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Approximate Capacity of the Gaussian Interference Channel with Noisy Channel-Output Feedback

Víctor Quintero, Samir M. Perlaza, Iñaki Esnaola and Jean-Marie Gorce

Abstract—In this paper, an achievability region and a converse region for the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) are presented. The achievability region is obtained using a random coding argument and three well-known techniques: rate splitting, superposition coding and backward decoding. The converse region is obtained using some of the existing perfect-output feedback outer-bounds as well as a set of new outer-bounds that are obtained by using genie-aided models of the original G-IC-NOF. Finally, it is shown that the achievable region and the converse region approximate the capacity region of the G-IC-NOF to within a constant gap in bits per channel use.

Index Terms—Capacity, Interference Channel, Noisy Channel-Output Feedback.

I. NOTATION

Throughout this paper, \( (\cdot)^+ \) denotes the positive part operator, i.e., \( (\cdot)^+ = \max(\cdot,0) \) and \( \mathbb{E}_X[\cdot] \) denotes the expectation with respect to the distribution of the random variable \( X \). The logarithm function \( \log \) is assumed to be base \( 2 \).

II. SYSTEM MODEL

Consider the two-user G-IC-NOF in Figure 1. Transmitter \( i \), with \( i \in \{1, 2\} \), communicates with receiver \( i \) subject to the interference produced by transmitter \( j \), with \( j \in \{1, 2\} \setminus \{i\} \). There are two independent and uniformly distributed messages, \( W_i \in W_i \), with \( W_i = \{1, 2, \ldots, 2^{N R_i}\} \), where \( N \) denotes the block-length in channel uses and \( R_i \) is the transmission rate in bits per channel use. At each block, transmitter \( i \) sends the codeword \( X_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,N})^T \in \mathcal{X}_1^N \), where \( \mathcal{X}_i \) and \( \mathcal{X}_j^N \) are respectively the channel-input alphabet and the codebook of transmitter \( i \).

The channel coefficient from transmitter \( j \) to receiver \( i \) is denoted by \( h_{ij} \); the channel coefficient from transmitter \( i \) to receiver \( i \) is denoted by \( h_{ii} \); and the channel coefficient from channel-output \( i \) to transmitter \( i \) is denoted by \( \bar{h}_{ii} \). All channel coefficients are assumed to be non-negative real numbers. At a given channel use \( n \in \{1, 2, \ldots, N\} \), the channel output at receiver \( i \) is denoted by \( Y_{i,n} \). During channel use \( n \), the input-output relation of the channel model is given by

\[
Y_{i,n} = h_{ii}X_{i,n} + h_{ij}X_{j,n} + \bar{Z}_{i,n}, \quad (1)
\]

where \( \bar{Z}_{i,n} \) is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver \( i \). Let \( d > 0 \) be the finite feedback delay measured in channel uses. At the end of channel use \( n \), transmitter \( i \) observes \( \hat{Y}_{i,n} \), which consists of a scaled and noisy version of \( Y_{i,n-d} \). More specifically,

\[
\hat{Y}_{i,n} = \left\{ \begin{array}{ll}
Z_{i,n} & \text{for } n \in \{1, 2, \ldots, d\} \\
\frac{1}{d} Y_{i,n-d} + \bar{Z}_{i,n}, & \text{for } n \in \{d+1, d+2, \ldots, N\},
\end{array} \right.
\]

where \( Z_{i,n} \) is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair \( i \). The random variables \( \bar{Z}_{i,n} \) and \( Z_{i,n} \) are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use, i.e., \( d = 1 \). The encoder of transmitter \( i \) is defined by a set of deterministic functions \( f_i^{(1)}, \ldots, f_i^{(N)} \), with \( f_i^{(1)} : W_i \rightarrow \mathcal{X}_i \) and for all \( n \in \{2, 3, \ldots, N\} \), \( f_i^{(n)} : W_i \times \mathbb{R}^{n-1} \rightarrow \mathcal{X}_i \), such that

\[
X_{i,1} = f_i^{(1)}(W_i), \quad (3a)
\]

\[
X_{i,n} = f_i^{(n)}(W_i, \hat{Y}_{i,1}, \ldots, \hat{Y}_{i,n-1}). \quad (3b)
\]

