number of elements of the smallest positive feedback vertex set [1]. Thus, to have many fixed points, a lot of "rather disjoint" positive cycles are necessary.

A More on transformation T_1

Proposition 2 *For every AND-NOT-net G we have* $|\mathrm{FP}(G)| = |\mathrm{FP}(G^1)|$.

PROOF. By Lemma 10 we have $|\mathrm{FP}(G)| \leq |\mathrm{FP}(G^1)|$. Suppose that $G \neq G^1$, and let us prove that $|\mathrm{FP}(G^1)| \leq |\mathrm{FP}(G)|$. Consider the function that maps every configuration $x \in \mathrm{FP}(G^1)$ to the configuration $\tilde{x}$ of G defined by:

$$\tilde{x}(v) = \begin{cases} x(v) & \text{if } v \in V(G) \setminus V_{\mathrm{cst}}(G) \\ 1 & \text{if } v \in V_{\mathrm{cst1}}(G) \\ 0 & \text{if } v \in V_{\mathrm{cst0}}(G) \end{cases}$$

Since $x \mapsto \tilde{x}$ is clearly an injection, it is sufficient to show that $\tilde{x} \in \mathrm{FP}(G)$ for all $x \in \mathrm{FP}(G^1)$. Let $x \in \mathrm{FP}(G^1)$, let $v \in V(G) \setminus V_{\mathrm{cst}}(G)$, and let us prove that $f_v^G(\tilde{x}) = \tilde{x}(v)$.

(1) Suppose that $x(v) = 1 = \tilde{x}(v)$ and, for a contradiction, that $f_v^G(\tilde{x}) = 0$.

Then there exists $u \in P^+_G(v)$ with $\tilde{x}(u) = 0$ or there exists $u \in P^-_G(v)$ with $\tilde{x}(u) = 1$. Suppose first that there exists $u \in P^+_G(v)$ with $\tilde{x}(u) = 0$. Then $u \notin V_{\text{cst0}}(G)$ since otherwise $v \in V_{\text{cst0}}(G)$. Thus $u \notin V_{\text{cst}}(G)$, and we deduce that $u \in P^+_{G^1}(v)$. Since $x(u) = \tilde{x}(u) = 0$ we have $f^{G^1}_v(x) = 0$, a contradiction. If there exists $u \in P^-_G(v)$ with $\tilde{x}(u) = 1$, then we obtain a contradiction in a similar way. Thus $f^G_v(\tilde{x}) = 1 = \tilde{x}(v)$.

(2) Suppose that $x(v) = 0 = \tilde{x}(v)$. If there exists $u \in P^+_{G^1}(v)$ with $x(u) = 0$, then $u \neq v^*$, thus $u \in P^+_G(v)$. Since $\tilde{x}(u) = x(u) = 0$, we deduce that $f^G_v(\tilde{x}) = 0$. Similarly, if there exists $u \in P^-_{G^1}(v)$ with $x(u) = 1$, then $u \neq v^*$, thus $u \in P^-_G(v)$. Since $\tilde{x}(u) = x(u) = 1$, we deduce that $f^G_v(\tilde{x}) = 0$. Thus $f^G(\tilde{x}) = 0$ in all cases.

So we have proved that $f^G_v(\tilde{x}) = \tilde{x}(v)$ for all $v \in V(G) \setminus V_{\text{cst}}(G)$. We now prove, by induction (following the inductive definition of $V_{\text{cst}}(G)$), that the same equality holds for $v \in V_{\text{cst}}(G)$. Actually, we only prove the base case, since the induction step is straightforward. Let $x \in \text{FP}(G^1)$.

(1) Suppose that there exists strong components H and H' in $G \setminus E^-(G)$ such that G has both a positive and a negative arc from $V(H')$ to $V(H)$ (so $V(H) \subseteq V_{\text{cst0}}(G)$). Let $v \in V(H)$. If H is not trivial, v has a positive predecessor $u \in V(H)$. Thus $\tilde{x}(u) = 0$ and we deduce that $f^G_v(\tilde{x}) = 0 = \tilde{x}(v)$. So suppose that H is trivial. If $V(H') \cap V_{\text{cst0}}(G) \neq \emptyset$ then $V(H') \subseteq V_{\text{cst0}}(G)$ thus $\tilde{x}(u) = 0$ for all $u \in V(H')$, and since v has a positive predecessor in $V(H')$ we have $f^G_v(\tilde{x}) = 0 = \tilde{x}(v)$. Similarly, if $V(H') \cap V_{\text{cst1}}(G) \neq \emptyset$ then $V(H') \subseteq V_{\text{cst1}}(G)$ thus $\tilde{x}(u) = 1$ for all $u \in V(H')$, and since v has a negative predecessor in $V(H')$ we have $f^G_v(\tilde{x}) = 0 = \tilde{x}(v)$. So finally, suppose that $V(H') \cap V_{\text{cst}}(G) = \emptyset$. Then $f^G_u(\tilde{x}) = \tilde{x}(u)$ for all $u \in V(H')$, and we deduce, as in Lemma 6, that there exists $c \in \{0, 1\}$ such that $\tilde{x}(u) = c$ for all $u \in V(H')$. If $c = 0$ then $f^G_v(\tilde{x}) = 0 = \tilde{x}(v)$, because v has a positive predecessor in $V(H')$, and if $c = 1$ then $f^G_v(\tilde{x}) = 0 = \tilde{x}(v)$, because v has a negative predecessor in $V(H')$. Thus $f^G_v(\tilde{x}) = \tilde{x}(v)$ in all cases.

