Plurisubharmonic functions with singularities and affine invariants for finite sets in n

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1 Introduction

Let $S \subset \mathbf{C}^n$ be a finite set. In this paper we consider the class of plurisubharmonic functions on $\mathbf{C}^n$ which have logarithmic poles in S and logarithmic growth at infinity. In particular we are interested in the subclass of such functions which are also maximal outside of S. The functions in this subclass are called pluricomplex Green functions on $\mathbf{C}^n$ with poles in S.

Entire plurisubharmonic functions with logarithmic poles in S, and in particular the ones which are maximal on $\mathbf{C}^n \setminus S$, have to satisfy certain growth conditions at infinity. We are dealing here with the problem of finding the minimal growth that they can have. In spite of its simple formulation, this problem is non-trivial and it is related to the algebraic geometric properties of S.

Using the growth of such plurisubharmonic functions we introduce and study two numbers $\gamma(S) \geq \widetilde{\gamma}(S)$ associated to S, which are invariant under affine automorphisms of $\mathbf{C}^n$. These numbers give information about the position of the points of S. More precisely, if $S \subset \mathbf{C}^2$ $\gamma(S)$ can detect when large subsets of S lie on curves of low degree, or it can give information about the minimal degree of the curves containing S.

We recall that for a bounded domain $D \subset \mathbf{C}^n$, the pluricomplex Green function of D with poles in the finite subset S of D, is defined by $g_D(z, S) = \sup u(z)$, where the supremum is taken over the class of negative plurisubharmonic functions u in D which have a logarithmic pole at each $p \in S$ (see [K], [L]). In [D1] and [L] it is shown that if D is hyperconvex then $g_D(\cdot, S)$ is the unique solution to the following Dirichlet problem for the complex Monge-Ampère operator: $u \in PSH(D) \bigcap C(\overline{D} \setminus S)$, $u(z) - \log \|z - p\| = O(1)$ as $z \to p \in S$, $(dd^c u)^n = \sum_{p \in S} \delta_p$, $u = 0$ on ∂D. Here, as well as in the sequel, $PSH(D)$ denotes the class of plurisubharmonic functions on D, $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2\pi i}(\partial - \overline{\partial})$, and δ_p is the Dirac mass at p.

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We now fix a finite subset $S=\{p_1,\dots,p_k\}\subset \mathbf{C}^n$ and denote by $E(S)$ the set of solutions to the following problem for the Monge-Ampère operator:

$$\begin{cases} u\in PSH(\mathbf{C}^n)\cap L^\infty_{loc}(\mathbf{C}^n\setminus S)\ , \\ u(z)-\log\|z-p_j\|=O(1) \text{ as } \ z\to p_j\ , \\ \exists\ \gamma_u=\limsup_{\|z\|\to\infty}(u(z)/\log\|z\|)\in(0,+\infty)\ , \\ (dd^c u)^n=\sum_{j=1}^k \delta_{p_j}\ . \end{cases} \tag{1.1}$$

We associate to S a number $\gamma(S)$ defined by

$$\gamma(S)=\inf\Gamma(S)\ ,\ \ \text{where } \Gamma(S)=\{\gamma_u:\ u\in E(S)\}\ . \tag{1.2}$$

It follows easily from this definition that if F is an affine automorphism of $\mathbf{C}^n$ then $\gamma(S)=\gamma(F(S))$, so $\gamma(S)$ is an affine invariant.

We also consider the class $\widetilde{E}(S)\supset E(S)$ which consists of the plurisubharmonic functions which satisfy only the first three requirements of (1.1). We set

$$\widetilde{\gamma}(S)=\inf\widetilde{\Gamma}(S)\ ,\ \ \text{where } \widetilde{\Gamma}(S)=\{\gamma_u:\ u\in \widetilde{E}(S)\}\ . \tag{1.3}$$

Then $\widetilde{\gamma}(S)\le\gamma(S)$ and $\widetilde{\gamma}(S)$ is also an affine invariant. Since $u\in\widetilde{E}(S)$, where $u(z)=\sum_{p\in S}\log\|z-p\|$, we have $\widetilde{\gamma}(S)\le|S|$.

