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Approximate Capacity of the Gaussian Interference Channel with Noisy Channel-Output Feedback

Victor Quintero, Samir M. Perlaza, Iñaki Esnaola and Jean-Marie Gorce

Abstract—In this paper, an achievability region and a converse region for the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) are presented. The achievability region is obtained using a random coding argument and three well-known techniques: rate splitting, superposition coding and backward decoding. The converse region is obtained using some of the existing perfect-output feedback outer-bounds as well as a set of new outer-bounds that are obtained by using genie-aided models of the original G-IC-NOF. Finally, it is shown that the achievability region and the converse region approximate the capacity region of the G-IC-NOF to within a constant gap in bits per channel use.

Index Terms—Capacity, Interference Channel, Noisy Channel-Output Feedback

I. NOTATION

Throughout this paper, $(\cdot)^+$ denotes the positive part operator, i.e., $(\cdot)^+ = \max(\cdot, 0)$ and $\mathbb{E}[\cdot]$ denotes the expectation with respect to the distribution of the random variable $X$. The logarithm function $\log$ is assumed to be base 2.

II. SYSTEM MODEL

Consider the two-user G-IC-NOF in Figure 1. Transmitter $i$, with $i \in \{1, 2\}$, communicates with receiver $i$ subject to the interference produced by transmitter $j$, with $j \in \{1, 2\} \setminus \{i\}$. There are two independent and uniformly distributed messages, $W_i \in \mathcal{W}_i$, with $\mathcal{W}_i = \{1, 2, \ldots, 2^{N R_i}\}$, where $N$ denotes the block-length in channel uses and $R_i$ is the transmission rate in bits per channel use. At each block, transmitter $i$ sends the codeword $X_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,N})^T \in \mathcal{X}_i^N$, where $\mathcal{X}_i$ and $\mathcal{X}_j^N$ are respectively the channel-input alphabet and the codebook of transmitter $i$. The channel coefficient from transmitter $j$ to receiver $i$ is denoted by $h_{ij}$; the channel coefficient from transmitter $i$ to receiver $i$ is denoted by $h_{ii}$; and the channel coefficient from channel-output $i$ to transmitter $i$ is denoted by $h_{ii}$. All channel coefficients are assumed to be non-negative real numbers. At a given channel use $n \in \{1, 2, \ldots, N\}$, the channel output at receiver $i$ is denoted by $Y_{i,n}$. During channel use $n$, the input-output relation of the channel model is given by

\begin{equation}
Y_{i,n} = h_{ii} X_{i,n} + h_{ij} X_{j,n} + Z_{i,n},
\end{equation}

where $Z_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver $i$. Let $d > 0$ be the finite feedback delay measured in channel uses. At the end of channel use $n$, transmitter $i$ observes $Y_{i,n}$, which consists of a scaled and noisy version of $Y_{i,n-d}$. More specifically,

\begin{equation}
Y_{i,n} = \begin{cases}
\frac{Z_{i,n}}{h_{ii}} Y_{i,n-d} + Z_{i,n}, & \text{for } n \in \{1, 2, \ldots, d\} \\
\frac{Z_{i,n}}{h_{ii}} Y_{i,n-d} + Z_{i,n}, & \text{for } n \in \{d+1, d+2, \ldots, N\},
\end{cases}
\end{equation}

where $Z_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair $i$. The random variables $Z_{i,n}$ and $\bar{Z}_{i,n}$ are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use, i.e., $d = 1$. The encoder of transmitter $i$ is defined by a set of deterministic functions $f_{i}^{(1)}(\cdot), \ldots, f_{i}^{(N)}(\cdot)$, with $f_{i}^{(1)}(W_i) : \mathcal{X}_i \rightarrow \mathcal{X}_i$ and for all $n \in \{2, \ldots, N\}$, $f_{i}^{(n)}(W_i) : \mathcal{X}_i \times \mathbb{R}^{n-1} \rightarrow \mathcal{X}_i$, such that

\begin{align}
X_{i,1} &= f_{i}^{(1)}(W_i), \\
X_{i,n} &= f_{i}^{(n)}(W_i, Y_{i,1}, \ldots, Y_{i,n-1}).
\end{align}