(2) If v is a source, then $v \in V_{\text{cst1}}(G)$ thus $f^G_v = \text{cst} = 1 = \tilde{x}(v)$. □

B More on transformation T_2

Proposition 3 *If G is an AND-NOT-net satisfying the property P_1, then* $|\text{FP}(G)| = |\text{FP}(G^2)|$.

PROOF. By Lemma 13 we have $|\text{FP}(G)| \leq |\text{FP}(G^2)|$. Let us prove that $|\text{FP}(G^2)| \leq |\text{FP}(G)|$. Let $H_1, \dots, H_r$ be the strongly connected components of

$G \setminus E^-(G)$. For every $1 \leq k \leq r$, let v_k be the smallest vertices in $V(H_k)$. Let $U = V(G) \setminus \{v_1, v_2, \dots v_r\}$. Consider the permutation mapping each configuration x of G to the configuration $\tilde{x}$ of G defined by:

$$\tilde{x}(u) = \begin{cases} 1 - x(u) & \text{if } u \in U \\ x(u) & \text{ortherwise.} \end{cases}$$

We prove that $\tilde{x} \in \mathrm{FP}(G)$ for each $x \in \mathrm{FP}(G^2)$. Let $x \in \mathrm{FP}(G^2)$ and $1 \leq k \leq r$. If $u \in V(H_k) \setminus v_k$, then $f_u^{G^2}(x) = 1 - x(v_k)$ and thus $x(u) \neq x(v_k)$. Consequently

$$\forall u \in V(H_k), \qquad \tilde{x}(u) = x(v_k).$$

We first prove that $f_u^G(\tilde{x}) = \tilde{x}(u)$ given any $u \in V(H_k) \setminus v_k$. Suppose that $\tilde{x}(u) = 0$. Since there exists $w \in P_G^+(u) \cap V(H_k)$, and since $\tilde{x}(w) = \tilde{x}(u) = 0$, we have $f_u^G(\tilde{x}) = 0 = \tilde{x}(u)$. Suppose now that $\tilde{x}(u) = 1$, and suppose, for a contradiction, that $f_u^G(\tilde{x}) = 0$. Then one of the following two conditions holds:

(1) There exists $w \in P_G^+(u)$ with $\tilde{x}(w) = 0$. Then $\tilde{x}(w) \neq \tilde{x}(u)$ so $w \notin V(H_k)$ thus $w \in V(H_l)$ for some $l \neq k$. Then, $v_l \in P_{G^2}^+(v_k)$ and $x(v_l) = \tilde{x}(w) = 0$ thus $f_{v_k}^{G^2}(x) = 0 = x(v_k) \neq \tilde{x}(u)$, a contradiction.

(2) There exists $w \in P_G^-(u)$ with $x(w) = 1$. Since G satisfies the condition Q_1, $w \notin V(H_k)$ thus $w \in V(H_l)$ for some $l \neq k$. Then, $v_l \in P_{G^2}^-(v_k)$ and $x(v_l) = \tilde{x}(w) = 1$ thus $f_{v_k}^{G^2}(x) = 0 = \tilde{x}(u)$, a contradiction.

Since there is a contradiction in both cases, $f_u^G(\tilde{x}) = 1 = \tilde{x}(u)$.

We now prove that $f_{v_k}^G(\tilde{x}) = \tilde{x}(v_k)$. Suppose first that $\tilde{x}(v_k) = 0$. If H_k is not trivial then there exists $u \in P_G^+(v_k) \cap V(H_k)$ and since $\tilde{x}(u) = x(v_k) = 0$ we deduce that $f_{v_k}^G(\tilde{x}) = 0$. Suppose that H_k is trivial. Since $x(v_k) = \tilde{x}(v_k) = 0$, one of the following two condition holds:

(1) There exists $u \in P_{G^2}^+(v_k)$ with $x(u) = 0$. Then $u = v_l$ for some $l \neq k$, so there exists $w \in P_G^+(v_k) \cap V(H_l)$. Since $\tilde{x}(w) = x(v_l) = 0$ we deduce that $f_{v_k}^G(\tilde{x}) = 0$.