We let

$$\omega(S)=\sup\left\{\frac{\sum_{p\in S} ord(P,p)}{\deg P}\ :\ P\in\mathbf{C}[z_1,\dots,z_n]\right\}\ , \tag{1.4}$$

where $ord(P,p)$ denotes the vanishing order of P at p. Then $\omega(S)$ is an affine invariant, related to the singular degree of S introduced by Waldschmidt [W] (see also [Ch] and Section 2). At the other extreme, we can consider algebraic curves containing subsets of S. We let

$$\begin{aligned} m_j(S)&=\max\{|S\cap C|:\ C \text{ algebraic curve, } \deg C=j\}\ , \\ m(S)&=\max\left\{\frac{m_j(S)}{j}\ :\ j\ge 1\right\}\ . \end{aligned} \tag{1.5}$$

Then $m(S)$ is an affine invariant. In dimension two we clearly have $m(S)\le\omega(S)$.

The paper is organized as follows. In Section 2 we introduce some notation and recall some definitions and facts from algebraic geometry. In Section 3 we use pluripotential theory to obtain some estimates for $\widetilde{\gamma}(S)$. We show that if V is a pure m-dimensional algebraic variety in $\mathbf{C}^n$ then

$$\sum_{p\in S}\nu(V,p)\le\widetilde{\gamma}(S)^m\deg V\ ,$$

where $\nu(V,p)$ is the Lelong number of V at p. This implies that

$$\omega(S)\le\widetilde{\gamma}(S)^{n-1} \text{ and } m(S)\le\widetilde{\gamma}(S)\ .$$

We also prove that

$$|S|^{1/n} \le \widetilde{\gamma}(S) \le \gamma(S) \le |S|\ ,$$

for every finite set $S \subset \mathbf{C}^n$, and that both the lower and the upper bound are sharp. An interesting question is whether $\gamma(S) = \widetilde{\gamma}(S)$. This is the case in all the examples analyzed throughout the paper. We conclude Section 3 with some results showing that standard upper-envelope methods do not work well in constructing elements of $E(S)$.

In Section 4 we consider the two dimensional case $n = 2$. From previous results we have $\sqrt{|S|} \le \widetilde{\gamma}(S) \le \gamma(S) \le |S|$ for every $S \subset \mathbf{C}^2$. We compute $\gamma(S)$ and $\widetilde{\gamma}(S)$ for special sets S. In particular we obtain the values of $\gamma(S)$ and $\widetilde{\gamma}(S)$ for all sets S with $|S| \le 6$. For such sets we have $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = m(S)$. We show that $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = 8/3$ for any set S with $|S| = 7$, $m_1(S) = 2$, $m_2(S) = 5$. We also prove that $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = 17/6$ for generic sets S with $|S| = 8$.

In Section 5 we consider the more special classes $M_n(S) \subset PSH(\mathbf{C}^n)$, which were studied in [Co]. They are obtained by replacing the growth condition at infinity from (1.1) with the stronger requirement that

$$\lim_{\|z\| \to \infty} (u(z)/ \log \|z\|) \in (0, +\infty)$$

exists. Then its value has to be $|S|^{1/n}$. We have $M_2(S) = \emptyset$ if $|S| \in \{2, 3, 5, 6\}$. Moreover a complete description of $M_2(S) \ne \emptyset$ was given when $|S| = d^2$ for some integer $d > 0$ and S is the complete intersection of two algebraic curves of degree d (see [Co]). Here we obtain some results regarding the converse of this. We show that if $u \in M_2(S)$ has the property that its restriction to some one dimensional subvariety of $\mathbf{C}^2$ is harmonic away from S, then $|S| = d^2$ for some positive integer d. Moreover, if $|S| = d^2$, $M_2(S) \ne \emptyset$, and $H(S, 2d-3) < d^2$ (see Section 2 for the definition of the Hilbert function H), we prove that S is the complete intersection of curves of degree d. We also show that $M_2(S) = \emptyset$ for $|S| = 7$ or $|S| = 8$.

