# A note on sharpness of the local Kato-smoothing property for dispersive wave equations

Shu-Ming Sun, Emmanuel Trélat, Bing-Yu Zhang, Ning Zhong

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Shu-Ming Sun
Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061
email: sun@math.vt.edu

Emmanuel Trélat
Sorbonne Universités, UPMC Univ Paris 06
CNRS UMR 7598, Laboratoire Jacques-Louis Lions
Institut Universitaire de France
4 place Jussieu, 75005, Paris, France
email: emmanuel.trelat@upmc.fr

Bing-Yu Zhang
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
email: zhangb@ucmail.uc.edu

Ning Zhong
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221
email: zhongn@ucmail.uc.edu

**Abstract**

It is well known that solutions of the Cauchy problem for general dispersive equations

$$w_t + iP(D)w = 0, \quad w(x,0) = q(x), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

enjoy the local smoothing property

$$q \in H^s(\mathbb{R}^n) \implies w \in L^2\Big(-T,T; H^{s+\frac{m-1}{2}}_{\text{loc}}(\mathbb{R}^n)\Big),$$

where $m$ is the order of the pseudo-differential operator $P(D)$. This property, called local Kato smoothing, was first discovered by Kato for the KdV equation and implicitly shown later for linear Schrödinger equations. In this paper, we show that the local Kato smoothing property possessed by solutions general dispersive equations in the 1D case is sharp, meaning that there exist initial data $q \in H^s(\mathbb{R})$ such that the corresponding solution $w$ does not belong to the space $L^2\Big(-T,T; H^{s+\frac{m-1}{2}+\epsilon}_{\text{loc}}(\mathbb{R})\Big)$ for any $\epsilon > 0$.

# 1 Introduction

Consider the Cauchy problems for the one-dimensional Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \quad x \in \mathbb{R},\ t \in \mathbb{R},$$

and for the one-dimensional linear Schrödinger equation

$$iv_t + v_{xx} = 0, \quad v(x,0) = \psi(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}. \tag{1.1}$$

While their solutions are known to be as smooth as their initial values in the sense that

$$u(\cdot,t) \in H^s(\mathbb{R}) \text{ if and only if } \phi \in H^s(\mathbb{R}) \text{ for any } t \in \mathbb{R},$$

and

$$v(\cdot,t) \in H^s(\mathbb{R}) \text{ if and only if } \psi \in H^s(\mathbb{R}) \text{ for any } t \in \mathbb{R},$$

they possess a local smoothing property:

$$\phi \in H^s(\mathbb{R}) \implies u \in L^2(-T,T; H^{s+1}_{\text{loc}}(\mathbb{R})) \text{ for any } T > 0,$$

and

$$\psi \in H^s(\mathbb{R}) \implies v \in L^2(-T,T; H^{s+1/2}_{\text{loc}}(\mathbb{R})) \text{ for any } T > 0.$$

This local smoothing property was first discovered by Kato [14] for the KdV equation and implicitly shown later by Sjölin [22] for the linear Schrödinger equation. It is now referred to as the *local Kato-smoothing property* and has been proved by Constantin and Saut [9] to be a common feature of dispersive-wave systems. Indeed, Constantin and Saut studied the following general dispersive-wave equation

$$w_t + iP(D)w = 0, \quad w(x,0) = q(x), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{1.2}$$

where $D = \frac{1}{i}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $P(D)w$ is the pseudo-differential operator

$$P(D)w = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(\xi)\hat{w}(\xi)d\xi$$

defined with a real symbol $p(\xi)$, and $\hat{w}$ is the Fourier transformation of $w$ with respect the spatial variable $x$. The symbol $p(\xi)$ is assumed to satisfy

(i) $p \in L^\infty_{\text{loc}}(\mathbb{R}^n, \mathbb{R})$ and is continuously differentiable for $|\xi| > R$ with some $R \geq 0$,

(ii) there exist $m > 1, C_1 > 0, C_2 > 0$ such that

$$|p(\xi)| \leq C_1(1 + |\xi|)^m \qquad \text{for all} \quad \xi \in \mathbb{R}^n,$$

and

$$|\partial p(\xi)/\partial \xi_j| \geq C_2(1 + |\xi|)^{m-1}|\xi_j|/|\xi|,$$

for all $\xi \in \mathbb{R}^n$ and $|\xi| > R,\ j = 0, 1, 2, \ldots, n$.

