Question order experimental constraints on quantum-like models of judgement

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Abstract

In this paper, we extend the critical approach of Boyer-Kassem et al. (Boyer-Kassem, Thomas, Duchêne, Sbastien, Guerci Eric (2016), "Testing quantum-like models of judgment for question order effect", Mathematical Social Sciences 80: 33-46.) to degenerate quantum models.

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1 Introduction

Question order experiments (as reported in [1–3] for example) have called for non-classical probabilistic interpretative models. Many quantum-like models have appeared to account for such experiments and other paradoxical cognitive behavior such as conjunction fallacies, violation of the sure-thing principle, asymmetries in similarity, (see [4,5] for two recent reviews). However, in a recent convincing study, Boyer-Kassem et al. [6] have shown how question order experiments provide constraints that rule out all non-degenerate quantum-like models of judgement. In this paper, we extend this critical approach to degenerate quantum models.

2 Quantum formalism for cognitive studies on human judgement

The mathematical formalism of traditional quantum mechanics has been advocated as an effective phenomenological model of human judgement. In this context, the Hilbert space of quantum mechanics represents the set of an individual's states of belief. The algebra of observables is the set of questions that can be asked, and the outcome of a measurement is the answer given to such a question.

Let us consider a finite Hilbert space, $\mathcal{H}$, of dimension N, as large as necessary, and call it the "belief space". Let A and B be two "yes and no" questions, that is to say, they are Hermitian operators acting on $\mathcal{H}$, having at most two eigenspaces corresponding to distinct eigenvalues. Let us denote by E_A, (respectively E_B), the eigenspace of A, (respectively B), corresponding to the answer "yes" for example, and n_A, (respectively n_B), its dimension, (both dimensions are supposed to be non zero: $n_A, n_B > 0$). We have $\mathcal{H} = E_A \oplus E_A^{\perp} = E_B \oplus E_B^{\perp}$, where $E_I^{\perp}$ denotes the orthogonal complement of E_I, of dimension $(N - n_I)$. Let us assume without loss of generality that $n_A \geq n_B$. It is well-known that one can find by bi-orthogonalization, an orthonormal basis set of E_A,

$(a_i)_{i\in\{1,\cdots,n_A\}}$ and an orthonormal basis set of E_B, $(b_i)_{i\in\{1,\cdots,n_B\}}$ such that,

$$\forall i \in \{1,\cdots,n_A\},\ \forall j \in \{1,\cdots,n_B\}\ \langle a_i|b_j\rangle = \delta_{i,j} cos\theta_j \tag{1}$$

where $\delta_{i,j}$ is the Krönecker symbol, and the θ_j's are the Araki's angles [7,8]. These two basis sets can be completed by basis sets of their ortho-complements denoted by $(a_i)_{i\in\{n_A+1,\cdots,N\}}$ and $(b_i)_{i\in\{n_B+1,\cdots,N\}}$ to build two basis sets of the whole belief space.

3 Reciprocity constraints

Let $|\psi\rangle \in \mathcal{H}$ be a state of belief. It can be expressed in two different manners according to the chosen basis set:

$$|\psi\rangle = \sum_{i\in\{1,\cdots,N\}} \alpha_i|a_i\rangle = \sum_{i\in\{1,\cdots,N\}} \beta_i|b_i\rangle \tag{2}$$

the coefficients α_i's and β_i's being complex numbers. We will assume that the belief state is normalized,

$$\langle\psi|\psi\rangle = \sum_{i\in\{1,\cdots,N\}} |\alpha_i|^2 = \sum_{i\in\{1,\cdots,N\}} |\beta_i|^2 = 1. \tag{3}$$