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2 Preliminaries

Throughout the paper S will denote a finite subset of $\mathbf{C}^n$ and $|S|$ is the cardinal of S. For $z \in \mathbf{C}^n$ we write $z = (z_1, \dots, z_n)$, and we use the standard embedding $\mathbf{C}^n \to \mathbf{P}^n$, $z \to [z : 1]$, where $[z : t] = [z_1 : \cdots : z_n : t]$, $(z, t) \in \mathbf{C}^{n+1} \setminus \{0\}$, are the homogeneous coordinates on $\mathbf{P}^n$. Any (holomorphic) polynomial P of degree d on $\mathbf{C}^n$ gives rise to a homogeneous polynomial $\widetilde{P}$ on $\mathbf{C}^{n+1}$, $\widetilde{P}(z, t) = t^d P\left(\frac{z}{t}\right)$. If P_1, P_2 are polynomials on $\mathbf{C}^2$ we shall denote by $(P_1 \cdot P_2)_z$ the intersection number of the algebraic curves

$V_j = \{P_j = 0\}$, $j = 1, 2$, at $z \in V_1 \cap V_2$. Moreover, if f is a holomorphic function defined in a neighborhood of $0 \in \mathbf{C}^n$ we denote by $ord(f, 0)$ the order of f at 0. Then $f = h_l(f) + h.o.t.$ near 0, where $l = ord(f, 0)$, $h_l(f)$ denotes the homogeneous part of degree l of the Taylor expansion of f at 0, and $h.o.t.$ means "higher order terms".

We discuss now the connection between the number $\omega(S)$ defined in (1.4) and other affine invariants of the set S. If $l > 0$ is an integer we define

$$\Omega(S, l) = \min\{\deg P : \ P \in \mathbf{C}[z_1, \ldots, z_n], \ ord(P, p) \geq l, \ \forall p \in S\} \ .$$

The number $\Omega(S) = \Omega(S, 1)$ is sometimes called the degree of S (see [Ch]). We clearly have $\Omega(S, l_1 + l_2) \leq \Omega(S, l_1) + \Omega(S, l_2)$, and in particular $\Omega(S, l) \leq \Omega(S)l$. The limit $\Omega_0(S) = \lim_{l \to +\infty} \frac{\Omega(S,l)}{l}$ exists and is called the singular degree of S. We have $\Omega(S, l) \geq \Omega_0(S)l$ for all $l \geq 1$, $\Omega_0(S) \leq |S|^{1/n}$ and $\Omega(S) \geq \Omega_0(S) \geq \Omega(S)/n$. The number $\Omega_0(S)$ was introduced by Waldschmidt[W]. We refer to [Ch] for further properties of the singular degree and also for the definition of the very singular degree $\widehat{\Omega}_0(S)$ of S. We have in fact $\widehat{\Omega}_0(S) = |S|/\omega(S) \leq \Omega_0(S)$ for every set S.

A conjecture of Nagata [N] states that if $k > 9$ then $\Omega(S, l) > l\sqrt{k}$, $\forall l \geq 1$, holds for the generic set $S \subset \mathbf{C}^2$ with $|S| = k$. Nagata proved his conjecture when k is a square [N]. Moreover this statement does not hold for $k \leq 9$ (see e.g. [Ch]). A more general version of the above conjecture is that given $k > 9$ then for the generic set $S \subset \mathbf{C}^2$ with $|S| = k$ one has $\sum_{p \in S} ord(P, p) < \sqrt{k} \deg P$, for every $P \in \mathbf{C}[z_1, z_2]$. By a result in this paper we have $\omega(S) \leq \widetilde{\gamma}(S)$ for every set $S \in \mathbf{C}^2$. Hence proving that $\widetilde{\gamma}(S) = \sqrt{k}$ for the generic set $S \in \mathbf{C}^2$ with $|S| = k$ ($k > 9$ not a square) would imply the latter conjecture.