**Theorem A** *(Constantin and Saut [9]) Let $s \geq -\frac{m-1}{2}$ be given. Then for any $q \in H^s(\mathbb{R}^n)$, the corresponding solution $w$ of (1.2) belongs to the space $C(\mathbb{R}; H^{s+\frac{m-1}{2}}_{loc}(\mathbb{R}^n))$ and moreover, for any given $T > 0$ and $R > 0$, there exists a constant $C$ depending only on $s$, $T$ and $R$ such that*

$$\int_{-T}^{T}\int_{|x|\leq R}\left|(I-\Delta)^{(m-1+2s)/4}w(x,t)\right|^2 dxdt \leq C\|q\|^2_{H^s(\mathbb{R}^n)}. \tag{1.3}$$

This local smoothing effect, as pointed out by Constantin and Saut in [9], is due to the dispersive nature of the linear part of the equation and the gain of the regularity, $(m-1)/2$, depends only on the order $m$ of the equation and has nothing to do, in particular, with the dimension of the spatial domain $\mathbb{R}^n$. The discovery of the local Kato-smoothing property has stimulated enthusiasm in seeking various smoothing properties of dispersive-wave equations which, in turn, has greatly enhanced the study of mathematical theory of nonlinear dispersive-wave equations, in particular, the well-posedness of their Cauchy problems in low regularity spaces (see [4, 5, 7, 15, 16, 17, 18, 19] and see the references therein).

A question arises naturally: *Does the solution $w$ of (1.2) gain more regularity than $\frac{m-1}{2}$ by comparing it to its initial value $q$?* More precisely, one may wonder if the estimate (1.3) is sharp in the following sense:

*Is it possible to find $\epsilon > 0$ such that, for any given $T > 0$ and $R > 0$, there exists a constant $C$ depending only on $s$, $T$ and $R$ such that*

$$\int_{-T}^{T}\int_{|x|\leq R}\left|(I-\Delta)^{(m-1+2s+\epsilon)/4}w(x,t)\right|^2 dxdt \leq C\|q\|^2_{H^s(\mathbb{R}^n)}? \tag{1.4}$$

In particular, applying Theorem A to the linear KdV equation (Airy equation)

$$u_t + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1.5}$$

and to the linear Schrödinger equation (1.1) leads to the following estimates for their solutions: given $s \in \mathbb{R}$ and $R,\ T > 0$, there exists a constant $C > 0$ such that for any $\phi,\ \psi \in H^s(\mathbb{R})$, the corresponding solutions $u$ of (1.5) and $v$ of (1.1) satisfy

$$\int_{-T}^{T}\int_{|x|\le R} \left|\Lambda^{s+1}u(x,t)\right|^2 dxdt \le C\|\phi\|^2_{H^s(\mathbb{R})}, \tag{1.6}$$

and

$$\int_{-T}^{T}\int_{|x|\le R} \left|\Lambda^{s+1/2}v(x,t)\right|^2 dxdt \le C\|\psi\|^2_{H^s(\mathbb{R})}, \tag{1.7}$$

respectively, where $\Lambda = (I - \partial_x^2)^{\frac{1}{2}}$.

Is it possible to find an $\epsilon > 0$ such that

$$u \in L^2(-T,T; H^{s+1+\epsilon}_{\text{loc}}(\mathbb{R})),\ v \in L^2(-T,T; H^{s+/12+\epsilon}_{\text{loc}}(\mathbb{R}))$$

and moreover, for any given $T > 0$ and $R > 0$,

$$\int_{-T}^{T}\int_{|x|\le R} \left|\Lambda^{s+1+\epsilon}u(x,t)\right|^2 dxdt \le C\|\phi\|^2_{H^s(\mathbb{R})}, \tag{1.8}$$

and

$$\int_{-T}^{T}\int_{|x|\le R} \left|\Lambda^{s+1/2+\epsilon}v(x,t)\right|^2 dxdt \le C\|\psi\|^2_{H^s(\mathbb{R})}? \tag{1.9}$$

Here the constant $C$ is assumed to depend only on $s,\ R$, and $T$.

As far as we know, this question has not been addressed in the literature. On the other hand, it has been proved by Kenig, Ponce and Vega [16, 17] that solutions of (1.5) and (1.1) possess the following smoothing property.

**Theorem B** (Kenig, Ponce and Vega [16, 17]) *For any $s \in \mathbb{R}$, there exists a constant $C_s > 0$ depending only on $s$ such that for any $\phi,\ \psi \in H^s(\mathbb{R})$, the corresponding solutions $u$ of (1.5) and the solution $v$ of (1.1) satisfy*

$$\sup_{x\in\mathbb{R}} \|\partial_x^{s+1}u(x,\cdot)\|_{L^2_t(\mathbb{R})} \le C_s\|\phi\|_{H^s(\mathbb{R})}, \tag{1.10}$$

*and*

$$\sup_{x\in\mathbb{R}} \|\partial_x^{s+1/2}v(x,\cdot)\|_{L^2_t(\mathbb{R})} \le C_s\|\psi\|_{H^s(\mathbb{R})}. \tag{1.11}$$

This property is referred to as *sharp Kato-smoothing*. It has become an important tool in studying the non-homogeneous boundary value problems [1, 2, 10, 11] and control theory of the KdV equation and the Schrödinger equation (see [6, 12, 20, 21] and see the references therein).