The components of the input vector \( X_i \) are real numbers subject to an average power constraint:

\[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(X_{i,n}^2) \leq 1. \quad (4)
\]

where the expectation is taken over the joint distribution of the message indexes \( W_1, W_2 \), and the noise terms, i.e., \( \bar{Z}_1, \bar{Z}_2, Z_1, \) and \( Z_2 \). The dependence of \( X_{i,n} \) on \( W_1, W_2 \), and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).

Victor Quintero, Samir M. Perlaza and Jean-Marie Gorce are with the CITI Laboratory, a joint lab between the Institut National de Recherche en Informatique et en Automatique (INRIA), Université de Lyon and Institut National de Sciences Appliquées (INSA) de Lyon, 6 Av. des Arts 69621 Villeurbanne, France. (victor.quintero-florez@inria.fr).

Iñaki Esnaola is with the University of Sheffield, Dep. of Automatic Control and Systems Engineering, Mappin Street, Sheffield, S1 3JD, United Kingdom.

Víctor Quintero is also with Universidad del Cauca, Popayán, Colombia.

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Assume that during a given communication, $T$ blocks are transmitted. Hence, the decoder of receiver $i$ is defined by a deterministic function $\psi_i : \mathbb{R}_i^{NT} \rightarrow \mathbb{W}_i^T$. At the end of the communication, receiver $i$ uses the vector $(\hat{W}_{i,1}, \hat{W}_{i,2}, \ldots, \hat{W}_{i,NT})$ to obtain an estimate of the message indices

$$(\hat{W}_{i}^{(1)}, \hat{W}_{i}^{(2)}, \ldots, \hat{W}_{i}^{(T)}) = \psi_i (\hat{Y}_{i,1}, \hat{Y}_{i,2}, \ldots, \hat{Y}_{i,NT}) ,$$

where $\hat{W}_{i}^{(t)}$ is an estimate of the message index sent during block $t \in \{1, 2, \ldots, T\}$. The decoding error probability in the two-user G-IC-NOF during block $t$ of a codebook of block-length $N$, denoted by $P_e^{(t)}(N)$, is given by

$$P_e^{(t)}(N) = \max \left( \Pr \left[ \hat{W}_{i}^{(t)} \neq W_{i}^{(t)} \right], \Pr \left[ \hat{W}_{2}^{(t)} \neq W_{2}^{(t)} \right] \right).$$

The definition of an achievable rate pair $(R_1, R_2) \in \mathbb{R}^2_+$ is given below.

**Definition 1 (Achievable Rate Pairs):** A rate pair $(R_1, R_2) \in \mathbb{R}^2_+$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of length $N$, and the corresponding encoding functions $f_1^{(n)}$, $f_2^{(n)}$, and $f_2^{(N)}$, such that the decoding error probability $P_e^{(t)}(N)$ can be made arbitrarily small by letting the block-length $N$ grow to infinity, for all blocks $t \in \{1, 2, \ldots, T\}$.

The two-user G-IC-NOF in Figure 1 can be fully described by six parameters: $\text{SNR}_{i}$, $\hat{\text{SNR}}_{i}$, and $\text{INR}_{ij}$, with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$, which are defined as follows:

$$\text{SNR}_{i} = \frac{h_i}{\hat{h}_i},$$

$$\text{INR}_{ij} = h_{ij}^2$$

and

$$\hat{\text{SNR}}_{i} = \hat{h}_i \left( \hat{h}_i + 2 \hat{h}_{ij} + 2 \hat{h}_{ij} + 1 \right).$$

### III. MAIN RESULTS

This section introduces an achievable region (Theorem 1) and a converse region (Theorem 2), denoted by $\mathcal{C}_{G-\text{IC-NOF}}$ and $\mathcal{C}_{G-\text{IC-NOF}}$ respectively, for the two-user G-IC-NOF with fixed parameters $\text{SNR}_1$, $\text{SNR}_2$, $\text{INR}_{12}$, $\text{INR}_{21}$, $\hat{\text{SNR}}_1$, and $\hat{\text{SNR}}_2$. In general, the capacity region of a given multi-user channel is said to be approximated to within a constant gap according to the following definition.

