Transient cluster formation for generalized Hegselmann-Krause opinion dynamics

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Abstract— **We analyse the generalized Hegselmann-Krause model of opinion dynamics. The asymptotic state of such a system has been well studied in the literature, however the transient state is still poorly understood. Predicting which groups of agents will form clusters remains to be studied. We present sufficient conditions to detect cluster formation in the transient phase of the multi-agent system. We also give a procedure to know how much time a cluster stays consistent, *i.e.,* before it merges with other agents in the system. Our criterion can be computed locally using variables obtained from the initial conditions. Finally our results are illustrated by a numerical example.**

I. Introduction

The analysis and design of cooperative behaviours in networked dynamic systems has recently received a lot of attention. It finds application in technical fields such as cooperative robotics [1], mobile sensor networks [2] or distributed algorithms [3] and takes inspiration from collective motion in nature [4]–[6].

A more recent direction of research is to use multi-agent systems as a tool to model social networks. The consensus system has been widely applied to model opinion dynamics. Regarding the linear consensus system, several conditions for convergence towards global agreement have been developed [7]–[15].

By contrast, when the communication strength between agents depends on their current opinions, as in the Hegselmann-Krause bounded confidence model [16], [17], it has been shown that clusters may asymptotically form. Groups of agents reach local consensuses but disagree with the rest of the agents [18]–[22]. The study of cluster formation is of particular relevance in social systems since it is linked to the formation and preservation of local norms and cultures (see [23] and references therein). Cluster patterns may also lead to conflict between distant clusters [24]. From a technical point of view, cluster formation in multi-agent systems allows to apply model reduction tools and simplify the analysis of the system [25].

In static networks of interactions, the detection of clusters or community patterns have been largely explored [26], [27]. However, interactions and opinions evolve in time. Methods to predict the birth of opinion clusters remains to be developed. Moreover, results on the convergence to clusters of opinions do not provide a way to predict a priori which groups of agents will converge towards a cluster. Finally, previous analyses of opinion dynamics always focus on the asymptotic behaviour of the systems, be it a global consensus or local clusters. However, simulations have shown that despite global consensus may occur asymptotically, meta-stable clusters may form in temporarily the transient phase [16]. This is in particular the case for the generalized Hegselmann-Krause model where the communication range is not a bounded set [19] (see Section II-B for an illustration of this fact). It is of interest to focus on the transient state rather than the asymptotic state because in many real world applications, the time frame of interest (political election, war, etc.) is finite. Finite time analysis has analogously been reported in epidemiology in [28], where extinction is the stable asymptotic state, but the study rather focuses on quasi-equilibrium occurring between the emergence of a disease and its extinction.

In this paper, we provide a new criterion to detect the formation of opinion clusters in systems of opinion dynamics. The criterion depends only on the initial set of opinions and can be computed locally. The results apply to various forms of opinion-dependent communication strengths, possibly with unbounded support such as the generalized Hegselmann-Krause model. The criterion we offer allows to detect cluster formation in the transient dynamics. It also gives a duration on how much time we can differentiate a local cluster from the rest of the agents before global consensus occurs.

This paper is organized as follows. In Section II we formalise the multi-agent system we use in this paper by introducing notation and definitions. We also describe experimental observations that display transient cluster formation. In Section III, several assumptions are stated in order to obtain simple bounds on the dynamics of the system. These bounds are used in Section IV to give conditions to detect transient cluster formation. We then give a numerical illustration in Section V to show how cluster formation is detected using our main result. Finally, we conclude in Section VI.

Throughout the paper, the notation $\mathbb{R}$ will denote the set of real numbers, $\mathbb{R}^+$ the set of the positive real (zero included) and $\mathbb{N}$ the set of integers. The notation $|.|$ will denote the cardinality of a set of integers, and the absolute value for a real number.

II. Problem formulation

Let us consider a system of n agents, numbered from 1 to n and forming a discrete-time multi-agent system. The opinion or state of agent i at time $t \in \mathbb{N}$ is a scalar denoted

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$x_i(t) \in \mathbb{R}$. When there is no ambiguity, we drop the time notation and write $x_i = x_i(t)$ and $x_i^+ = x_i(t+1)$, the opinion at time $t+1$, which is obtained according to the following consensus update :

$$x_i^+ = x_i + \sum_{j=1}^{n} \frac{\alpha_{ij}}{\sum_{k=1}^{n} \alpha_{ik}} (x_j - x_i), \quad (1)$$

where α_{ij} is the weight associated with the influence of agent j over i. In this paper we will use state dependent weights of the form

$$\alpha_{ij} = f(|x_i - x_j|), \quad (2)$$

where the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is called the influence function. Examples of suitable influence functions are provided in Section III-B.

A. Notation and Definitions

This subsection gives two definitions used to describe the multi-agent system. Namely, two key distances are defined for a given group of agents : the diameter of the group, and the distance separating the group from the other agents of the system. For the rest of the paper, let $A \subseteq \{1, \dots, n\}$ be a group of agents. The group A will be the cluster candidate.

Definition 1 (Group diameter δ): The diameter of the group A is the maximum distance between two agents in A

$$\delta = \max_{(i,j)\in A^2} |x_i - x_j|. \quad (3)$$

Definition 2: The distance Δ is defined as the minimum distance between all agents in group A and all the other agents in the system, which can be formally written as

$$\Delta = \min_{i\in A,\ j\notin A} |x_i - x_j|. \quad (4)$$

We just defined the main parameters of a group of agents, and we will then look at how these parameters evolve.