Since the estimates (1.6) and (1.7) follow easily from (1.10) and (1.11), it remains a doubt if estimates (1.6) and (1.7) are sharp: *Does there exist $\epsilon > 0$ such that (1.8) and (1.9) hold for all solutions of (1.5) and (1.1), respectively?*

In this paper, we will first consider the Cauchy problem for general one-dimensional evolutionary partial differential equations

$$u_t = Q(\partial_x)u, \qquad u(x,0) = \phi(x), \qquad x \in \mathbb{R}, \quad t > 0 \tag{1.12}$$

where

$$Q(D) = Q_1(\partial_x) + Q_2(\partial_x)$$

is a $(2N+1)$-order differential operator, with

$$Q_1(\partial_x) = \sum_{j=0}^{N} a_j \partial_x^{2j+1}, \quad \textit{and} \qquad Q_2(\partial_x) = \sum_{k=1}^{N} b_k \partial_x^{2k},$$

where $a_j, j = 0, 1, \cdots, N$ are real constants and

$$b_k = \alpha_k + i\beta_k, \quad k = 1, 2, \cdots, N,$$

are complex constants. Assume that there exists an integer $0 \le \nu \le N$ such that either

$$(-1)^N a_N > 0, \qquad \alpha_k = 0,\ k = \nu+1, \cdots, N, \quad (-1)^\nu \alpha_\nu < 0, \tag{1.13}$$

or

$$a_N = 0, \quad (-1)^N \alpha_N < 0. \tag{1.14}$$

Note that, if $(-1)^N a_N < 0$, then we can make a change of variable $x \to -x$ to change the equation so that $(-1)^N a_N > 0$. Then it can be verified that the solution $u$ of (1.12) possesses the local Kato-smoothing property:

*Given $s \in \mathbb{R}$ and $R,\ T > 0$, there exists $C > 0$ such that for any $\phi \in H^s(\mathbb{R})$, the corresponding solution $u$ of (1.12) satisfies*

$$\int_0^T \int_{|x| \le R} \left|\Lambda^{s+N} u(x,t)\right|^2 dxdt \le C\|\phi\|^2_{H^s(\mathbb{R})}. \tag{1.15}$$

We show that the local Kato smoothing property for (1.12) given by the estimate (1.15) is sharp.

**Theorem 1.1.** *For any $\epsilon > 0$, the estimate*

$$\int_0^T \int_{|x| \le R} \left|\Lambda^{s+N+\epsilon} u(x,t)\right|^2 dxdt \le C\|\phi\|^2_{H^s(\mathbb{R})} \tag{1.16}$$

*fails to be true for all solutions of the Cauchy problem (1.12).*

This result applies not only to the Airy equation, but also to the linear KdV-Burgers equation

$$u_t + u_x + u_{xxx} - u_{xx} = 0.$$

which is not cannot be written in the form of the dispersive equation (1.2). Moreover, the parabolic equation

$$u_t = (-1)^{m+1}\partial_x^{2m} u, \qquad u(x,0) = \phi(x), \qquad x \in \mathbb{R}, \quad t \geq 0,$$

and the (linear) Ginzburg-Landau type equation

$$u_t = (\alpha + i\beta)(-1)^{m+1}\partial_x^{2m} u, \qquad u(x,0) = \phi(x), \qquad x \in \mathbb{R}, \quad t \geq 0,$$

with $\alpha > 0$ are special cases of (1.12). Their solutions possess the following global dissipative smoothing property:

$$\phi \in H^s(\mathbb{R}) \implies u \in C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+m}(\mathbb{R}))$$

and there exists a constant $C$ depending only on $T$ and $s$ such that

$$\|u\|_{L^2(0,T;H^{s+m}(\mathbb{R}))} \leq C\|\phi\|_{H^s(\mathbb{R})}. \tag{1.17}$$

Similarly, one may wonder if this type of dissipative smoothing property is sharp: does the solution $u$ belong to $L^2(0,T; H^{s+m+\epsilon}_{\text{loc}}(\mathbb{R}))$ for some $\epsilon > 0$ in general when $\phi \in H^s(\mathbb{R})$, in general? Theorem 1.1 provides a negative answer to this question and shows that the dissipative smoothing property (1.17) is sharp.