**Definition 2 (Approximation to within $\xi$ units):** A closed and convex set $\mathcal{T} \subset \mathbb{R}_m^+$ is approximated to within $\xi$ units by the sets $\mathcal{L}$ and $\mathcal{T}$ if $\mathcal{L} \subset \mathcal{T} \subset \mathcal{L}$ and for all $t = (t_1, \ldots, t_m) \in \mathcal{T}$ then $(t_1 - \xi, \ldots, (t_m - \xi)^+) \in \mathcal{L}$.

Denote by $\mathcal{C}_{G-\text{IC-NOF}}$ the capacity region of the 2-user G-IC-NOF. The achievable region $\mathcal{C}_{G-\text{IC-NOF}}$ and the converse region $\mathcal{C}_{G-\text{IC-NOF}}$ approximate the capacity region $\mathcal{C}_{G-\text{IC-NOF}}$ to within 4.4 bits per channel use (Theorem 3).

#### A. An Achievable Region for the Two-User G-IC-NOF

The description of the achievable region $\mathcal{C}_{G-\text{IC-NOF}}$ is presented using the constants $a_{1,i}$, the functions $a_{2,i} : [0, 1] \rightarrow \mathbb{R}$, $a_{2,i} : [0, 1] \rightarrow \mathbb{R}$, $a_{2,i} : [0, 1] \rightarrow \mathbb{R}$, with $l \in \{3, \ldots, 6\}$; and $a_{2,i} : [0, 1] \rightarrow \mathbb{R}$, which are defined as follows, for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

$$a_{1,i} = \frac{1}{2} \log \left( 2 + \frac{\text{SNR}_i}{\text{INR}_{ii}} \right) - \frac{1}{2},$$

$$a_{2,i} = \frac{1}{2} \log \left( b_{1,i}(\rho) + 1 \right) - \frac{1}{2},$$

$$a_{3,i} = \frac{1}{2} \log \left( \frac{\text{SNR}_i}{\text{SNR}_i(1-\mu) + b_{1,i}(1+1)} \right),$$

$$a_{4,i} = \frac{1}{2} \log \left( \left( 1 - \mu \right) b_{2,i}(\rho) + 2 \right) - \frac{1}{2},$$

$$a_{5,i} = \frac{1}{2} \log \left( 2 + \frac{\text{SNR}_i}{\text{INR}_{ii} + (1 - \mu) b_{2,i}(\rho)} - \frac{1}{2} \right),$$

$$a_{6,i} = \frac{1}{2} \log \left( \frac{\text{SNR}_i}{\text{INR}_{ij}} \left( (1 - \mu) b_{2,i}(\rho) + 2 \right) - \frac{1}{2} \right),$$

and

$$a_{7,i} = \frac{1}{2} \log \left( \frac{\text{SNR}_i}{\text{INR}_{ij}} \left( 1 - \mu \right) b_{2,i}(\rho) + 1 \right) + \left( 1 - \mu \right) b_{2,i}(\rho) + 2 - \frac{1}{2}.$$
Theorem 1: The capacity region $C_{\text{GIC–NOF}}$ contains the region $C_{\text{G–IC–NOF}}$ given by the closure of the set of all possible non-negative achievable rate pairs $(R_1, R_2)$ that satisfy