B. Experimental observations

As it has been emphasized in the introduction, the asymptotic behaviour of such a system is well known. In particular, if the influence function is bounded above and remains positive over $\mathbb{R}^+$, the system always asymptotically converges towards consensus [15]. Nevertheless, to characterize the transient behaviour remains an open question. One typical observation is that before merging toward a unique opinion, agents converge into some temporary local agreement. Figure 1 displays an instance of opinion dynamics (1) when the influence function is the decreasing exponential $f = x \mapsto \exp(-x)$. It first shows that two clusters of similar opinions rapidly form, these clusters remain far appart over a relatively long period of time while they get closer. Finally the two groups merged into one to achieve the expected consensus.

We will now provide a definition to characterize this transient cluster formation. During the first phase when the agents form clusters, the cluster formation is faster than the dynamics which tends to bring the clusters together. Formally speaking and using the previous definition, we can foresee the evolution of the diameter δ of a group is decreasing faster than the distance Δ between the two groups.

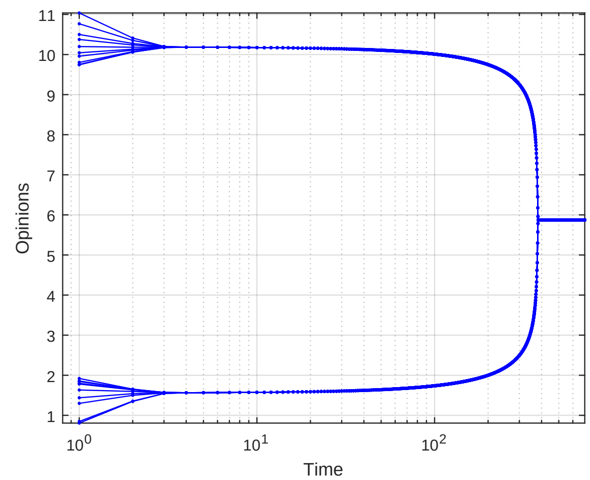

Fig. 1. Simulation over 700 iterations of a multi-agent system with an influence function $f(x) = \exp(-|x|)$ and with 20 agents, with 10 agents initially above 9 and the others below 2, which eventually merge and converge to one final opinion, the consensus. One can see 3 phases : rapid convergence of agents into clusters, clusters remaining far apart and rapid convergence of clusters toward consensus. Note the log x-scale.

C. Problem statement

From the previous observations, the problem of the detection of transient clusters arises. Once a cluster is detected, it is also important to know for how long the cluster stays before merging with other agents.

The criterion we present to answer this question is based only on the evolution over time of the couple (δ, Δ) defined in equations (3) and (4). In accordance with the previous observation, we say that cluster is forming when a group of agents has a diameter decreasing faster than the distance separating the group from the other agents of the system.

Definition 3 (Cluster formation): Given a group of agents A, this group is forming a cluster at time t if its ratio δ/Δ is decreasing from t to $t+1$, *i.e.* if it verifies

$$\frac{\delta(t+1)}{\Delta(t+1)} \leq \frac{\delta(t)}{\Delta(t)}. \quad (5)$$

We propose in the rest of this paper an upper bound of the evolution on the ratio δ/Δ, by finding an upper bound on the evolution of δ (Section III-D), and then a lower bound on the evolution of the distance Δ (Section III-E). As those bounds are obtained by considering worst case scenarios, we can then form a worst case system whose evolution bounds the evolution of the real system. By mean of this worst case system, which is easily computable locally, we can predict how long a group matches the definition of a cluster stated above (Theorem 1 in Section IV).

III. Preliminaries

A. Assumptions

In this subsection, several assumptions are stated for use in later lemmas and propositions. We first introduce the function h defined as follows to lighten the equations :

$$h = \left\{ \begin{array}{ccc} \mathbb{R}^+ & \to & \mathbb{R}^+ \\ x & \mapsto & x\, f(x) \end{array} \right. , \tag{6}$$

with f the influence function defined in (2).

Assumption 1: Function f defined in (2) is non-negative, non-increasing, of class $\mathcal{C}^2$ on $\mathbb{R}^+$ and verifies $f(0) > 0$.

The non-increasing and non-negative properties of the influence function is common in terms of opinion dynamics because the further apart two opinions are, the less they tend to influence each others. The classical Hegselmann-Krause model from [16] does not satisfy the previous assumption because of the discontinuity of its influence function, hence it is not $\mathcal{C}^2$. However we propose in this paper smoothed versions of the Hegselmann-Krause function to match the previous assumption.

The next assumption is more technical and allows us to get locally computable results, but it is not essential to guaranty the occurrence of transient cluster formations.

Assumption 2: Function h defined in (6) has the following properties :

- there exists $\widetilde{x}$ such that h is concave on $[0, \widetilde{x}]$ and convex on $[\widetilde{x}, +\infty)$,
- $\lim_{x\to+\infty} h(x)$ is finite,
- and for all $\widehat{x} \in \arg\max_{x\in\mathbb{R}^+} h(x)$, we have $h''(\widehat{x}) < 0$.

According to the previous assumption, $\widetilde{x}$ is an inflexion point of function h. Aiming at preserving the order of agents, we consider the following assumption.

Assumption 3: Function f is $\log$ concave, with $\log$ the natural logarithm, meaning function $\log(f)$ is concave.