We next consider the Cauchy problem (1.2) in the 1D case

$$w_t + iP(D)w = 0, \quad w(x,0) = q(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \tag{1.18}$$

and we show that the estimate (1.3) is sharp.

**Theorem 1.2.** *The local Kato-smoothing property of the Cauchy problem (1.2) posed in $\mathbb{R}$ as described by (1.3) is sharp in the sense that, for any $\epsilon > 0$, the estimate*

$$\int_0^T \int_{|x|\leq R} \left|\Lambda^{s+(m-1)/2+\epsilon} w(x,t)\right|^2 dxdt \leq C\|q\|^2_{H^s(\mathbb{R})} \tag{1.19}$$

*fails to be true for all solutions of (1.2).*

To prove Theorems 1.1 and 1.2, given $s$ and $\epsilon > 0$, we will construct bounded sequences $\{\phi_n\}$ and $\{\psi_n\}$ in $H^s(\mathbb{R})$ such that the corresponding solutions $u_n$ of (1.5), and $v_n$ of (1.1), satisfy respectively

$$\lim_{n\to+\infty} \int_{-T}^T \int_{|x|\leq R} \left|\Lambda^{s+1+\epsilon} u_n(x,t)\right|^2 dxdt = +\infty,$$

$$\lim_{n\to+\infty} \int_{-T}^T \int_{|x|\leq R} \left|\Lambda^{s+1/2+\epsilon} v_n(x,t)\right|^2 dxdt = +\infty,$$

for some $T > 0$ and $R > 0$.

## 2 Proof of Theorem 1.1

By contradiction, assume that the estimate (1.16) holds true for some $\epsilon > 0$. Without loss of generality, we assume that $s = 0$ and $0 < \epsilon < \frac{1}{2}$. Let $p$ be a given $C^\infty$ smooth function with compact support, such that $0 \le p(x) \le 1$ for any $x \in \mathbb{R}$, and

$$p(x) = \begin{cases} 1 & \text{for any } x \in [-99, -5], \\ 0 & \text{for any } x \notin [-110, -1]. \end{cases}$$

Applying $\Lambda^\epsilon$ to the equation in (1.5) yields

$$(\Lambda^\epsilon u)_t + Q(\partial_x)[(\Lambda^\epsilon u)] = 0, \qquad (\Lambda^\epsilon u)(x,0) = (\Lambda^\epsilon \phi)(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}. \tag{2.1}$$

Multiplying the equation in (2.1) by $p(x)\Lambda^\epsilon u$ and integrating the resulting equation over $\mathbb{R}$ with respect to $x$ gives

$$\begin{aligned}\left(\int_{-\infty}^{\infty} p(x)\frac{(\Lambda^\epsilon u)^2}{2}dx\right)_t &= \sum_{j=0}^{N}(-1)^{j+1}a_j\left(j+\frac{1}{2}\right)\int_{-\infty}^{\infty} p'(x)(\partial_x^j(\Lambda^\epsilon u)^2 dx \\ &\quad + \sum_{j=0}^{N}\sum_{l=2}^{j+1}\begin{pmatrix} l \\ j+1 \end{pmatrix}\int_{-\infty}^{\infty} p^{(l)}(x)\partial_x^{j+1-l}(\Lambda^\epsilon u)\partial_x^j(\Lambda^\epsilon u)dx \\ &\quad + \sum_{k=1}^{N}(-1)^k b_k \int_{-\infty}^{\infty}\partial_x^k(p(x)\Lambda^\epsilon u)\partial_x^k(\Lambda^\epsilon u)dx\end{aligned}$$

Integrating both sides of the above equation with respect to $t$ over $(0,T)$ leads to

$$\begin{aligned}&\int_{-\infty}^{\infty} p(x)(\Lambda^\epsilon\phi)^2 dx - \int_{-\infty}^{\infty} p(x)\left(\Lambda^\epsilon u(x,T)\right)^2 dx \\ &\quad = \sum_{j=0}^{N}(-1)^{j+1}a_j\left(j+\frac{1}{2}\right)\int_0^T\int_{-\infty}^{\infty} p'(x)(\partial_x^j(\Lambda^\epsilon u)^2 dxdt \\ &\qquad + \sum_{j=0}^{N}\sum_{l=2}^{j+1}\begin{pmatrix} l \\ j+1 \end{pmatrix}\int_0^T\int_{-\infty}^{\infty} p^{(l)}(x)\partial_x^{j+1-l}(\Lambda^\epsilon u)\partial_x^j(\Lambda^\epsilon u)dxdt \\ &\qquad + \sum_{k=1}^{N}(-1)^k b_k\int_0^T\int_{-\infty}^{\infty}\partial_x^k(p(x)\Lambda^\epsilon u)\partial_x^k(\Lambda^\epsilon u)dxdt \end{aligned} \tag{2.2}$$