\begin{align}
R_1 & \leq \min \left\{ a_{2,1}(\rho), a_{6,1}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1), a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1) \right\}, \\
R_2 & \leq \min \left\{ a_{2,2}(\rho), a_{6,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{4,1}(\rho, \mu_2) + a_{1,2} \right\}, \\
R_1 + R_2 & \leq \min \left\{ a_{2,1}(\rho) + 1 + a_{2,1}(\rho) + a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), \\
& a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{1,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1), a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} \right\}, \\
2R_1 + R_2 & \leq \min \left\{ a_{2,1}(\rho) + 1 + a_{2,3}(\rho, \mu_1) + a_{7,2}(\rho, \mu_1, \mu_2), \\
& a_{3,1}(\rho, \mu_2) + a_{2,3}(\rho, \mu_1) + a_{7,1}(\rho, \mu_1, \mu_2) + 2a_{3,2}(\rho, \mu_1), a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho, \mu_1) + a_{5,2}(\rho, \mu_1) \right\}, \\
R_1 + 2R_2 & \leq \min \left\{ a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{2,2}(\rho) + a_{1,2}, a_{3,1}(\rho, \mu_2) + a_{7,1}(\rho, \mu_1, \mu_2) + a_{2,2}(\rho) + a_{1,2}, \\
& 2a_{3,1}(\rho, \mu_2) + a_{5,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{1,2} + a_{7,2}(\rho, \mu_1, \mu_2) \right\},
\end{align}

with $(\rho, \mu_1, \mu_2) \in [0, 1 - \max\left(\frac{1}{\text{INR}_1}, \frac{1}{\text{INR}_2}\right)]^+ \times [0, 1] \times [0, 1].$

are described by two events denoted by $S_{1,i}$ and $S_{2,i}$, where $(l_1, l_2) \in \{1, \ldots, 5\}^2$. The events are defined as follows:

\begin{align}
S_{1,i} : \quad & \text{SNR}_j < \min(\text{INR}_{ij}, \text{INR}_{ji}), \\
S_{2,i} : \quad & \text{INR}_{ij} \leq \text{SNR}_j < \text{INR}_{ij}, \\
S_{3,i} : \quad & \text{INR}_{ij} \leq \text{SNR}_j < \text{INR}_{ji}, \\
S_{4,i} : \quad & \max(\text{INR}_{ij}, \text{INR}_{ji}) \leq \text{SNR}_j < \text{INR}_{ij}\text{INR}_{ji}, \\
S_{5,i} : \quad & \text{SNR}_j \geq \max(\text{INR}_{ij}, \text{INR}_{ji}, \text{INR}_{ij}, \text{INR}_{ji}).
\end{align}

Note that for all $i \in \{1, 2\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ are mutually exclusive. This observation shows that given any 4-tuple ($\text{SNR}_1, \text{SNR}_2, \text{INR}_{12}, \text{INR}_{21}$), there always exists one and only one pair of events $(S_{1,i}, S_{2,i})$, with $(l_1, l_2) \in \{1, \ldots, 5\}^2$, that identifies a unique scenario. Note also that the pairs of events $(S_{2,1}, S_{2,2})$ and $(S_{3,1}, S_{3,2})$ are not feasible. In view of this, twenty-three different scenarios can be identified using the events in (13). Once the exact scenario is identified, the converse region is described using the functions $\kappa_{i,j} : [0, 1] \to \mathbb{R}_+$, with $(i,j) \in \{1, \ldots, 3\} \times \{1, 2\}$; $\kappa_{i} : [0, 1] \to \mathbb{R}_+$, with $i \in \{4, 5\}$; $\kappa_{6,1} : [0, 1] \to \mathbb{R}_+$, with $i \in \{1, \ldots, 4\}$; and $\kappa_{7,i,j} : [0, 1] \to \mathbb{R}_+$, with $(i,j) \in \{1, 2\}$.

These functions are defined as follows for all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$:

\begin{align}
\kappa_{1,i}(\rho) &= \frac{1}{2} \log \left( b_{1,1}(\rho) + 1 \right) + \frac{1}{2} \log \left( 1 + b_{4,2}(\rho) \right), \\
\kappa_{2,i}(\rho) &= \frac{1}{2} \log \left( 1 + b_{5,1}(\rho) \right) + \frac{1}{2} \log \left( 1 + b_{4,2}(\rho) \right), \\
\kappa_{3,i}(\rho) &= \frac{1}{2} \log \left( \frac{\text{SNR}_j (b_{4,1}(\rho) + b_{5,1}(\rho) + 1)}{(b_{1,1}(\rho) + 1)(b_{4,1}(\rho) + 1)} + 1 \right) + \frac{1}{2} \log \left( b_{1,1}(\rho) + 1 \right), \\
\kappa_{4}(\rho) &= \frac{1}{2} \log \left( 1 + \frac{b_{4,1}(\rho)}{1 + b_{5,2}(\rho)} \right) + \frac{1}{2} \log \left( b_{1,2}(\rho) + 1 \right).
\end{align}