Let now $S = \{p_1, \ldots, p_k\} \subset \mathbf{C}^2$, $k = |S|$, let $I(S)$ be the ideal of polynomials in z_1, z_2 vanishing on S, and let $I_m(S) = I(S) \cap \mathcal{P}_m$, $m \in \mathbf{N}$. Here $\mathcal{P}_m$ denotes the vector space of polynomials in $\mathbf{C}[z_1, z_2]$ of degree at most m and $\dim \mathcal{P}_m = (m + 1)(m + 2)/2$. We write $e_m(S) = \dim I_m(S)$ and note that $I_m(S) = \ker E_m$, where $E_m : \mathcal{P}_m \to \mathbf{C}^k$, $E_m(P) = (P(p_1), \ldots, P(p_k))$. The Hilbert function $H(S, \cdot)$ of S is defined by $H(S, m) = \dim E_m(\mathcal{P}_m)$; so $H(S, m) + e_m(S) = (m + 1)(m + 2)/2$. We have that $H(S, m)$ increases with m, $H(S, m) \leq k$ and $H(S, m) = k$ if $m \geq k - 1$. The following theorem is a particular case of a result of [EP]:

Theorem 2.1 (EP) *Let $S \subset \mathbf{P}^2$ with $|S| = d^2$ for some $d \in \mathbf{N}$. Assume that $H(S, 2d - 3) < d^2$ and that S is not the complete intersection of two curves of degree d. Then there exists an algebraic curve $V \subset \mathbf{P}^2$ of degree m such that $0 < m < d$ and $|V \cap S| \geq m(2d - m)$.*

3 Estimates for $\widetilde{\gamma}(S)$

For $a > 1$ let $f_a : \mathbf{R} \to \mathbf{R}$ be the convex increasing function defined by $f_a(t) = t$ if $t < 0$, $f_a(t) = at$ if $t \geq 0$. If $u \in \widetilde{E}(S)$ then $f_a \circ u \in \widetilde{E}(S)$ and $\gamma_{f_a \circ u} = a\gamma_u$. This shows that $(\widetilde{\gamma}(S), +\infty) \subseteq \widetilde{\Gamma}(S) \subseteq [\widetilde{\gamma}(S), +\infty)$.

Proposition 3.1 *If $S' \subseteq S \subset \mathbf{C}^n$ then $\widetilde{\gamma}(S') \leq \widetilde{\gamma}(S) \leq \widetilde{\gamma}(S') + |S \setminus S'|$.*

Proof. Let $\epsilon > 0$ and $u \in \widetilde{E}(S)$ such that $\gamma_u < \widetilde{\gamma}(S) + \epsilon$. We fix $r > 0$ such that the balls $\overline{B}(p, r_0)$, $p \in S$, are pairwise disjoint. For $r \in (0, r_0)$ let $K_r = \bigcup\{B(p,r) : p \in S \setminus S'\}$, and fix r such that $u < 0$ on $\overline{K}_r$. Let $M = \inf\{u(z) : \ z \in \overline{K}_r \setminus K_{r/2}\} \in (-\infty, 0)$ and consider the function u' defined by $u'(z) = u(z)$ for $z \in \mathbf{C}^n \setminus K_r$, $u'(z) = \max\{u(z), 2M\}$ for $z \in K_r$. Then $u' \in \widetilde{E}(S')$ and $\gamma_{u'} = \gamma_u$. It follows that $\widetilde{\gamma}(S') \leq \widetilde{\gamma}(S)$.

Let now $u' \in \widetilde{E}(S')$ and define $u(z) = u'(z) + \sum_{p \in S \setminus S'} \log \|z - p\|$. Then $u \in \widetilde{E}(S)$ and $\gamma_u = \gamma_{u'} + |S \setminus S'|$, which proves the second inequality. $\Box$

We will need the following result which generalizes a lemma in [T]:

Proposition 3.2 *Let T be a closed positive current of bidimension (l,l), $l > 0$, on $\mathbf{C}^n$, let $u, v \in PSH(\mathbf{C}^n)$ be locally bounded outside a compact set such that $\lim_{\|z\| \to \infty} v(z) = +\infty$ and $\limsup_{\|z\| \to \infty}(u(z)/v(z)) = \alpha \in (0, +\infty)$. Then*

$$\int_{\mathbf{C}^n} (dd^c u)^l \wedge T \leq \alpha^l \int_{\mathbf{C}^n} (dd^c v)^l \wedge T \ .$$

Equality holds in the above if $\lim_{\|z\| \to \infty}(u(z)/v(z)) = \alpha$.