Since $p$ has a compact support, and since (1.16) holds true, it follows that the right-hand side of (2.2) estimated by $C_T\|\phi\|^2_{L^2(\mathbb{R})}$, and thus

$$\int_{-\infty}^{\infty} p(x)(\Lambda^\epsilon\phi)^2 dx - \left|\int_{-\infty}^{\infty} p(x)\left(\Lambda^\epsilon u(x,T)\right)^2 dx\right| \le C_T\|\phi\|^2_{L^2(\mathbb{R})} \tag{2.3}$$

Let us prove that (2.3) cannot hold true, by constructing a sequence $\{\phi_n\}_{n=1}^\infty$ in $L^2(\mathbb{R})$ such that

$$\sup_{0<n<+\infty}\|\phi_n\|_{L^2(\mathbb{R})} < +\infty, \tag{2.4}$$

$$\lim_{n\to+\infty}\int_{-\infty}^{\infty} p(x)(\Lambda^\epsilon\phi_n(x))^2dx = +\infty, \tag{2.5}$$

and

$$\sup_{0<n<+\infty}\left|\int_{-\infty}^{\infty} p(x)\left(\Lambda^\epsilon u_n(x,T)\right)^2 dx\right| < +\infty, \tag{2.6}$$

where $u_n$ is the corresponding solution of (1.12) with $u_n(x,0)=\phi_n(x)$.

Let $\eta$ be the function whose Fourier transform is

$$\hat{\eta}(\xi) = \frac{1}{(1+\xi^2)^{1/4}(1+\ln(1+\xi^2))^2}.$$

The function $\eta$ belongs to the space $L^2(\mathbb{R})$ and has the following properties:

(i) $\Lambda^\epsilon\eta \notin L^2(c,d)$ for any $(c,d)$ with $0\in(c,d)$,

(ii) $\Lambda^\epsilon\eta$ is well-defined for any $x\neq 0$,

(iii) both $\eta(x)$ and $(\Lambda^\epsilon\eta)(x)$ decay faster than any algebraic order when $|x|\to+\infty$.

Let $\chi_n(\xi)$ be an even $C^\infty$ cut-off function satisfying $\chi_n(\xi)=1$ for $\xi\in[0,n]$ and $\chi_n(\xi)=0$ for $\xi\in[n+1,+\infty)$. Note that $\chi_n(\xi)$ is a monotone decreasing function and keeps the same shape for $\xi\in[n,n+1]$ as $n$ changes. For any $n\geq 1$, let $\phi_n\in L^2(\mathbb{R})$ be the function whose Fourier transform is

$$\hat{\phi}_n(\xi) = \frac{e^{i50\xi}\chi_n(\xi)}{(1+\xi^2)^{1/4}(1+\ln(1+\xi^2))^2} = e^{i50\xi}\chi_n(\xi)\hat{\eta}(\xi).$$

Then we have

$$\|\phi_n\|_{L^2(\mathbb{R})} = \|\hat{\phi}_n\|_{L^2(\mathbb{R})} \leq C\|\eta\|_{L^2(\mathbb{R})} < +\infty$$

for any $n\geq 1$. Since $\Lambda^\epsilon\eta(x+50)\notin L^2(c,d)$ for any $(c,d)$ with $-50\in(c,d)$ and

$$\max_{|x+50|\geq 10}\left(|\phi_n(x)|+|\Lambda^\epsilon\phi_n(x)|\right) \leq C(1+|x|)^{-2} \text{ for } n=1,2,\cdots$$

where $C$ is independent of $n$, we have that for any $n\geq 1$,

$$\int_{-\infty}^{\infty}(1-p(x))(\Lambda^\epsilon\phi_n)^2dx \leq C_0 < +\infty,$$

and, henceforth, as $n\to+\infty$,

$$\begin{aligned}\int_{-\infty}^{\infty} p(x)(\Lambda^\epsilon\phi_n)^2dx &= \int_{-\infty}^{\infty}(\Lambda^\epsilon\phi_n)^2dx - \int_{-\infty}^{\infty}(1-p(x))(\Lambda^\epsilon\phi_n)^2dx \\ &= \int_{-\infty}^{\infty}\Big((1+\xi^2)^{\epsilon/2}\chi_n(\xi)\hat{\eta}(\xi)\Big)^2d\xi - \int_{-\infty}^{\infty}(1-p(x))(\Lambda^\epsilon\phi_n)^2dx \to +\infty\end{aligned}$$