Assumption 3 ensures that the multi-agent system is order preserving, as shown in [29, Th 2]. It guarantees that agents are staying in the same order over time. More formally the order preservation property is stated in the following lemma.

Lemma 1: Suppose that Assumption 3 holds. The multi-agent system (1) is order preserving, i.e., for any agent i and j in the multi-agent system, $x_i \leq x_j$ implies $x_i^+ \leq x_j^+$.

In the sequel of the paper, a renumbering allows to write without loss of generality $x_1 \leq x_2 \leq \cdots \leq x_n$. It is also natural to consider for candidate clusters, groups of consecutive opinions, that is two integers $m, M \in \{1, \ldots, n\}$ such that $m \leq M$ and $A = \{m, \cdots, M\}$. One immediate consequence is that under Assumption 3 we have for all time

$$\delta = x_M - x_m. \tag{7}$$

Examples of influence functions satisfying Assumptions 1-3 are detailed in the following subsection.

B. Examples of suitable influence functions

Here we give two examples of functions satisfying Assumptions 1-3. The class of decreasing exponentials satisfies Assumptions 1-3 and is a classical class of influence function [19].

Another class of functions satisfying Assumptions 1-3 is the one of smoothed generalized Hegselmann-Krause influence functions on infinite support. We choose to consider such functions using sigmoid according to the formula

$$f_{GHK}(x) = \frac{1 - \mathrm{sig}(\alpha(x-1))}{1 - \mathrm{sig}(-\alpha)}, \ \mathrm{sig}(x) = \frac{1}{1 + \mathrm{e}^{-x}}$$

where α is a parameter to control the smoothness of the curve. Figure 2 shows that the larger the α the closer we are to an original Hegselmann-Krause function as in [16], with a communication range of 1. Note that the Hegselmann-Krause model cannot be considered here due to its discontinuity, which does not verify Assumption 1.

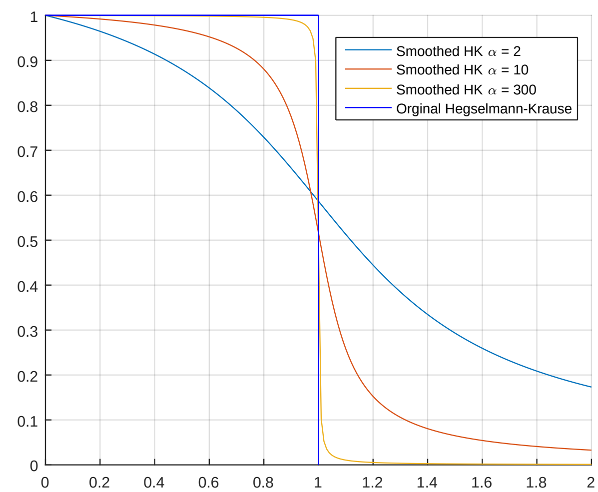

Fig. 2. Generalized Hegselmann-Krause influence functions satisfying Assumption 1-3 and Hegselmann-Krause influence function.

C. Basic properties

This subsection gives properties of function h defined in (6) and will be used in the proofs of this paper.

Lemma 2 (Function h): Under Assumptions 1 and 2 the following properties hold for function h defined in (6) :

- $\arg\max_{x\in\mathbb{R}^+} h(x)$ is a singleton and its unique element is denoted $\widehat{x}$,
- h is non-decreasing on $[0, \widehat{x}]$ and non-increasing on $[\widehat{x}, +\infty)$.

Proof: Under Assumption 2, function h is convex on interval $[\widetilde{x}, +\infty)$. We then prove that h is non-increasing on the same interval. Assume the opposite is true. Then, there exist $x, y \in [\widetilde{x}, +\infty)$ with $x < y$ such that $h(x) < h(y)$. By the mean value theorem, there exits $z \in [\widetilde{x}, +\infty)$ such that

$h'(z) > 0$. Since h is convex over this interval, the graph of h stays above its tangents. As a consequence, function h would diverge if it were not non-increasing.

As h is non-increasing on $[\widetilde{x}, +\infty)$, it falls that all elements of $\arg\max_{x\in\mathbb{R}^+} h(x)$ are in $[0, \widetilde{x}]$. Let us pick an element $\widehat{x} \in \arg\max_{x\in\mathbb{R}^+} h(x)$ and show that it is unique. By concavity of h over $[0, \widetilde{x}]$, h' is non-increasing over the same interval. Since $h'(\widehat{x}) = 0$, h' is non-negative over $[0, \widehat{x}]$ and non-positive over $[\widehat{x}, \widetilde{x}]$. This implies that h is non-decreasing over $[0, \widehat{x}]$ and non-increasing over $[\widehat{x}, \widetilde{x}]$ and also over $[\widehat{x}, \infty)$.