Proof. As $v > 0$ outside a compact set we have $\int_{\mathbf{C}^n} (dd^c v)^l \wedge T = \int_{\mathbf{C}^n} (dd^c v_+)^l \wedge T$, so replacing v by v_+ we may assume that $v \geq 0$ on $\mathbf{C}^n$. Moreover, replacing v by αv, we may also assume $\alpha = 1$.

For $\epsilon > 0$, $R > 0$, $M > 0$ fixed we let $u_M = \max\{u, -M\}$, $w_m = \max\{(1+\epsilon)v - m, u_M\}$, where m is sufficiently large so that $w_m = u_M$ on the ball $B(0, 2R)$. There exists $R' > 2R$ so that $w_m = (1+\epsilon)v - m$ for $\|z\| > R'$. We let $\phi \in C_0^\infty(\mathbf{C}^n)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $B(0, 2R')$. Then

$$\int_{B(0,2R)} (dd^c u_M)^l \wedge T = \int_{B(0,2R)} (dd^c w_m)^l \wedge T \leq$$
$$\leq \int_{\mathbf{C}^n} \phi (dd^c w_m)^l \wedge T = \int_{\mathbf{C}^n} w_m dd^c \phi \wedge (dd^c w_m)^{l-1} \wedge T \ .$$

As the support of $dd^c \phi$ lies in the open set $\|z\| > R'$ where $w_m = (1+\epsilon)v - m$, the last integral is equal to $(1+\epsilon)^l \int_{\mathbf{C}^n} \phi (dd^c v)^l \wedge T$. We conclude that

$$\int_{B(0,2R)} (dd^c u_M)^l \wedge T \leq (1+\epsilon)^l \int_{\mathbf{C}^n} (dd^c v)^l \wedge T \ .$$

By the results of [BT] and [D2] on the continuity of the Monge-Ampère operator with respect to decreasing sequences, the measures $(dd^c u_M)^l \wedge T$ converge weakly to $(dd^c u)^l \wedge T$ as $M \nearrow \infty$. Using a cut-off function $\phi \in C_0^\infty(B(0,2R))$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $B(0,R)$ it follows from the previous estimate that

$$\int_{B(0,R)} (dd^c u)^l \wedge T \leq (1+\epsilon)^l \int_{\mathbf{C}^n} (dd^c v)^l \wedge T \, .$$

The proof is finished by letting $R \nearrow +\infty$, $\epsilon \searrow 0$.

In the case when $\lim_{\|z\|\to\infty}(u(z)/v(z)) = \alpha$ the conclusion follows by interchanging the roles of u and v in the above argument. $\square$

For $u \in PSH(\mathbf{C}^n)$ we let

$$\delta_u = \liminf_{\|z\|\to\infty} \frac{u(z)}{\log \|z\|} \quad , \quad \gamma_u = \limsup_{\|z\|\to\infty} \frac{u(z)}{\log \|z\|} \, .$$

Corollary 3.3 *We have* $\delta_u \leq \left[\int_{\mathbf{C}^n} (dd^c u)^n\right]^{1/n} \leq \gamma_u$. *In particular* $\delta_u \leq |S|^{1/n} \leq \gamma_u$ *for* $u \in E(S)$.