\begin{align}
\kappa_{5}(\rho) &= \frac{1}{2} \log \left( 1 + \frac{b_{4,2}(\rho)}{1 + b_{5,1}(\rho)} \right) + \frac{1}{2} \log \left( b_{1,1}(\rho) + 1 \right), \\
\kappa_{6,1}(\rho) &= \text{if } (S_{1,1} \lor S_{2,2} \lor S_{5,2}) \\
& \lor (S_{1,1} \lor S_{2,1} \lor S_{5,1}), \\
\kappa_{6,2}(\rho) &= \text{if } (S_{1,2} \lor S_{2,2} \lor S_{5,2}) \\
& \lor (S_{3,1} \lor S_{4,1}), \\
\kappa_{6,3}(\rho) &= \text{if } (S_{3,2} \lor S_{4,2}) \\
& \lor (S_{3,1} \lor S_{4,1}), \\
\kappa_{6,4}(\rho) &= \text{if } (S_{3,2} \lor S_{4,2}) \land (S_{3,1} \lor S_{4,1}), \\
\kappa_{7,i,1}(\rho) &= \text{if } (S_{1,i} \lor S_{2,i} \lor S_{5,i}), \\
\kappa_{7,i,2}(\rho) &= \text{if } (S_{3,i} \lor S_{4,i}).
\end{align}

where

\begin{align}
\kappa_{6,1}(\rho) &= \frac{1}{2} \log \left( b_{1,1}(\rho) + b_{5,1}(\rho) \right) + \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
& + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \text{SNR}_2}{b_{1,1}(\rho) + 1} \right) + \log(2\pi e), \\
\kappa_{6,2}(\rho) &= \frac{1}{2} \log \left( b_{2,1}(\rho) + b_{5,1}(\rho) \right) + \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
& + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \text{SNR}_2}{b_{1,1}(\rho) + 1} \right) + \log(2\pi e), \\
\kappa_{6,3}(\rho) &= \frac{1}{2} \log \left( b_{2,2}(\rho) + b_{5,1}(\rho) \right) + \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) \\
& + \frac{1}{2} \log \left( 1 + \frac{b_{5,2}(\rho) \text{SNR}_2}{b_{1,1}(\rho) + 1} \right) + \log(2\pi e),
\end{align}
\[ \kappa_{6,i} (\rho) = \frac{1}{2} \log \left( b_{6,i} (\rho) + \frac{b_{6,1} (\rho) \text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{6,1} + b_{3,1}} \right) \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) - \frac{1}{2} \log \left( 1 + \frac{b_{5,2} (\rho)}{b_{1,2}(1) + 1} \right) + \frac{1}{2} \log \left( 1 + b_{5,1} (\rho) \frac{\text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{5,1} + b_{3,1}} \right) \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,1} (\rho) \text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{5,1} + b_{3,1}} \right) \right) + \log (2\pi e), \quad (15c) \]

\[ \kappa_{6,i} (\rho) = \frac{1}{2} \log \left( b_{6,i} (\rho) + \frac{b_{6,1} (\rho) \text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{6,1} + b_{3,1}} \right) \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{12} \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{21} \right) + \frac{1}{2} \log \left( 1 + b_{5,2} (\rho) \frac{\text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{5,2} + b_{3,2}} \right) \right) + \frac{1}{2} \log \left( 1 + b_{5,1} (\rho) \frac{\text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{5,1} + b_{3,1}} \right) \right) + \frac{1}{2} \log \left( 1 + \frac{b_{6,2} (\rho) + b_{5,1} (\rho) \text{INR}_{12}}{\text{SNR}_2} \left( \text{SNR}_2 + b_{3,2} \right) \right) + \frac{1}{2} \log \left( 1 + b_{5,1} (\rho) \frac{\text{INR}_{21}}{\text{SNR}_1} \left( \frac{\text{SNR}_1}{b_{5,1} + b_{3,1}} \right) \right) + \log (2\pi e), \quad (15d) \]