The components of the input vector $X_i$ are real numbers subject to an average power constraint:

\begin{equation}
\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[X_{i,n}^2] \leq 1,
\end{equation}

where the expectation is taken over the joint distribution of the message indexes $W_1, W_2$, and the noise terms, i.e., $\bar{Z}_1$, $\bar{Z}_2$, $\bar{Z}_1$, and $\bar{Z}_2$. The dependence of $X_{i,n}$ on $W_1, W_2$, and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).
Assume that during a given communication, $T$ blocks are transmitted. Hence, the decoder of receiver $i$ is defined by a deterministic function $\psi_i : \mathbb{R}^{NT}_i \rightarrow \mathbb{W}_i^T$. At the end of the communication, receiver $i$ uses the vector $(\hat{Y}_{i,1}, \hat{Y}_{i,2}, \ldots, \hat{Y}_{i,NT})$ to obtain an estimate of the message indices

$$(\hat{W}_i^{(1)}, \hat{W}_i^{(2)}, \ldots, \hat{W}_i^{(T)}) = \psi_i (\hat{Y}_{i,1}, \hat{Y}_{i,2}, \ldots, \hat{Y}_{i,NT}),$$  

where $\hat{W}_i^{(t)}$ is an estimate of the message index sent during block $t \in \{1, 2, \ldots, T\}$. The decoding error probability in the two-user G-IC-NOF during block $t$ of a codebook of block-length $N$, denoted by $P_e^t(N)$, is given by

$$P_e^t(N) = \max \left( \Pr [\hat{W}_1^{(t)} \neq W_1^{(t)}], \Pr [\hat{W}_2^{(t)} \neq W_2^{(t)}] \right).$$

The definition of an achievable rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is given below.

**Definition 1 (Achievable Rate Pairs):** A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $X_1^N$ and $X_2^N$ with codewords of length $N$, and the corresponding encoding functions $f_1^{(1)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, \ldots, f_2^{(N)}$ such that the decoding error probability $P_e^t(N)$ can be made arbitrarily small by letting the block-length $N$ grow to infinity, for all blocks $t \in \{1, 2, \ldots, T\}$.

The two-user G-IC-NOF in Figure 1 can be fully described by six parameters: $\SNR_{i}, \tilde{\SNR}_{i}$, and $\INR_{ij}$, with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, which are defined as follows:

$$\SNR_{i} = \frac{1}{\rho_i},$$

$$\INR_{ij} = h_{ij}^2,$$

$$\tilde{\SNR}_{i} = \frac{1}{\rho_i^2} \left( h_{ii}^2 + 2 \hat{h}_{ii} h_{ij} + h_{ij}^2 + 1 \right).$$

### III. MAIN RESULTS

This section introduces an achievable region (Theorem 1) and a converse region (Theorem 2), denoted by $\mathcal{C}_{G-IC-NOF}$ and $\mathcal{C}_{IC-NOF}$ respectively, for the two-user G-IC-NOF with fixed parameters $\SNR_1$, $\SNR_2$, $\INR_{12}$, $\INR_{21}$, $\SNR_1$, and $\tilde{\SNR}_2$. In general, the capacity region of a given multi-user channel is said to be approximated to within a constant gap according to the following definition.

**Definition 2 (Approximation to within $\epsilon$ units):** A closed and convex set $\mathcal{T} \subseteq \mathbb{R}^{m}$ is approximated to within $\epsilon$ units by the sets $\mathcal{I}$ and $\mathcal{T}$ if $\mathcal{I} \subseteq \mathcal{T} \subseteq \mathcal{I}$ and for all $t = (t_1, \ldots, t_m) \in \mathcal{T}$ then $(t_1 - \epsilon, \ldots, (t_m - \epsilon)^+) \in \mathcal{I}$.

Denote by $\mathcal{C}_{GIC-NOF}$ the capacity region of the 2-user G-IC-NOF. The achievable region $\mathcal{C}_{G-IC-NOF}$ and the converse region $\overline{\mathcal{C}}_{G-IC-NOF}$ approximate the capacity region $\mathcal{C}_{GIC-NOF}$ to within 4.4 bits per channel use (Theorem 3).