Under Assumption 1, h is $\mathcal{C}^2$ and its Taylor development up to the order 2 at $\widehat{x}$ is

$$h(x) = h(\widehat{x}) + \frac{(x-\widehat{x})^2}{2} h''(\widehat{x}) + o(x-\widehat{x})^2,$$

where the first order term is zero because, by definition, $\widehat{x}$ is an extremum of function h, so $h'(\widehat{x}) = 0$. Then by dividing the previous equation by the quantity $(x-\widehat{x})^2$, we get for $x \neq \widehat{x}$

$$\frac{h(x) - h(\widehat{x})}{(x-\widehat{x})^2} = \frac{h''(\widehat{x})}{2} + o(1).$$

Let $\alpha = -\frac{1}{4}h''(\widehat{x}) > 0$, by Assumption 2. From definition of $o(1)$ it comes that it exists $\eta > 0$ such that for $x \in [\widehat{x} - \eta \,,\, \widehat{x} + \eta]\backslash\{\widehat{x}\}$ we have

$$\frac{h(x) - h(\widehat{x})}{(x-\widehat{x})^2} \leq \frac{h''(\widehat{x})}{2} - \alpha = \frac{h''(\widehat{x})}{4} < 0,$$

according to Assumption 2, then as a consequence,

$$\forall x \in [\widehat{x} - \eta, \widehat{x} + \eta]\backslash\{\widehat{x}\},\ h(x) < h(\widehat{x}). \tag{8}$$

Since h is non-decreasing over $[0, \widehat{x}]$ and non-inscreasing over $[\widehat{x}, +\infty)$, using equation (8), we obtain

$$\forall x \in [0, \widetilde{x}]\backslash\{\widehat{x}\},\ h(x) < h(\widehat{x}). \qquad \blacksquare$$

D. Upper bound on group diameter evolution

In this subsection we give an upper bound on the evolution of the diameter δ defined in (3) of a selected group of agents A, over two iterations. This bound is defined as the minimum of two bounds obtained via different approaches, one being less conservative for small values of δ.

Proposition 1: Under Assumptions 1-3 and for $\Delta \geq \widetilde{x}$, we have

$$\delta^+ \leq \mu(\delta, \Delta)\, \delta\,, \tag{9}$$

where

$$\mu(\delta,\Delta) = 1 - \frac{|A|\, f(\delta)}{s_A(\delta,\Delta)} + \frac{(n-|A|)}{s_B(\delta,\Delta)}\, \mu_B(\delta,\Delta), \tag{10}$$

with

$$s_A(\delta,\Delta) = f(\delta) + (|A|-1)f(0) + (n-|A|)\, f(\Delta), \tag{11}$$

$$s_B = s_B(\delta,\Delta) = f(0) + (|A|-1)f(\delta) + f(\Delta), \tag{12}$$

$$\mu_B(\delta,\Delta) = \frac{1}{\delta}\Big(h(\Delta) + \min\{Mh(\Delta) - h(\Delta+\delta), 0\}\Big), \tag{13}$$

$$M = M(\delta,\Delta) = \begin{cases} \dfrac{\varepsilon_0}{(s_B - \varepsilon_0)} & \text{if } \varepsilon_0 \leq \dfrac{s_B}{2} \\ 1 & \text{otherwise} \end{cases}, \tag{14}$$

$$\varepsilon_0 = \varepsilon_0(\delta,\Delta) = \delta\Big(|A| \max_{[0,\delta]} |f'| + (n-|A|) \max_{[\Delta,+\infty)} |f'|\Big), \tag{15}$$

with function h defined in (6).

Remark 1: The previous equations are easy to compute locally because the only variables encountered are cardinalities and distances δ and Δ.

This result can be interpreted as follows : if the computation of $\mu(\delta,\Delta)$ gives a value smaller than 1, it gives the information that the group of agents is contracting, meaning its diameter is decreasing.

Proof: First, let us detail the expression of δ^+ given in (7), where we split the global sum into three, each corresponding to a part of the system : the selected group A, the agents below A put in a group named B_1, and the agents above A grouped in B_2. More formally we have $B_1 = \{1, \cdots, m-1\}$ and $B_2 = \{M+1, \cdots, n\}$. Then we can write

$$\delta^+ \;=\; \delta - d_A + d_B\ , \tag{16}$$

where

$$d_A = \sum_{i\in A} \left(\frac{h(x_M - x_i)}{\sigma_M} + \frac{h(x_i - x_m)}{\sigma_m} \right), \tag{17}$$

with the notation $d_B = d_{B_1} + d_{B_2}$ and

$$d_{B_1} = \sum_{i\in B_1} \left(\frac{h(x_m - x_i)}{\sigma_m} - \frac{h(x_M - x_i)}{\sigma_M} \right), \tag{18}$$

$$d_{B_2} = \sum_{i\in B_2} \left(\frac{h(x_i - x_M)}{\sigma_M} - \frac{h(x_i - x_m)}{\sigma_m} \right), \tag{19}$$

and

$$\sigma_i = \sum_{j=1}^{N} f(|x_i - x_j|). \tag{20}$$

To obtain an upper bound on δ^+, we will seek separate bounds on previously defined quantities d_A and d_B .

It can be first verified that the quantities s_A and s_B respectively defined in (11) and (12) satisfy

$$s_A(\delta,\Delta) \geq \max\{\sigma_m, \sigma_M\}, \text{ and } s_B(\delta,\Delta) \leq \min\{\sigma_m, \sigma_M\},$$

where σ_i has been defined in (20). Such bounds deal with the denominators of d_A and d_B.