According to the rules of traditional quantum theory and the measurement postulate, the probability to obtain the answer "yes" to question A will be, in self-explanatory notation,

$$p(A_y) = \sum_{i\in\{1,\cdots,n_A\}} \langle\psi|a_i\rangle\langle a_i|\psi\rangle = \sum_{i\in\{1,\cdots,n_A\}} |\alpha_i|^2, \tag{4}$$

the belief state being projected onto

$$|\psi_{A_y}\rangle = \frac{1}{\sqrt{\sum_{k\in\{1,\cdots,n_A\}} |\alpha_k|^2}} \sum_{i\in\{1,\cdots,n_A\}} \alpha_i|a_i\rangle, \tag{5}$$

(where the prefactor in front of the sum is just a normalization factor), and the probability to obtain "no" to A is,

$$p(A_n) = \sum_{i\in\{n_A+1,\cdots,N\}} \langle\psi|a_i\rangle\langle a_i|\psi\rangle = \sum_{i\in\{n_A+1,\cdots,N\}} |\alpha_i|^2, \tag{6}$$

the state of belief becoming

$$|\psi_{A_n}\rangle = \frac{1}{\sqrt{\sum\limits_{k\in\{n_A+1,\cdots,N\}} |\alpha_k|^2}} \sum_{i\in\{n_A+1,\cdots,N\}} \alpha_i|a_i\rangle, \tag{7}$$

after the answer is given.

Similarly for question B we have,

$$p(B_y) = \sum_{i\in\{1,\cdots,n_B\}} \langle\psi|b_i\rangle\langle b_i|\psi\rangle = \sum_{i\in\{1,\cdots,n_B\}} |\beta_i|^2, \tag{8}$$

$$|\psi_{B_y}\rangle = \frac{1}{\sqrt{\sum\limits_{k\in\{1,\cdots,n_B\}} |\beta_k|^2}} \sum_{i\in\{1,\cdots,n_B\}} \beta_i|b_i\rangle, \tag{9}$$

$$p(B_n) = \sum_{i\in\{n_B+1,\cdots,N\}} \langle\psi|b_i\rangle\langle b_i|\psi\rangle = \sum_{i\in\{n_B+1,\cdots,N\}} |\beta_i|^2, \tag{10}$$

$$|\psi_{B_n}\rangle = \frac{1}{\sqrt{\sum\limits_{k\in\{n_B+1,\cdots,N\}} |\beta_k|^2}} \sum_{i\in\{n_B+1,\cdots,N\}} \beta_i|b_i\rangle. \tag{11}$$

Now, the probability of obtaining the answer "yes" to B knowing that the answer to A was "yes" can be easily derived by substituting $|\psi_{A_y}\rangle$ to $|\psi\rangle$ into Eq.(8):

$$\begin{aligned} p(B_y|A_y) &= \frac{1}{\sum\limits_{k\in\{1,\cdots,n_A\}} |\alpha_k|^2} \sum_{j\in\{1,\cdots,n_B\}} \sum_{i\in\{1,\cdots,n_A\}} |\alpha_i|^2\langle a_i|b_j\rangle\langle b_j|a_i\rangle \\ &= \frac{1}{\sum\limits_{k\in\{1,\cdots,n_A\}} |\alpha_k|^2} \sum_{j\in\{1,\cdots,n_B\}} \sum_{i\in\{1,\cdots,n_A\}} |\alpha_i|^2\delta_{i,j}cos^2\theta_j \\ &= \frac{1}{\sum\limits_{k\in\{1,\cdots,n_A\}} |\alpha_k|^2} \sum_{j\in\{1,\cdots,n_B\}} |\alpha_j|^2cos^2\theta_j. \end{aligned} \tag{12}$$

Similarly, the probability of obtaining the answer "yes" to A knowing that the answer to B was "yes" is

$$p(B_y|A_y) = \frac{1}{\sum\limits_{k\in\{1,\cdots,n_B\}} |\beta_k|^2} \sum_{j\in\{1,\cdots,n_B\}} |\beta_j|^2 cos^2\theta_j. \tag{13}$$