It remains to prove (2.6). To this end, note that if $\alpha_k = 0,\ k = \nu+1, \cdots, N, \quad (-1)^\nu \alpha_\nu < 0$ for some $1 \le \nu \le N$, then for any $x \in \mathbb{R}$,

$$\begin{aligned}|\Lambda^\epsilon u_n(x,T)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|e^{Q(i\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)\right|d\xi\\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{Q_2(i\xi)T}\left|(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)\right|d\xi\\ &\le \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{2Q_2(i\xi)T}(1+\xi^2)^{\epsilon}d\xi\right)^{\frac{1}{2}}\|\phi_n\|_{L^2(\mathbb{R})}\end{aligned}$$

Then

$$\begin{aligned}\left|\int_{-\infty}^{\infty} p(x)\left(\Lambda^\epsilon u_n(x,T)\right)^2 dx\right| &\le \frac{1}{2\pi}\int_{-110}^{-1} p(x)\int_{-\infty}^{\infty} e^{2Q_2(i\xi)T}(1+\xi^2)^{\epsilon}d\xi\|\phi_n\|^2_{L^2(\mathbb{R})}dx\\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2Q_2(i\xi)T}(1+\xi^2)^{\epsilon}d\xi\|\phi_n\|^2_{L^2(\mathbb{R})}\int_{-110}^{-1} p(x)dx < C < +\infty\end{aligned}$$

for any $n$. On the other hand, If $\alpha_k = 0,\ k = 1, 2, \cdots, N$, then

$$Q(i\xi) = i\widetilde{Q}(\xi)$$

where $\widetilde{Q}(\xi)$ is a real valued polynomial of order $2N+1$ with the coefficient of highest order term satisfying $(-1)^N a_N > 0$. If $T > 0$ is given, by our assumption, there exist $M > 0$ and $C_1 > 0$ such that $\widetilde{Q}'(\xi)T \ge C_1\xi^{2N}$ whenever $|\xi| > M$. Then, for $x \le -1$ and $T > 0$,

$$\begin{aligned}\Lambda^\epsilon u_n(x,T) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\widetilde{Q}(\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)d\xi\\ &=\frac{1}{2\pi}\int_{|\xi|>M} e^{i\widetilde{Q}(\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)d\xi + \frac{1}{2\pi}\int_{|\xi|\le M} e^{i\widetilde{Q}(\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)d\xi\,,\end{aligned}$$

where

$$\begin{aligned}&\frac{1}{2\pi}\left|\int_{|\xi|\le M} e^{i\widetilde{Q}(\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)d\xi\right| \le \frac{1}{2\pi}(2M)^{1/2}(1+M^2)^{\epsilon/2}\|\phi_n\|_{L^2(\mathbb{R})},\\ &\frac{1}{2\pi}\int_{|\xi|>M} e^{i\widetilde{Q}(\xi)T-ix\xi}(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)d\xi = \frac{1}{2\pi}\int_{|\xi|>M}\left(\frac{(1+\xi^2)^{\epsilon/2}\hat{\phi}_n(\xi)}{i(\widetilde{Q}'(\xi)T-x)}\right)_\xi e^{i(\widetilde{Q}(\xi)T-x\xi)}d\xi\\ &\qquad = \frac{1}{2\pi}\int_{|\xi|>M}\left(\frac{(1+\xi^2)^{\epsilon/2}e^{i50\xi}\chi_n(\xi)}{i(\widetilde{Q}'(\xi)T-x)(1+\xi^2)^{1/4}(1+\ln(1+\xi^2))^2}\right)_\xi e^{-i(\widetilde{Q}(\xi)T-x\xi)}d\xi\end{aligned}$$

For $|\xi| > M$, we have

$$\begin{aligned}|w_n(\xi)| :=& \left|\left(\frac{(1+\xi^2)^{\epsilon/2}e^{i50\xi}\chi_n(\xi)}{i(\widetilde{Q}'(\xi)T-x)(1+\xi^2)^{1/4}(1+\ln(1+\xi^2))^2}\right)_\xi e^{-i(\widetilde{Q}(\xi)T-x\xi)}\right|\\ \le& \frac{C}{(C_1\xi^{2m}T+|x|)(1+\xi^2)^{\frac{1}{4}-\frac{\epsilon}{2}}(1+\ln(1+\xi^2))^2} \le \frac{C}{\xi^{2m}+1},\end{aligned}$$

where $\frac{1}{4}-\frac{\epsilon}{2}\geq 0$ and $C>0$ is independent of $n$. Hence,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} p(x)\left(\Lambda^{\epsilon}u_n(x,T)\right)^2 dx\right| \\
&\qquad \leq C\int_{-110}^{-1} p(x)\left(\int_{|\xi|>M}|w_n(\xi)|d\xi+\frac{(2M)^{1/2}}{2\pi}(1+M^2)^{\epsilon/2}\|\phi_n\|_{L^2(\mathbb{R})}\right)^2 dx \\
&\qquad \leq C\int_{-110}^{-1} p(x)dx<+\infty
\end{aligned}$$