To obtain a lower bound on d_A defined in (17), we first remark that, for $i \in A$

$$f(x_M - x_i) \geq f(\delta) \text{ and } f(x_i - x_m) \geq f(\delta),$$

leading to

$$h(x_M - x_i) \geq (x_M - x_i) f(\delta),$$

and

$$h(x_i - x_m) \geq (x_i - x_m) f(\delta).$$

By summing the two previous equations it comes

$$h(x_M - x_i) + h(x_i - x_m) \geq h(\delta),$$

allowing us to find the following lower bound for d_A

$$d_A \geq \frac{|A|\, h(\delta)}{s_A(\delta,\Delta)}.$$

To cope with d_B, we propose the following upper bound

$$d_B \;\leq\; (n - |A|) \left(\max_{u\in J} \frac{h(u)}{s_B(\delta,\Delta)} - \min_{u\in J} \frac{h(u+\delta)}{s_A(\delta,\Delta)} \right),$$

with u an appropriate substitution and $J = [\Delta, +\infty)$. Thanks to Lemma 2, for $\Delta \geq \widehat{x}$ we have $\max_{u\in J} h(u) = h(\Delta)$ and $\inf_{u\in J} h(u+\delta) = 0$. This allows to get a part of the expression of $\mu(\delta,\Delta)$ in (10) by combining the two previous equations.

To obtain equation (14) defining $M(\delta,\Delta)$, we rewrite the terms of the sum d_{B_1} in (18), knowing the reasoning is the same for d_{B_2} defined in (19). We define, for an agent $j \in B_1$, the quantity Q_j as

$$Q_j = \frac{h(x_m - x_j)}{\sigma_M + \varepsilon} - \frac{h(x_M - x_j)}{\sigma_M},$$

where $\varepsilon = \sigma_m - \sigma_M$, and the previous equation can be rewritten as

$$Q_j = \frac{1}{\sigma_M}\Big(h(x_m - x_j) - h(x_M - x_j)\Big) + \left(\frac{1}{\sigma_M + \varepsilon} - \frac{1}{\sigma_M}\right) h(x_m - x_j).$$

At this point and in order to bound d_B, we want to obtain an upper bound on Q_j for an agent $j \in B_1$.

First, under the assumption $\Delta \geq \widehat{x}$ and according to Lemma 2, we have $h(x_m - x_j) \leq h(\Delta)$ because with $j \in B_1$, $x_m - x_j \geq \Delta$.

Next we suppose $\Delta \geq \widetilde{x}$, with $\widetilde{x}$ the inflexion point of function h defined in Assumption 2. Then for agent $i \in B_1$ we remark that $x_m - x_j \geq \Delta$ and $x_M - x_j \geq \Delta + \delta$. Then using Assumption 2, we also know that function h is convex on $[\widetilde{x}, +\infty)$, implying that its first derivative h' is non-decreasing, allowing us to write

$$h'(x_m - x_j) - h'(x_M - x_j) \leq 0.$$

This means that $u \mapsto h'(u) - h'(u+\delta)$ is non-increasing on $[\widetilde{x}, +\infty)$ and allows us to get the following upper bound

$$h(x_m - x_j) - h(x_M - x_j) \leq h(\Delta) - h(\Delta + \delta).$$

Finally, we need to find an upper bound of $\left(\frac{1}{\sigma_M+\varepsilon} - \frac{1}{\sigma_M}\right)$ and for that we need to bound the absolute value of ε, which is, by definition of σ_m and σ_M in (20), given by

$$\varepsilon = \sum_{i=1}^{n} f(|x_i - x_m|) - f(|x_i - x_M|).$$

Then by using the triangle inequality and the mean value theorem, $|\varepsilon|$ can be bounded by

$$|\varepsilon| \leq \delta \left(|A| \max_{[0,\delta]} |f'| + (n - |A|) \max_{[\Delta,+\infty)} |f'| \right).$$

The right member of the previous equation is the definition of $\varepsilon_0(\delta,\Delta)$ given in (15). Finally, let us show that

$$\left(\frac{1}{\sigma_M + \varepsilon} - \frac{1}{\sigma_M}\right) \leq \frac{M(\delta,\Delta)}{s_B(\delta,\Delta)},$$

where $M(\delta,\Delta)$ is defined in (14). It comes directly that

$$\left(\frac{1}{\sigma_M + \varepsilon} - \frac{1}{\sigma_M}\right) \leq \frac{1}{s_B(\delta,\Delta)}.$$

Then we show that if $\varepsilon_0 < s_B$ we have

$$\left(\frac{1}{\sigma_M + \varepsilon} - \frac{1}{\sigma_M}\right) \leq \frac{\varepsilon_0}{s_B(s_B - \varepsilon_0)}.$$

For that we consider the case where $\varepsilon < 0$, the opposite case giving a null upper bound that does not add precision. So for $\varepsilon < 0$ we have

$$\left(\frac{1}{\sigma_M + \varepsilon} - \frac{1}{\sigma_M}\right) = \frac{|\varepsilon|}{\sigma_M(\sigma_M - |\varepsilon|)},$$

and then for $\varepsilon_0 < \sigma_M$ such that $\varepsilon \leq |\varepsilon_0|$ that

$$\begin{aligned} \frac{|\varepsilon|}{\sigma_M(\sigma_M - |\varepsilon|)} &\leq \frac{\varepsilon_0}{\sigma_M(\sigma_M - \varepsilon_0)} \\ &\leq \frac{\varepsilon_0}{s_B(s_B - \varepsilon_0)}. \end{aligned}$$

By observing that for $s_B \in]\varepsilon_0, 2\varepsilon_0]$ we have

$$\frac{1}{s_B} \leq \frac{\varepsilon_0}{s_B(s_B - \varepsilon_0)},$$

and we obtain the quantity $M(\delta,\Delta)$ as in (14) and with it the quantity $\mu(\delta,\Delta)$ in (10) by combination of the different results we just provided. ■

E. Lower bound on the evolution of inter-group distance

This subsection gives a lower bound on the evolution of the distance Δ related to a selected group of agents, Δ defined in (4) being the minimum distance separating the agents in the group from the others.