Proof. Proposition 3.2 applied to $u(z)$, $\log \|z\|$, and $T = 1$, yields the estimate for γ_u. For the estimate on δ_u we may assume $\delta_u > 0$. Then $\lim_{\|z\|\to\infty} u(z) = +\infty$, $\limsup_{\|z\|\to\infty}(\log \|z\|/u(z)) = 1/\delta_u$, so the estimate follows from Proposition 3.2. $\square$

For an algebraic variety $V \subset \mathbf{C}^n$ we denote by $\nu(V,p)$ the Lelong number of V at p and by $\deg V$ its degree (provided V is pure dimensional). We recall that $\deg V = \max |V \cap H|$, where the maximum is taken over all planes $H \subset \mathbf{C}^n$ such that $\dim V + \dim H = n$ and $V \cap H$ is discrete. Our next result relates $\widetilde{\gamma}(S)$ to the singularities that algebraic varieties have at points of S:

Theorem 3.4 *Let* $S \subset \mathbf{C}^n$ *be a finite set and let* $V \subset \mathbf{C}^n$ *be an algebraic variety of pure dimension* m. *Then*

$$|S \cap V| \leq \sum_{p \in S \cap V} \nu(V,p) \leq \widetilde{\gamma}(S)^m \deg V \, .$$

Proof. Let $u \in \widetilde{E}(S)$. Since u has logarithmic poles in S it follows from a comparison theorem for Lelong numbers with weights due to Demailly [D2] that

$$\lim_{r \searrow 0} \int_{B(p,r)} (dd^c u)^m \wedge [V] = \nu_u(V,p) = \nu(V,p)$$

for $p \in S \cap V$. Here $[V]$ denotes the current of integration on V and $\nu_u(V,p)$ is the Lelong number of V at p with weight u (see [D2]). It follows that

$$\int_{\mathbf{C}^n} (dd^c u)^m \wedge [V] \geq \sum_{p \in S \cap V} \nu(V,p) \geq |S \cap V| \, ,$$

since $\nu(V,p) \geq 1$. Proposition 3.2 now implies

$$\int_{\mathbf{C}^n} (dd^c u)^m \wedge [V] \leq \gamma_u^m \int_{\mathbf{C}^n} (dd^c \log \|z\|)^m \wedge [V] \, .$$

By classical results on Lelong numbers (see e.g. [LG], Corollary 5.21) we have

$$\int_{\mathbf{C}^n} (dd^c \log \|z\|)^m \wedge [V] = \lim_{R\to\infty} \int_{B(0,R)} (dd^c \log \|z\|)^m \wedge [V] =$$
$$= \lim_{R\to\infty} \frac{\sigma_V(B(0,R))}{\tau_{2m} R^{2m}} = \deg V \, ,$$

where $\tau_{2m} = \pi^m/m!$ is the volume of the unit ball of $\mathbf{C}^m$ and σ_V is the trace measure of $[V]$. We conclude that

$$\sum_{p\in S\cap V} \nu(V,p) \leq \gamma_u^m \deg V \, ,$$

and the theorem follows. □

Corollary 3.5 *$\omega(S) \leq \widetilde{\gamma}(S)^{n-1}$, $m(S) \leq \widetilde{\gamma}(S)$.*

Proof. Let $P \in \mathbf{C}[z_1,\dots,z_n]$ and let $V = \{P = 0\}$. The first inequality follows since $\nu(V,p) = ord(P,p)$. The second one follows directly from Theorem 3.4. □

We remark that all of the above estimates clearly hold with $\gamma(S)$ instead of $\widetilde{\gamma}(S)$, since $\widetilde{\gamma}(S) \leq \gamma(S)$.

Theorem 3.6 *For every $S \subset \mathbf{C}^n$ we have $|S|^{1/n} \leq \widetilde{\gamma}(S) \leq \gamma(S) \leq |S|$. Moreover, $\widetilde{\gamma}(S) = |S|$ if and only if $m_1(S) = |S|$ (i.e. S is contained in a complex line).*

Remark. The lower bound $|S|^{1/n}$ is also sharp: Indeed, assume that $|S| = d^n$ and $S = P^{-1}(0)$, where $P : \mathbf{C}^n \to \mathbf{C}^n$ is a polynomial mapping such that 0 is a regular value and

$$0 < \liminf_{\|z\|\to\infty} \frac{\|P(z)\|}{\|z\|^d} \leq \limsup_{\|z\|\to\infty} \frac{\|P(z)\|}{\|z\|^d} < +\infty \, .$$

Then $v(z) = \log \|P(z)\|$ is in $E(S)$, so $\gamma(S) = d = |S|^{1/n}$ (see [Co]).