### A. An Achievable Region for the Two-User G-IC-NOF

The description of the achievable region $\mathcal{C}_{G-IC-NOF}$ is presented using the constants $a_{1,i}$; the functions $a_{2,1,i} : [0, 1] \rightarrow \mathbb{R}_+$, $a_{1,i} : [0, 1]^2 \rightarrow \mathbb{R}_+$, with $l \in \{3, \ldots, 6\}$; and $a_{2,1,i} : [0, 1]^3 \rightarrow \mathbb{R}_+$, which are defined as follows, for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$a_{1,i} = \frac{1}{2} \log \left( 2 + \frac{\SNR_{i}}{\INR_{ji}} \right) - \frac{1}{2},$$

$$a_{2,1,i} = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2},$$

$$a_{3,i} = \frac{1}{2} \log \left( \frac{\SNR_{i}}{\INR_{ji}}(1 - \mu) b_{2,i}(\rho) + 2b_{1,i}(1) + 1 \right),$$

$$a_{4,i} = \frac{1}{2} \log \left( (1 - \mu) b_{2,i}(\rho) + 2 \right) - \frac{1}{2},$$

$$a_{5,i} = \frac{1}{2} \log \left( 2 + \frac{\SNR_{i}}{\INR_{ji}} + (1 - \mu) b_{2,i}(\rho) \right) - \frac{1}{2},$$

$$a_{6,i} = \frac{1}{2} \log \left( \frac{\SNR_{i}}{\INR_{ji}}((1 - \mu)b_{2,i}(\rho)+1)^2 \right) - \frac{1}{2},$$

$$(10a)$$

$$(10b)$$

$$(10c)$$

$$(10d)$$

$$(10e)$$

$$(10f)$$

$$(10g)$$

where the functions $b_{1,i} : [0, 1] \rightarrow \mathbb{R}_+$, with $(l, i) \in \{1, 2\}^2$ are defined as follows:

$$b_{1,i}(\rho) = \SNR_{i} + 2\rho \sqrt{\SNR_{i} \INR_{ij} + \INR_{ij}},$$

$$b_{2,i}(\rho) = (1 - \rho) \INR_{ij} - 1,$$

with $j \in \{1, 2\} \setminus \{i\}$.

Note that the functions in (10) and (11) depend on $\SNR_{1}$, $\SNR_{2}$, $\INR_{12}$, $\INR_{21}$, $\tilde{\SNR}_{1}$, and $\tilde{\SNR}_{2}$; however as these parameters are fixed in this analysis, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 1 is presented on the next page.

### Proof: The proof of Theorem 1 is presented in [1].

### B. Comments on the Achievability

The achievable region is obtained using a random coding argument and combining three classical tools: rate splitting, superposition coding, and backward decoding. This coding scheme is described in [1] and it is especially designed for the two-user IC-NOF. Consequently, only the strictly needed number of superposition code-layers is used. Other achievable schemes, as reported in [2], can also be obtained as special cases of the more general scheme presented in [3]. However, in this more general case, the resulting code for the IC-NOF contains a handful of unnecessary superposing code-layers, which complicates the error probability analysis.

### C. A Converse Region for the Two-User G-IC-NOF

The description of the converse region $\overline{\mathcal{C}}_{G-IC-NOF}$ is determined by the ratios $\frac{\INR_{1j}}{\SNR_{1}}$, and $\frac{\INR_{2j}}{\SNR_{2}}$, for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$. All relevant scenarios regarding these ratios
Theorem 1: The capacity region $C_{\text{GCIC--NOF}}$ contains the region $C_{\text{G--IC--NOF}}$ given by the closure of the set of all possible non-negative achievable rate pairs $(R_1, R_2)$ that satisfy

$$R_1 \leq \min \left( a_{2,1}(\rho), a_{6,1}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1), a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1) \right),$$

$$R_2 \leq \min \left( a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{6,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{4,1}(\rho, \mu_2) + a_{1,2} \right),$$

$$R_1 + R_2 \leq \min \left( a_{2,1}(\rho) + a_{1,2} + a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{1,2} + a_{5,2}(\rho, \mu_1, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right),$$

$$2R_1 + R_2 \leq \min \left( a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{7,1}(\rho, \mu_1, \mu_2) + 2a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1) + a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1) \right),$$

$$R_1 + 2R_2 \leq \min \left( a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{2,2}(\rho) + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{2,2}(\rho) + a_{1,2}, 2a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} + a_{7,2}(\rho, \mu_1, \mu_2) \right),$$

with $(\rho, \mu_1, \mu_2) \in \left[ 0, 1 - \max \left( \frac{1}{\text{INR}_1}, \frac{1}{\text{INR}_2} \right) \right]^+ \times [0, 1] \times [0, 1].$

are described by two events denoted by $S_{1,i}$ and $S_{i,2}$, where $(l_1, l_2) \in \{1, \ldots, 5\}^2$. The events are defined as follows:

$$S_{1,i}: \quad \text{SNR}_{ij} < \min (\text{INR}_{ij}, \text{INR}_{ji}),$$

$$S_{2,i}: \quad \text{INR}_{ij} \leq \text{SNR}_{ij} < \text{INR}_{ji},$$

$$S_{3,i}: \quad \text{INR}_{ij} \leq \text{SNR}_{ij} < \text{INR}_{ji},$$

$$S_{4,i}: \quad \max (\text{INR}_{ij}, \text{INR}_{ji}) \leq \text{SNR}_{ij} < \text{INR}_{ij}\text{INR}_{ji},$$

$$S_{5,i}: \quad \text{SNR}_{ij} \geq \max (\text{INR}_{ij}, \text{INR}_{ji}, \text{INR}_{ij}\text{INR}_{ji}).$$

Note that for all $i \in \{1, 2\}$, the events $S_{1,i}, S_{2,i}, S_{3,i}, S_{4,i},$ and $S_{5,i}$ are mutually exclusive. This observation shows that given any 4-tuple $(\text{SNR}_1, \text{SNR}_2, \text{INR}_1, \text{INR}_2)$, there always exists one and only one pair of events $(S_{1,i}, S_{2,i})$, with $(l_1, l_2) \in \{1, \ldots, 5\}^2$, that identifies a unique scenario. Note also that the pairs of events $(S_{2,1}, S_{2,2})$ and $(S_{3,1}, S_{3,2})$ are not feasible. In view of this, twenty-three different scenarios can be identified using the events in (13). Once the exact scenario is identified, the converse region is described using the functions $\kappa_{ij}: [0, 1] \rightarrow \mathbb{R}_+$, with $(l, i) \in \{1, \ldots, 3\} \times \{1, 2\}$; $\kappa_l: [0, 1] \rightarrow \mathbb{R}_+$, with $l \in \{4, 5\}$; and $\kappa_{6,i}: [0, 1] \rightarrow \mathbb{R}_+$, with $(i, l) \in \{1, 2\}^2$. These functions are defined as follows for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$\kappa_{1,i}(\rho) = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right),$$

$$\kappa_{2,i}(\rho) = \frac{1}{2} \log \left( 1 + b_{5,j}(\rho) \right) + \frac{1}{2} \log \left( 1 + \frac{b_{4,i}(\rho)}{b_{5,j}(\rho)} \right),$$

$$\kappa_{3,i}(\rho) = \frac{1}{2} \log \left( \frac{\text{SNR}_j(b_{4,i}(\rho) + b_{5,j}(\rho) + 1)}{(b_{1,j}(1) + 1)(b_{4,i}(\rho) + 1) + 1} \right) + \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right),$$

$$\kappa_{4}(\rho) = \frac{1}{2} \log \left( 1 + \frac{b_{4,i}(\rho)}{1 + b_{5,j}(\rho)} \right) + \frac{1}{2} \log \left( b_{1,2}(\rho) + 1 \right).$$

By substituting these functions, we obtain

$$\kappa_{5}(\rho) = \frac{1}{2} \log \left( 1 + \frac{b_{4,2}(\rho)}{1 + b_{5,1}(\rho)} \right) + \frac{1}{2} \log \left( b_{1,1}(\rho) + 1 \right),$$

$$\kappa_{6,1}(\rho) = \begin{cases} \kappa_{6,1}(\rho) & \text{if } (S_{1,1} \lor S_{2,2} \lor S_{5,2}) \\ \land (S_{1,1} \lor S_{2,1} \lor S_{5,1}) & \end{cases},$$

$$\kappa_{6,2}(\rho) = \begin{cases} \kappa_{6,2}(\rho) & \text{if } (S_{2,2} \lor S_{5,2}) \\ \land (S_{3,1} \lor S_{4,1}) & \end{cases},$$

$$\kappa_{6,3}(\rho) = \begin{cases} \kappa_{6,3}(\rho) & \text{if } (S_{5,2} \lor S_{4,2}) \\ \land (S_{1,1} \lor S_{2,1} \lor S_{5,1}) & \end{cases},$$

$$\kappa_{6,4}(\rho) = \begin{cases} \kappa_{6,4}(\rho) & \text{if } (S_{3,2} \lor S_{4,2}) \land (S_{3,1} \lor S_{4,1}) \end{cases},$$

$$\kappa_{7,i,1}(\rho) = \begin{cases} \kappa_{7,i,1}(\rho) & \text{if } (S_{1,1} \lor S_{2,1} \lor S_{5,1}) \end{cases},$$