Proposition 2: Under Assumptions 1-3 and for $\Delta \geq \widehat{x}$ we have

$$\Delta^+ \geq \eta(\delta,\Delta)\, \Delta, \tag{21}$$

with

$$\eta(\delta,\Delta) = 1 - |A|\, \eta_A(\delta,\Delta) - (n - |A|)\, \eta_B(\delta,\Delta), \tag{22}$$

where

$$\eta_A(\delta,\Delta) = \frac{h(\Delta)}{\sigma_B(\delta,\Delta)}, \tag{23}$$

$$\eta_B(\delta,\Delta) = \frac{1}{\Delta} \max\Big\{ \frac{h(\Delta)}{\sigma_A(\delta,\Delta)}, \frac{h(2\Delta+\delta)}{\sigma_B(\delta,\Delta)} \Big\}, \tag{24}$$

$$\sigma_A(\delta,\Delta) = f(0) + (|A| - 1)\, f(\delta) + f(\Delta), \tag{25}$$

$$\sigma_B(\delta,\Delta) = f(0) + f(\Delta) + (|A| - 1)\, f(\delta + \Delta). \tag{26}$$

Remark 2: The previous equations are, just like the upper bound on δ^+, easy to compute locally because only local variables (specific to the selected group of agents A)

intervene. These variables are distances δ and Δ, and cardinalities.

Proof: We first introduce the following distance

$$\Delta_1 = \min_{i\in A, j\in B_1} |x_i - x_j|,$$

and Δ_2 for group B_2. We will prove the result dealing with Δ_1, knowing the expression is similar for Δ_2. We begin by writing the detailed expression of Δ_1^+

$$\Delta_1^+ = \Delta_1 + D_A + D_B$$

where

$$D_A = \sum_{j\in A}\left(\frac{h(x_j - x_m)}{\sigma_m} - \frac{h(x_j - x_{m-1})}{\sigma_{m-1}}\right),$$

and where $D_B = D_{B_1} + D_{B_2}$ with

$$D_{B_1} = \sum_{j\in B_1}\left(\frac{h(x_{m-1} - x_j)}{\sigma_{m-1}} - \frac{h(x_m - x_j)}{\sigma_m}\right),$$

$$D_{B_2} = \sum_{j\in B_2}\left(\frac{h(x_j - x_m)}{\sigma_m} - \frac{h(x_j - x_{m-1})}{\sigma_{m-1}}\right),$$

with σ_m and σ_M defined in (20).

It can be verified that quantities $\sigma_A(\delta,\Delta)$ and $\sigma_B(\delta,\Delta)$ defined in (25) and (26) satisfy

$$\sigma_A(\delta,\Delta) \le \sigma_m, \text{ and } \sigma_B(\delta,\Delta) \le \sigma_{m-1}.$$

Then by using Lemma 2, we can get the following lower bounds for $\Delta \ge \widetilde{x}$

$$\begin{aligned}
D_A &\ge -\frac{|A|\,h(\Delta)}{\sigma_B(\delta,\Delta)} = -|A|\,\eta_A(\delta,\Delta),\\
D_{B_1} &\ge -\frac{|B_1|\,h(\Delta)}{\sigma_A(\delta,\Delta)},\\
D_{B_2} &\ge -\frac{|B_2|\,h(2\Delta+\delta)}{\sigma_B(\delta,\Delta)},
\end{aligned}$$

giving

$$D_B \ge -(n-|A|)\max\left\{\frac{h(\Delta)}{\sigma_A(\delta,\Delta)}, \frac{h(2\Delta+\delta)}{\sigma_B(\delta,\Delta)}\right\},$$

leading to (24) and finally to (22). ■

F. Upper bound on evolution of ratio δ/Δ

In this subsection, we combine the two previous bounds on δ and Δ to bound the ratio of these two quantities.

Proposition 3: Under Assumptions 1-3 and for $\Delta \ge \widetilde{x}$ we have

$$\frac{\delta^+}{\Delta^+} \le \rho(\delta,\Delta)\,\frac{\delta}{\Delta}\ \text{, where } \rho(\delta,\Delta) = \frac{\mu(\delta,\Delta)}{\eta(\delta,\Delta)}, \tag{27}$$

with $\mu(\delta,\Delta)$ and $\eta(\delta,\Delta)$ defined in Propositions 1 and 2, from which this proposition falls directly.

IV. Main result

Before introducing the main result allowing to detect cluster formation, we introduce a few notation in order to ease its formulation.