In order to prove Theorem 3.6 we need the following interpolation theorem:

Theorem 3.7 *Let $p_1,\dots,p_k$ ($k \geq 2$) be distinct points in $\mathbf{C}^n$ and let $\nu_1,\dots,\nu_k$ be k positive integers. There exist n polynomials $P_1,\dots,P_n \in \mathbf{C}[z_1,\dots,z_n]$ such that :*

(i) The common zeros (in $\mathbf{C}^n$) of $P_1,\dots,P_n$ are exactly the points $p_1,\dots,p_k$.

(ii) If $P = (P_1,\dots,P_n) : \mathbf{C}^n \to \mathbf{C}^n$ then

$$\log \|P(z)\| = \nu_j \log \|z - p_j\| + O(1) \, , \quad as\ z \to p_j \, ,$$

for $j = 1,\dots,k$.

(iii) $\deg P_m \leq k(k-1) + \sum_{j=1}^k \nu_j$, for $m = 1,\dots,n$.

Proof. The proof is an explicit construction. Let us denote $p_j = (p_{j1}, \dots, p_{jn})$, for $j = 1, \dots, k$. We may assume, using a linear change of coordinates, that for every $m = 1, \dots, n$ the numbers $p_{1m}, \dots, p_{km}$ are distinct.

The theorem is evident in one complex variable: It suffices to consider the polynomial $P(z) = \prod_{j=1}^{k} (z - p_j)^{\nu_j}$. Suppose next that $n \geq 2$. For $m = 2, \dots, n$ and for $j = 1, \dots, k$, we consider the following polynomial $Q_j^m(z_1, \xi)$ of two complex variables:

$$Q_j^m(z_1, \xi) = \frac{\prod_{l=1, l \neq j}^{k} (z_1 - p_{l1})}{\prod_{l=1, l \neq j}^{k} (p_{j1} - p_{l1})} (\xi - p_{jm})^{\nu_j} + (z_1 - p_{j1})^{\nu_j + 1} .$$

This polynomial has degree $\nu_j + k - 1$ and verifies the following properties:

$$Q_j^m(p_{l1}, \xi) = (p_{l1} - p_{j1})^{\nu_j + 1} \neq 0, \text{ for every } \xi \in \mathbf{C} \text{ and } l \neq j \tag{3.1}$$

$$Q_j^m(p_{j1}, \xi) = (\xi - p_{jm})^{\nu_j} \tag{3.2}$$

$$Q_j^m(z_1, \xi) = (\xi - p_{jm})^{\nu_j} + (\xi - p_{jm})^{\nu_j} (z_1 - p_{j1}) R_j(z_1) + (z_1 - p_{j1})^{\nu_j + 1} \tag{3.3}$$

Here $R_j(z_1)$ is a polynomial of one complex variable.

We construct now the desired polynomials:

$$P_1(z) = P_1(z_1) = \prod_{j=1}^{k} (z_1 - p_{j1})^{\nu_j} ,$$

$$P_m(z) = P_m(z_1, z_m) = \prod_{j=1}^{k} Q_j^m(z_1, z_m), \text{ for } m = 2, \dots, n .$$

Then $\deg P_m = \sum_{j=1}^{k} \nu_j + k(k-1)$, so assertion (iii) of the theorem is satisfied. Next assume that $z \in \mathbf{C}^n$ is such that $P_1(z) = \cdots = P_n(z) = 0$. Then $P_1(z) = 0$ implies that z_1 equals one of the numbers $p_{11}, \dots, p_{k1}$. Suppose, without loss of generality, that $z_1 = p_{11}$. Using (3.1) and (3.2) the equation $P_m(p_{11}, z_m) = 0$ implies $z_m = p_{1m}$, for $m = 2, \dots, n$, so $z = p_1$. It follows that the common zeros of $P_1, \dots, P_n$ are exactly the points $p_1, \dots, p_k$.