and

\[ \kappa_{7,i,2} (\rho) = \frac{1}{2} \log \left( b_{1,i} (\rho) + 1 \right) - \frac{1}{2} \log \left( 1 + \text{INR}_{ij} \right) + \frac{1}{2} \log \left( 1 + \frac{b_{5,j} (\rho)}{b_{1,1}(1) + 1} \right) + \frac{1}{2} \log \left( b_{1,j} (\rho) + b_{5,i} (\rho) \text{INR}_{ij} \right) + \frac{1}{2} \log \left( 1 + b_{1,i} (\rho) + b_{5,j} (\rho) \right) - \frac{1}{2} \log \left( 1 + b_{5,j} (\rho) \right) + 2 \log (2\pi e), \quad \text{with } j \in \{1, 2\} \setminus \{i\}, \quad (16b) \]

where the functions \( b_{l,i} \), with \((l, i) \in \{1, 2\} \times \{1, 2\}\) are defined in (11); \( b_{3,i} \) are constants; and the functions \( b_{l,i} : [0, 1] \to \mathbb{R}_+ \), with \((l, i) \in \{4, 5, 6\} \times \{1, 2\}\) are defined as follows, with \( j \in \{1, 2\} \setminus \{i\} \):

\[ b_{3,i} = \text{SNR}_i - 2\sqrt{\text{SNR}_i \text{INR}_{ij} + \text{INR}_{ji}}, \quad (17a) \]

\[ b_{4,i} (\rho) = \left( 1 - \rho^2 \right) \text{SNR}_i, \quad (17b) \]

\[ b_{5,i} (\rho) = \left( 1 - \rho^2 \right) \text{INR}_{ij}, \quad (17c) \]

\[ b_{6,i} (\rho) = \text{SNR}_i + \text{INR}_{ij} + 2\rho \sqrt{\text{INR}_{ij}} \left( \sqrt{\text{SNR}_i} - \sqrt{\text{INR}_{ji}} \right) + \text{INR}_{ij} \sqrt{\text{INR}_{ji}} \left( \sqrt{\text{SNR}_i} - 2\sqrt{\text{SNR}_i} \right). \quad (17d) \]

Note that the functions in (14), (15), (16) and (17) depend on \( \text{SNR}_1, \text{SNR}_2, \text{INR}_{12}, \text{INR}_{21}, \text{SNR}_1, \) and \( \text{SNR}_2 \). However, these parameters are fixed in this analysis, and therefore, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 2 is presented below.

**Theorem 2:** The capacity region \( C_{\text{GIC-NOF}} \) is contained within the region \( \bar{C}_{\text{GIC-NOF}} \) given by the closure of the set of non-negative rate pairs \((R_1, R_2)\) that for all \( i \in \{1, 2\}\), with \( j \in \{1, 2\} \setminus \{i\} \) satisfy:

\[ R_1 \leq \min \left( \kappa_{1,i} (\rho), \kappa_{2,i} (\rho) \right), \quad (18a) \]

\[ R_i \leq \kappa_{3,i} (\rho), \quad (18b) \]

\[ R_1 + R_2 \leq \min \left( \kappa_{4}(\rho), \kappa_{5}(\rho) \right), \quad (18c) \]

\[ R_1 + R_2 \leq \kappa_{6} (\rho), \quad (18d) \]

\[ 2R_i + R_j \leq \kappa_{7,i} (\rho), \quad (18e) \]
Fig. 3. Gap between the converse region $\mathcal{C}_{G-IC-NOF}$ and the achievable region $\mathcal{C}_{G-IC-NOF}$ of the two-user G-IC-NOF, under symmetric channel conditions, i.e., $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, $\text{INR}_{12} = \text{INR}_{21} = \text{INR}$, and $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, as a function of $\alpha = \frac{\log \text{INR}}{\log \text{SNR}}$ and $\beta = \frac{\log \text{SNR}}{\log \text{SNR}}$.

with $\rho \in [0, 1]$.