$$\kappa_{7,i,2}(\rho) = \begin{cases} \kappa_{7,i,2}(\rho) & \text{if } (S_{5,2} \lor S_{4,1}) \end{cases}.$$
\[
\kappa_{6,1}(\rho) = \frac{1}{2} \log \left( b_{6,1}(\rho) + b_{5,1}(\rho) \text{INR}_{21} \left( \frac{\text{SNR}_1 + b_{3,1}}{\text{SNR}_1} \right) \right) \\
- \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \text{SNR}_2}{b_{1,2}(1) + 1} \right) \\
+ \frac{1}{2} \log \left( \frac{b_{5,1}(\rho) \text{SNR}_1}{b_{1,1}(1) + 1} \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\text{SNR}_1} \right) + \log(2\pi e), 
\]  

\[
\kappa_{6,4}(\rho) = \frac{1}{2} \log \left( b_{6,1}(\rho) + b_{5,1}(\rho) \text{INR}_{21} \left( \frac{\text{SNR}_1 + b_{3,1}}{\text{SNR}_1} \right) \right) \\
- \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
+ \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \text{SNR}_2}{b_{1,2}(1) + 1} \right) - \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\text{SNR}_1} \right) \\
+ \frac{1}{2} \log \left( \frac{b_{5,1}(\rho) \text{SNR}_1}{b_{1,1}(1) + 1} \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,1}(\rho) \text{INR}_{21}}{\text{SNR}_1} \right) + \log(2\pi e),
\]

and

\[
\kappa_{7,1,2}(\rho) = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{ij} \right) \\
- \frac{1}{2} \log \left( 1 + b_{5,j}(\rho) \right) + \frac{1}{2} \log \left( 1 + b_{4,i}(\rho) + b_{5,j}(\rho) \right) \\
+ \frac{1}{2} \log \left( \frac{b_{3,j} \text{SNR}_j}{b_{1,j}(1) + 1} \right) - \frac{1}{2} \log \left( 1 + \frac{b_{5,i}(\rho) \text{INR}_{ji}}{\text{SNR}_j} \right) \\
+ \frac{1}{2} \log \left( b_{6,j}(\rho) + \frac{b_{5,i}(\rho) \text{INR}_{ji}}{\text{SNR}_j} \right) + \log(2\pi e), 
\]

where the functions \( b_{l,i} \), with \((l, i) \in \{1, 2\}^2 \) are defined in (11); \( b_{3,i} \) are constants; and the functions \( b_{l,i} : [0, 1] \to \mathbb{R}_+ \), with \((l, i) \in \{4, 5, 6\} \times \{1, 2\} \) are defined as follows, with \( j \in \{1, 2\} \setminus \{i\} \):

\[
b_{3,i} = \text{SNR}_i - 2\sqrt{\text{SNR}_i \text{INR}_{ji} + \text{INR}_{ji}},
\]

\[
b_{4,i}(\rho) = \left( 1 - \rho^2 \right) \text{SNR}_i,
\]

\[
b_{5,i}(\rho) = \left( 1 - \rho^2 \right) \text{INR}_{ij},
\]

\[
b_{6,i}(\rho) = \text{SNR}_i + \text{INR}_{ij} + 2\rho \sqrt{\text{INR}_{ij}} \left( \sqrt{\text{SNR}_i} - \sqrt{\text{INR}_{ij}} \right) + \frac{\text{INR}_{ij} \sqrt{\text{INR}_{ij}}}{\text{SNR}_i} \left( \sqrt{\text{INR}_{ij}} - 2\sqrt{\text{SNR}_i} \right).
\]

Note that the functions in (14), (15), (16) and (17) depend on \( \text{SNR}_1, \text{SNR}_2, \text{INR}_{12}, \text{INR}_{21}, \frac{\text{SNR}_1}{\text{SNR}_2} \), and \( \frac{\text{SNR}_1}{\text{SNR}_2} \). However, these parameters are fixed in this analysis, and therefore, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 2 is presented below.