When $n_A = n_B = 1$ as in the non degenrate case investigated in [6], the latter two expressions reduce to $p(B_y|A_y) = \frac{1}{|\alpha_1|^2}|\alpha_1|^2 cos^2\theta_1 = cos^2\theta_1$ and $p(A_y|B_y) = \frac{1}{|\beta_1|^2}|\beta_1|^2 cos^2\theta_1 = cos^2\theta_1$, so we retrieve the reciprocity relation,

$$p(A_y|B_y) = p(B_y|A_y), \tag{14}$$

the three other reciprocity relations of [6], could be obtained in a analogous fashion by permuting the parts played by A and B on the one hand, and "yes" and "no" on the other hand, provided that $N - n_A = N - n_B = 1$. Note, however, that, Eq.(14) does not depend upon N to hold.

To extend the reciprocity relations to the degenerate case, there is no loss of generality in limiting the study to Eqs.(12) and (13), because here A and B, and "yes" and "no" are just pairs of interchangeable abstract symbols. Unfortunately, it is clear that as soon as $n_A > 1$ with $n_B = 1$, or as soon as $n_B > 1$, the reciprocity relation (14) does not have to be satisfied. Except for very particular cases, it is easy to find a belief state, $|\psi\rangle$, such that relation (14) is not verified. In the case $n_A > 1, n_B = 1$ and $\theta_1 \neq 0$, one can just take $\alpha_2 \neq 0$, and when $n_B > 1$ it suffices to choose two questions A and B such that all the Araki's angles except θ_1 are equal to $\frac{\pi}{2}$ to be in the same situation as in the previous case. However, there is a particular case of degenerate model where the reciprocity relation (14) is true. This is when all Araki's angles are zero, $\forall i,\ \theta_i = 0$, which means that the belief subspaces E_A and E_B are orthogonal. In such a case, answering "yes" to A implies answering "no" to B, so $p(B_y|A_y) = p(B_y|B_n) = 0$, and symmetrically $p(A_y|B_y) = p(A_y|A_n) = 0$. This particular case is not relevant to the experiments reported in [1–3] and re-analysed in [6].

4 More constraints

Other constraints can easily rule out all degenerate and non-degenerate quantum models alike. It is arguably not worth performing a real experiment to convince oneself that asking again question A or question B to a reasonable human being immediately after both questions have been asked in whatever order A then B or B than A, will produce the answer already given the first time (unless question A or B is something like: tell me randomly either "yes" or "no"?). So one expects for example that $p(B_y|(A_y|B_y)) = p(B_y|(B_y|A_y)) = 1$, or equivalently, $p(B_n|(A_y|B_y)) = p(B_n|(B_y|A_y)) = 1$. However, some of these equalities will be wrong in general in the quantum formalism we have introduced. We will have $p(B_y|(B_y|A_y)) = 1$ but not $p(B_y|(A_y|B_y)) = 1$ if A and B are non-commuting operators.

For a quantum-like formalism to make sense, this sort of chain of questions has to be avoided. In the similar way as only a Jordan subalgebra of the algebra of observables is relevant in quantum physics, only question products made of all-distinct factors should be considered as relevant in the modelling of human judgement, i.e. the "algebra of questions" should be restricted to a set of "words" (in the mathematical sense) that can be written with letters occuring at most once in their expression. The product law should be modified to cancel any word with repeated letters i.e. it should associate the "null question" to it.

The most challenging task in our opinion, is not to choose between degenerate or non-degenerate eigenspaces for question operators, but to define a meaningful structure for a subset of the "algebra of questions", which should be compatible with logical operations in some broaden sense. "Broaden" because for example the logical "and" cannot be related to successive question-answer events, since it is symmetrical in its arguments, whereas A then B and B then A are not, as found in question order experiments. What meaning could be granted to the product and the sum of two non-commuting question operators or to the multiplication of one such operator by a scalar, are still *a priori* open problems.

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