Proof: The proof of Theorem 2 is presented in [1].

D. Comments on the Converse Region

The outer bounds (18a) and (18c) correspond to the outer bounds for the case of perfect channel-output feedback [4]. The bounds (18b), (18d) and (18e) correspond to new outer bounds that generalize those presented in [2] for the two-user symmetric G-IC-NOF. These new outer-bounds were obtained using the genie-aided models shown in Figure 2.

E. A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap between the achievable region $\mathcal{C}_{G-IC-NOF}$ and the converse region $\mathcal{C}_{G-IC-NOF}$ using the approximation notion described in Definition 2.

Theorem 3: The capacity region of the two-user G-IC-NOF is approximated to within 4.4 bits per channel use by the achievable region $\mathcal{C}_{G-IC-NOF}$ and the converse region $\mathcal{C}_{G-IC-NOF}$.

Proof: The proof of Theorem 3 is presented in [1].

The gap, denoted by $\delta$, between the sets $\mathcal{C}_{G-IC-NOF}$ and $\mathcal{C}_{G-IC-NOF}$ can be approximated (Definition 2) as follows:

$$\delta \leq \max \left(\delta_{R_1}, \delta_{R_2}, \frac{\delta_{R_3}}{2}, \frac{\delta_{R_4}}{3}, \frac{\delta_{R_5}}{3}\right),$$

where

$$\delta_{R_1} \triangleq \min \left(\kappa_1(\rho), \kappa_2(\rho), \kappa_3(\rho)\right) - \min \left(\kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho)\right) - \min \left(a_{2,1}(\rho), a_{1,1} + a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{4,1}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1) + a_{4,1}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1)\right),$$

and

$$\delta_{R_2} \triangleq \min \left(\kappa_1(\rho), \kappa_2(\rho), \kappa_3(\rho)\right) - \min \left(\kappa_4(\rho), \kappa_5(\rho), \kappa_6(\rho)\right) - \min \left(a_{2,1}(\rho), a_{1,1} + a_{2,2}(\rho), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_2), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{4,1}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1), a_{3,1}(\rho, \mu_2) + a_{3,2}(\rho, \mu_1) + a_{3,2}(\rho, \mu_1) + a_{4,1}(\rho, \mu_1) + a_{4,2}(\rho, \mu_1)\right).$$

Note that $\delta_{R_1}$ and $\delta_{R_2}$ represent the gap between the active achievable single-rate bound and the active converse single-rate bound; $\delta_{R_3}$ represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and, $\delta_{R_4}$ and $\delta_{R_5}$ represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound.

Finally, it is important to highlight that, as suggested in [2], [4], and [5], the gap between $\mathcal{C}_{G-IC-NOF}$ and $\mathcal{C}_{G-IC-NOF}$ can be calculated more precisely. However, the choice in (19) eases the calculations at the expense of less precision. Figure 3 presents the exact gap existing between the achievable region $\mathcal{C}_{G-IC-NOF}$ and the converse region $\mathcal{C}_{G-IC-NOF}$ for the case in which $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$, $\text{INR}_{12} = \text{INR}_{21} = \text{INR}$, and $\text{SNR}_1 = \text{SNR}_2 = \text{SNR}$ as a function of $\alpha = \frac{\log \text{INR}}{\log \text{SNR}}$ and $\beta = \frac{\log \text{SNR}}{\log \text{SNR}}$. Note that in this case, the maximum gap is 1.1 bits per channel use and occurs when $\alpha = 1.05$ and $\beta = 1.2$.

IV. CONCLUSIONS

An achievable region and a converse region for the two-user G-IC-NOF have been introduced. It has been shown that these regions approximate the capacity region of the two-user G-IC-NOF to within 4.4 bits per channel use.

REFERENCES