**Theorem 2:** The capacity region \( C_{GIC-NOF} \) is contained within the region \( \tilde{C}_{GIC-NOF} \) given by the closure of the set of non-negative rate pairs \((R_1, R_2)\) that for all \( i \in \{1, 2\} \), with \( j \in \{1, 2\} \setminus \{i\} \) satisfy:

\[
R_i \leq \min \left( \kappa_{1,i}(\rho), \kappa_{2,i}(\rho) \right), \quad \text{(18a)}
\]

\[
R_i \leq \kappa_{3,i}(\rho), \quad \text{(18b)}
\]

\[
R_1 + R_2 \leq \min \left( \kappa_{4}(\rho), \kappa_{5}(\rho) \right), \quad \text{(18c)}
\]

\[
R_1 + R_2 \leq \kappa_{6}(\rho), \quad \text{(18d)}
\]

\[
2R_i + R_j \leq \kappa_{7,i}(\rho), \quad \text{(18e)}
\]
with \( \rho \in [0, 1] \).

**Proof**: The proof of Theorem 2 is presented in [1]. □

### D. Comments on the Converse Region

The outer bounds (18a) and (18c) correspond to the outer bounds for the case of perfect channel-output feedback [4]. The bounds (18b), (18d) and (18e) correspond to new outer bounds that generalize those presented in [2] for the two-user symmetric G-IC-NOF. These new outer-bounds were obtained using the genie-aided models shown in Figure 2.

### E. A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap between the achievable region \( \mathcal{C}_{G-IC-NOF} \) and the converse region \( \mathcal{C}_{G-IC-NOF} \) using the approximation notion described in Definition 2.

**Theorem 3**: The capacity region of the two-user G-IC-NOF is approximated to within 4.4 bits per channel use by the achievable region \( \mathcal{C}_{G-IC-NOF} \) and the converse region \( \mathcal{C}_{G-IC-NOF} \).

**Proof**: The proof of Theorem 3 is presented in [1]. □

The gap, denoted by \( \delta \), between the sets \( \mathcal{C}_{G-IC-NOF} \) and \( \mathcal{C}_{G-IC-NOF} \) can be approximated (Definition 2) as follows:

\[
\delta \leq \max \left( \delta_{R_1}, \delta_{R_2}, \frac{\delta_{R_3}}{3}, \frac{\delta_{R_4}}{3} \right),
\]

where

\[
\delta_{R_1} \overset{\Delta}{=} \min \left( \kappa_1(\rho), \kappa_2(\rho), \kappa_3(\rho) \right) - \min \left( a_{2,1}(\rho), a_{6,1}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1), a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1) \right),
\]

\[
\delta_{R_2} \overset{\Delta}{=} \min \left( \kappa_1(\rho), \kappa_2(\rho), \kappa_3(\rho) \right) - \min \left( a_{2,1}(\rho), a_{3,1}(\rho, \mu_2) + a_{6,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{4,1}(\rho, \mu_2) + a_{1,2} \right),
\]

\[
\delta_{R_3} \overset{\Delta}{=} \min \left( \kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho) \right) - \min \left( a_{2,1}(\rho) + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right),
\]

\[
\delta_{R_4} \overset{\Delta}{=} \min \left( \kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho) \right) - \min \left( a_{2,1}(\rho) + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right).
\]

Note that \( \delta_{R_1} \) and \( \delta_{R_2} \) represent the gap between the active achievable single-rate bound and the active converse single-rate bound; \( \delta_{R_2} \) represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and, \( \delta_{R_3} \) and \( \delta_{R_4} \) represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound.

Finally, it is important to highlight that, as suggested in [2], [4], and [5], the gap between \( \mathcal{C}_{G-IC-NOF} \) and \( \mathcal{C}_{G-IC-NOF} \) can be calculated more precisely. However, the choice in (19) eases the calculations at the expense of less precision.

Figure 3 presents the exact gap existing between the achievable region \( \mathcal{C}_{G-IC-NOF} \) and the converse region \( \mathcal{C}_{G-IC-NOF} \) for the case in which \( \text{SNR}_1 = \text{SNR}_2 = \text{SNR} \), \( \text{INR}_{12} = \text{INR}_{21} = \text{INR} \), and \( \text{SNR}_1 = \text{SNR}_2 = \text{SNR} \) as a function of \( \alpha = \frac{\log \text{SNR}}{\log \text{SNR}} \) and \( \beta = \frac{\log \text{SNR}}{\log \text{SNR}} \). Note that in this case, the maximum gap is 1.1 bits per channel use and occurs when \( \alpha = 1.05 \) and \( \beta = 1.2 \).

### IV. Conclusions

An achievable region and a converse region for the two-user G-IC-NOF have been introduced. It has been shown that these regions approximate the capacity region of the two-user G-IC-NOF to within 4.4 bits per channel use.

### REFERENCES