We first define the domain $\mathcal{D}$ of the plane (δ,Δ) where the quantity $\rho(\delta,\Delta)$ defined in (27) is smaller than one, meaning that the group diameter is decreasing faster than the distance in between the group and the other agents. More formally, this domain is defined as

$$\mathcal{D} = \{(\delta,\Delta) \in (\mathbb{R}^+)^2 \;/\; \rho(\delta,\Delta) \le 1\}, \tag{28}$$

where $\rho(\delta,\Delta)$ is defined in (27). We also introduce the worst case function W as

$$W = \left\{\begin{array}{ccc} (\mathbb{R}^+)^2 & \to & \mathbb{R}^2 \\ (\delta,\Delta) & \mapsto & \big(\mu(\delta,\Delta)\,\delta\ ,\ \eta(\delta,\Delta)\,\Delta\big) \end{array}\right. . \tag{29}$$

As domain $\mathcal{D}$ does not provide enough information about the duration of a transient cluster formation, we now define a sequence of decreasing (in the sense of inclusion) domains $\mathcal{D}_\tau$, for $\tau \in \mathbb{N}$, such that

$$\mathcal{D}_0 = \{(\delta_w,\Delta_w) \in \mathcal{D} \;/\; [0,\delta_w]\times[\Delta_w,+\infty) \subseteq \mathcal{D}\}, \tag{30}$$

and

$$\begin{aligned}\mathcal{D}_{\tau+1} = \{(\delta_w,\Delta_w) \in \mathcal{D}_\tau \cap W^{-1}(\mathcal{D}_\tau) \;/ \\ [0,\delta_w]\times[\Delta_w,+\infty) \subseteq \mathcal{D}_\tau \cap W^{-1}(\mathcal{D}_\tau)\},\end{aligned} \tag{31}$$

with $W^{-1}(\mathcal{D}_\tau)$ the pre-image of $\mathcal{D}_\tau$ by function W. The definition of $\mathcal{D}_\tau$ allows to show that when $W(\delta,\Delta) \in \mathcal{D}_\tau$ then $(\delta^+,\Delta^+) \in \mathcal{D}_\tau$, by Proposition 1 and 2. Iterating this property over time is the argument to prove the following theorem.

Theorem 1: Given a selected group of agents A satisfying $(\delta(0),\Delta(0)) \in \mathcal{D}_T$, for $T \in \mathbb{N}$ and $\mathcal{D}_T$ defined in (31), we have

$$\forall t \in \{0,\dots,T\}, \frac{\delta(t+1)}{\Delta(t+1)} \le \frac{\delta(t)}{\Delta(t)}, \tag{32}$$

in other words, the group A is forming a cluster according to Definition 3 during at least T iterations.

This theorem allows us to detect when a transient cluster is forming and gives how many iterations it will be guaranteed to remain a cluster before other agents merge with it.

Remark 3: Notice that it makes sense to detect cluster formation for a limited period of time only since the decrease of ratio δ/Δ may occur for some time but may stop when Δ becomes small enough. This is illustrated in Section V.

Proof: Using Proposition 3 and equation (28), statement (32) is implied by

$$\forall t \in \{0,\dots,T\}, (\delta(t),\Delta(t)) \in \mathcal{D}.$$

Instead of proving the previous equation, we show that for $t \in \{0,\dots,T\}$, $(\delta(t),\Delta(t)) \in \mathcal{D}_{T-t}$. We will prove it by induction. The basis case is obvious because if $(\delta(0), \Delta(0)) \in \mathcal{D}_T \subseteq \mathcal{D}$ then the inequality (32) stands for $t = 0$.

We prove the inductive step by taking $t \in \{0, \dots, T-1\}$ and assuming $(\delta(t), \Delta(t)) \in \mathcal{D}_{T-t}$, then we have

$$W(\delta(t), \Delta(t)) = \big(\delta_w(t+1), \Delta_w(t+1)\big) \in \mathcal{D}_{T-t-1},$$

where

$$\begin{aligned} \delta_w(t+1) &= \mu(\delta(t), \Delta(t))\, \delta(t), \\ \Delta_w(t+1) &= \eta(\delta(t), \Delta(t))\, \Delta(t). \end{aligned}$$

By Propositions 1 and 2,

$$\begin{aligned} \delta(t+1) &\leq \delta_w(t+1), \\ \Delta(t+1) &\geq \Delta_w(t+1), \end{aligned}$$

which ensures by definition of $\mathcal{D}_{T-t-1}$ that

$$(\delta(t+1), \Delta(t+1)) \in \mathcal{D}_{T-t-1}. \qquad \blacksquare$$

V. Numerical illustration

In the present section, we run numerical simulations to validate the main result of the paper. We choose an influence function $f = x \mapsto \exp(-|x|)$. The number of agents in the system is $n = 30$, and the number of agents in group A is set at $|A| = 10$. These parameters enable us to compute the domains $\mathcal{D}$ and $\mathcal{D}_0$ to $\mathcal{D}_{10}$ as defined in equations (28), (30) and (31), and drawn in Figure 3.

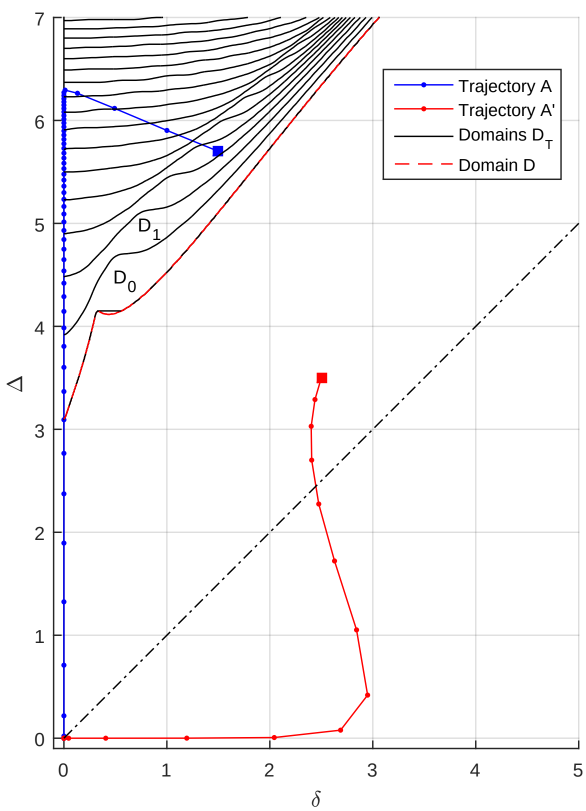

Fig. 3. Trajectories, in the (δ, Δ) plane, of groups A and A' composed of 10 agents in a system of 30 agents, whose time evolutions are shown in Figures 4 and 5.The domain $\mathcal{D}$ in dashed red and domains $\mathcal{D}_\tau$ in black are also drawn. The black dashed-dotted line stands for the first bisector.