Therefore, (2.3)-(2.6) yield a contradiction, which implies that there does not exist any $\epsilon>0$ such that (1.16) holds true. The theorem is proved. □

# 3 Proof of Theorem 1.2

By assumption, there exists a $N>0$ such $\tau=p(\xi)$ is invertible on $(N,+\infty)$ and

$$\pi(\xi)\sim\xi^m,\ \xi=\nu(\tau)\sim\tau^{1/m},\quad p'(\xi)\sim\tau^{m-1}\ as\ \tau\to+\infty.$$

By contradiction, assume that (1.19) holds true for some $\epsilon>0$. Without loss of generality, we assume that $s=0$ and that $0<\epsilon<\frac{1}{2}$. Let $\eta$ be the function as defined in the proof of Theorem 1.1. For any integer $n\geq 1$, let $\mu_n(\xi)$ be a $C^\infty$ cut-off function such that

$$\mu_n(\xi)=\begin{cases}1 & \xi\in[N,n+N]\\ 0 & \text{for } \xi\leq N \text{ and } \xi\geq n+N+1\end{cases}$$

and $\psi_n\in L^2(\mathbb{R})$ be the function whose Fourier transform is

$$\hat{\psi}_n(\xi)=\frac{\mu_n(\xi)}{(1+\xi^2)^{1/4}(1+\ln(1+\xi^2))^2}=\mu_n(\xi)\hat{\eta}(\xi).$$

Then, for any $n\geq 1$,

(i) $\Lambda^{\epsilon}\psi_n\in L^2(\mathbb{R})$ and $\lim_{n\to+\infty}\|\Lambda^{\epsilon}\psi_n\|_{L^2(\mathbb{R})}=\infty$,

(ii) there exists $C_0>0$ such that

$$\int_{-\infty}^{\infty}|\psi_n(x)|^2dx\leq C_0,\qquad \max_{|x|\geq 1}\left(|\psi_n(x)|+|\Lambda^{\epsilon}\psi_n(x)|\right)\leq C(1+|x|)^{-2}.$$

Let $v_n$ be the solution of (1.18) with $\psi$ replaced by $\psi_n$. Since (1.19) holds true, we must have

$$\int_{-T}^{T}\int_{|x|\leq R}\left|\Lambda^{(m-1)/2+\epsilon}v_n(x,t)\right|^2 dxdt\leq C\|\psi_n\|^2_{L^2(\mathbb{R})}<+\infty. \tag{3.1}$$

The left-hand side of (3.1) can be written as

$$\int_{-T}^{T}\int_{|x|\leq R}\left|\Lambda^{(m-1)/2+\epsilon}v_n(x,t)\right|^2 dxdt=I_n-J_n,$$

with

$$I_n = \int_{-R}^{R}\int_{-\infty}^{\infty} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt\,dx, \qquad J_n = \int_{-R}^{R}\left(\int_{T}^{\infty} + \int_{-\infty}^{-T}\right) \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt\,dx.$$

Let us prove that (3.1) cannot be true by showing that

$$\lim_{n\to+\infty} I_n = +\infty \quad \text{and} \quad \sup_{1\le n<+\infty} J_n < +\infty.$$

We first prove that, for any $n \ge 1$,

$$\int_{-\infty}^{\infty} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt = C\|\Lambda^{\epsilon}\psi_n\|^2_{L^2(\mathbb{R})} \quad \text{for any } x \in \mathbb{R}. \tag{3.2}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt &= \int_{-\infty}^{\infty} \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} (1+\xi^2)^{\frac{\epsilon}{2}+\frac{m-1}{4}} e^{-i(tp(\xi)-x\xi)}\hat{\psi}_n(\xi)d\xi\right|^2 dt \\
&= C\int_{-\infty}^{\infty} \left|\int_{N}^{\infty} (1+\xi^2)^{\frac{\epsilon}{2}+\frac{m-1}{4}} e^{-i(tp(\xi)-x\xi)}\mu_n(\xi)\hat{\eta}(\xi)d\xi\right|^2 dt \\
&= C\int_{-\infty}^{\infty} \left|\int_{M}^{\infty} (1+\nu^2(\tau))^{\frac{\epsilon}{2}+\frac{m-1}{4}} e^{-i(t\tau-x\nu(\tau))}\mu_n(\nu(\tau))\hat{\eta}(\nu(\tau))\nu'(\tau)d\tau\right|^2 dt \\
&= C\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} e^{-it\tau} w_n(x,\tau)d\tau\right|^2 dt
\end{aligned}$$