(2) There exists $u \in P_{G^2}^-(v_k)$ with $x(u) = 1$. Then $u = v_l$ for some $l \neq k$, so there exists $w \in P_G^-(v_k) \cap V(H_l)$. Since $\tilde{x}(w) = x(v_l) = 1$ we deduce that $f_{v_k}^G(\tilde{x}) = 0$.

So in all cases, $f_{v_k}^G(\tilde{x}) = 0 = \tilde{x}(v_k)$. Suppose now that $\tilde{x}(v_k) = 1$, and suppose, for a contradiction, that $f_{v_k}^G(\tilde{x}) = 0$. Then one of the following two conditions holds:

(1) There exists $u \in P_G^+(v_k)$ with $\tilde{x}(u) = 0$. Then $\tilde{x}(u) \neq \tilde{x}(v_k) = x(v_k)$

so $u \notin V(H_k)$ thus $u \in V(H_l)$ for some $l \neq k$. Then, $v_l \in P^+_{G^2}(v_k)$ and $x(v_l) = \tilde{x}(u) = 0$ thus $f^{G^2}_{v_k}(x) = 0 \neq \tilde{x}(v_k) = x(v_k)$, a contradiction.

(2) There exists $u \in P^-_G(v_k)$ with $\tilde{x}(u) = 1$. Since G satisfies the condition Q_1, $u \notin V(H_k)$ thus $u \in V(H_l)$ for some $l \neq k$. Then, $v_l \in P^-_{G^2}(v_k)$ and $x(v_l) = \tilde{x}(u) = 1$ thus $f^{G^2}_{v_k}(x) = 0 \neq \tilde{x}(v_k) = x(v_k)$, a contradiction.

Since there is a contradiction in both cases, $f^G_{v_k}(\tilde{x}) = 1 = \tilde{x}(v_k)$. □

C More on transformation T_3

Proposition 4 *If G is an AND-NOT-net with the property P_2 then* $\mathrm{FP}(G) = \mathrm{FP}(G^3)$.

PROOF. By Lemma 15 we have $\mathrm{FP}(G) \subseteq \mathrm{FP}(G^3)$, so we just prove that $\mathrm{FP}(G^3) \subseteq \mathrm{FP}(G)$. Let $x \in \mathrm{FP}(G^3)$ and $v \in V(G)$.

Suppose that $x(v) = 0$. Then there exists $u \in P^-_{G^3}(v)$ with $x(u) = 1$. If $u \in P^-_G(v)$ then $f^G_v(x) = 0 = x(v)$. Otherwise, there exists w such that $(u, w) \in E^-(G)$ and such that G has a path P from w to v with only positive arc. Then, $(u, t) \in E^-(G^3)$ for all $t \in V(P)$, and thus $f^{G_3}_t(x) = 0 = x(t)$ for all $t \in V(P)$. Thus, there exists $t \in P^+_G(v) \cap V(P)$ with $x(t) = 0$ and we deduce that $f^G_v(x) = 0 = x(v)$.

Suppose now that $x(v) = 1$, and suppose, for a contradiction, that $f^G_v(x) = 0$. Then one of the following two conditions hold:

(1) There exists $u \in P^+_G(v)$ with $x(u) = 0$. Let P be the longest path of G with only positive arcs, with u as final vertex, and such that $x(w) = 0$ for all $w \in V(P)$. Let w be the initial vertex of P. If there exists $t \in P^+_G(w)$ then $t \notin V(P)$ (since G has the property P_2) and $x(t) = 1$ (since otherwise P is not of maximal length). Thus $x(t) = 1$ for all $t \in P^+_G(w)$. Since $x(w) = 0$ we have, according to the arguments above, $f^G_w(x) = 0$. Thus, there exists $t \in P^-_G(w)$ with $x(t) = 1$. But then $(t, v) \in E^-(G^3)$ so $f^{G^3}_v(x) = 0 \neq x(v)$, a contradiction.

(2) There exists $u \in P^-_G(v)$ with $x(u) = 1$. Then $u \in P^-_{G^3}(v)$ thus $f^{G^3}_v(x) = 0 \neq x(v)$, a contradiction.

We deduce that $f^G_v(x) = 1 = x(v)$. □

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