We then consider a system with two different sets of initial conditions. In the first case a group of agent denoted A lies in set $\mathcal{D}$ while the other denoted A' do not. Groups A and A' are composed of 10 agents and are initially distributed over two extreme points respectively spaced by $\delta(0) = 1.5$ and $\delta(0) = 2.5$. All the remaining agents are located at one isolated point at distance $\Delta(0) = 5.7$ and $\Delta(0) = 3.5$ from A and A', respectively. These initial conditions are displayed for each group in Figure 3 with squares.

The time evolution of the two sets of initial conditions are depicted in Figure 4 and Figure 5 and also in Figure 3.

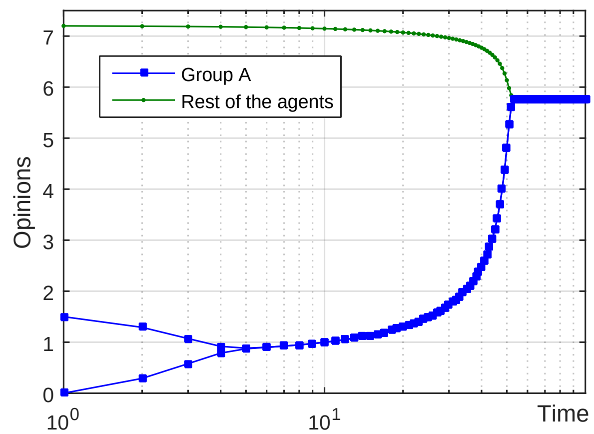

Fig. 4. Evolution of a system of $N = 30$ agents using influence function $f(x) = \exp(-|x|)$. Group A in blue is composed of 10 agents where 5 are initially at $x = 0$ and 5 at $x = 1.5$. The rest of the agents are in green and initially at $x = 7.2$. Note the log x-scale.

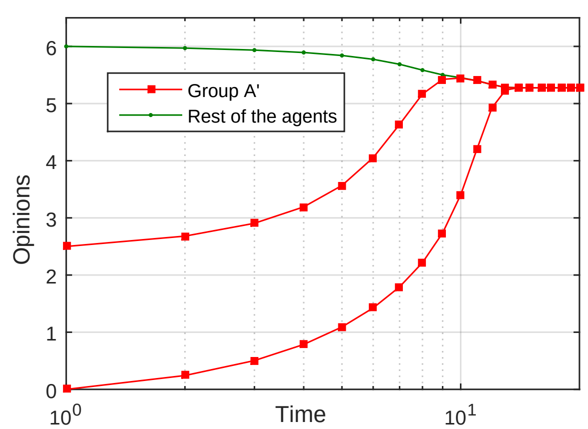

Fig. 5. Evolution of a system of $N = 30$ agents using influence function $f(x) = \exp(-|x|)$. Group A' in red is composed of 10 agents where 5 are initially at $x = 0$ and 5 at $x = 2.5$. The rest of the agents are in green and initially at $x = 6$. Note the log x-scale.

In the configuration displayed in Figure 4, group A quickly contracts to form a transient cluster that will merge with the rest of the agents after a greater amount of iterations. By contrast, in Figure 5, the group A' do not form a transient cluster despite having $\delta(0) < \Delta(0)$. This illustrates the following fact. The property of having a diameter smaller than the distance to the rest of the agents is not a sufficient criterion to detect a cluster.

Initially, the couple $(\delta(0), \Delta(0))$ for group A is in $\mathcal{D}_3$. Using Theorem 1, we know that group A is forming a cluster during at least 3 iterations, which is conservative compared the actual evolution of group A. This is because Theorem 1 deals with the worst case scenario. At time $t = 3$, the distance $\Delta(t)$ has increased and $(\delta(t), \Delta(t))$ for group A is now in domain $\mathcal{D}_9$. As a consequence, knowing the position of the couple $(\delta(t), \Delta(t))$ at time $t = 3$, Theorem 1 ensures that the transient cluster is forming for at least 9 more iterations.

Unlike group A, the dynamics of group A' do not present a transient cluster formation. This is consistent with the $(\delta(t), \Delta(t))$ trajectory of group A' as displayed in Figure 3 : group A' starts out of domain $\mathcal{D}$ and never reaches it. This underlines this importance of a formal criterion to detect whether a cluster is forming or not.

VI. Conclusions

In this paper we analysed transient cluster formation for general models of opinion dynamics including the generalized Hegselmann-Krause model. We proposed a criterion for detecting cluster formation. Our criterion can be computed locally using the initial conditions and only requires the knowledge of the diameter of the cluster candidate and its distance to the rest of the agents. Moreover, our main result provides a lower bound on how many iterations a cluster forms before merging with other agents. Finally our results are illustrated by a numerical example.

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