where

$$w_n(x,\tau) = ((1+\nu^2(\tau))^{\frac{\epsilon}{2}+\frac{m-1}{4}} e^{ix\nu(\tau))}\mu_n(\nu(\tau))\hat{\eta}(\nu(\tau))\nu'(\tau) \qquad \text{for} \qquad \tau \ge M$$

and $w_n(\tau) = 0$ for $\tau \le M$, by the Plancherel theorem, we have, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt &= C\int_{-\infty}^{\infty} |w_n(x,\tau)|^2 d\tau \\
&= C\int_{1}^{\infty} \left|(1+\nu^2(\tau))^{\frac{\epsilon}{2}+\frac{m-1}{4}}\mu_n(\nu(\tau))\hat{\eta}(\nu(\tau))\nu'(\tau)\right|^2 d\tau \\
&= C\int_{1}^{\infty} \left|(1+\xi^2)^{\frac{\epsilon}{2}+\frac{m-1}{4}}\mu_n(\xi)\hat{\eta}(\xi)\sqrt{\nu'(p(\xi))}\right|^2 d\xi \\
&= C\|\Lambda^{\epsilon}\psi_n\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

and therefore,

$$I_n = 2RC\|\Lambda^{\epsilon}\psi_n\|^2_{L^2(\mathbb{R})} \to +\infty \quad \text{as} \quad n \to +\infty.$$

It remains to show that $\sup_{1\le n<+\infty} J_n < +\infty$. We only consider the term

$$J_{n,T} := \int_{-R}^{R}\int_{T}^{\infty} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt\,dx,$$

the discussion for the other term

$$J_{n,-T} := \int_{-R}^{R}\int_{-\infty}^{-T} \left|\Lambda^{\epsilon+\frac{m-1}{2}} v_n(x,t)\right|^2 dt\, dx$$

being similar. Using integration by parts, we get that, for $|x| \le R$ and $T \ge R+1$ ,

$$\begin{aligned}\int_T^\infty \left|\Lambda^{\epsilon+\frac{1}{2}} v_n(x,t)\right|^2 dt &= \int_T^\infty \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\xi^2)^{\frac{\epsilon}{2}+\frac{1}{4}} e^{-i(tp(\xi)-x\xi)}\hat{\psi}_n(\xi)d\xi\right|^2 dt \\ &= C\int_T^\infty \left|\int_N^\infty (1+\xi^2)^{\frac{\epsilon}{2}+\frac{1}{4}} e^{-i(tp(\xi)-x\xi)}\mu_n(\xi)\hat{\eta}(\xi)d\xi\right|^2 dt \\ &= C\int_T^\infty \left|\int_N^\infty \left(\frac{(1+\xi^2)^{\frac{\epsilon}{2}+\frac{1}{4}}\mu_n(\xi)\hat{\eta}(\xi)}{-i(tp'(\xi)-x)}\right)_\xi e^{-i(tp(\xi)-x\xi)}d\xi\right|^2 dt \qquad (3.3)\end{aligned}$$

where

$$|tp'(\xi)-x| \ge |t||p'(\xi)| - R \ge |t| + T - R \ge |t|+1 \quad \text{or} \quad |tp'(\xi)-x| \ge \xi^{m-1},$$

for $\xi \ge N$. Therefore, by performing integrations by parts in (3.3) and noticing that each of them produces at least one factor of the order of $|\xi|^{-1}$, we see that the integral in (3.3) with respect to $\xi$ is finite and still has a term $tp'(\xi)-x$ in the denominator such that

$$\int_T^\infty \left|\Lambda^{\epsilon+\frac{1}{2}} v_n(x,t)\right|^2 dt \le C\int_T^\infty \left|\frac{1}{t+1}\right|^2 dt < +\infty,$$

or

$$\int_{-R}^{R}\int_T^\infty \left|\Lambda^{\epsilon+\frac{1}{2}} v_n(x,t)\right|^2 dt\, dx \le C < +\infty,$$

where $C > 0$ is independent of $n$, but may depend on $T$ and $R$. The theorem is proved. □

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