We fix $j \in \{1, \dots, k\}$ and estimate the map P near the point p_j. There exist two positive constants M_{j1} and M'_{j1}, such that

$$M'_{j1} |z_1 - p_{j1}|^{\nu_j} \leq |P_1(z)| \leq M_{j1} |z_1 - p_{j1}|^{\nu_j}.$$

Using (3.1) and (3.3), there exist positive constants $M_{j2}, \dots, M_{jn}$, $M'_{j2}, \dots, M'_{jn}$, $C_{j2}, \dots, C_{jn}$, such that for any $m = 2, \dots, n$, we have

$$M'_{jm} |z_m - p_{jm}|^{2\nu_j} - C_{jm} \|z - p_j\|^{2\nu_j + 1} \leq |P_m(z)|^2 \leq M_{jm} |z_m - p_{jm}|^{2\nu_j} + C_{jm} \|z - p_j\|^{2\nu_j + 1}.$$

Finally, there exist two positive constants M_j and M'_j such that

$$M'_j \|z - p_j\|^{2\nu_j} \leq \|P(z)\|^2 = \sum_{m=1}^{n} |P_m(z)|^2 \leq M_j \|z - p_j\|^{2\nu_j}$$

for z sufficiently close to p_j. Hence the map P verifies the second assertion of the theorem. $\square$

Proof of Theorem 3.6. Let $u \in \widetilde{E}(S)$. Since u has logarithmic poles in S we have by a result of [D1] that $\int_{\mathbf{C}^n}(dd^c u)^n \geq |S|$, hence $\widetilde{\gamma}(S) \geq |S|^{1/n}$ by Corollary 3.3. Let N be a positive integer and let $P_N = (P_1^N, \dots, P_n^N)$ be a polynomial mapping satisfying the conclusions of Theorem 3.7 for the set $S = \{p_1, \dots, p_k\}$ and with $\nu_1 = \cdots = \nu_k = N$. Then $v = \frac{1}{N}\log\|P_N\| \in E(S)$, since

$$N^n(dd^c v)^n = (P_N)^\star(dd^c \log\|\cdot\|)^n = 0$$

on $\mathbf{C}^n \setminus S$. Moreover $\deg P_j^N \leq kN + k(k-1)$, so $\gamma_v \leq k + k(k-1)/N$. Letting $N \to \infty$ we obtain $\gamma(S) \leq k = |S|$.

We assume now that $m_1(S) = |S|$. Since $m_1(S) \leq \widetilde{\gamma}(S)$ we obtain $\widetilde{\gamma}(S) = |S|$. The following lemma can be proved in a similar way as the corresponding two dimensional results (see Section 4):

Lemma 3.8 *If* $|S| \leq 3$ *then* $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$.

To complete the proof of Theorem 3.6 we assume that $\widetilde{\gamma}(S) = |S|$. If $|S| \leq 3$ then $m_1(S) = |S|$ by Lemma 3.8. If $|S| > 3$ and $m_1(S) < |S|$, we can find $S' \subset S$ with $|S'| = 3$ and $m_1(S') = 2$. By Proposition 3.1 we have $\widetilde{\gamma}(S) \leq \widetilde{\gamma}(S') + |S \setminus S'| = |S| - 1$, a contradiction. $\square$

Remark. Assume that S is contained in a complex line, say without loss of generality the z_1-axis. It is easy to see that there exists $u \in E(S)$ with $\gamma_u = \gamma(S) = |S|$. Indeed, let $a_1, \dots, a_k$ be the z_1-coordinates of the points of S. If $f(z) = (\prod_{j=1}^k (z_1 - a_j), z_2, \dots, z_n)$ and $u = \log\|f\|$, then $u \in E(S)$ and $\gamma_u = |S|$.

We conclude this section by addressing the following question. Given $u \in \widetilde{E}(S)$, is it possible to construct by upper-envelope methods a function $v \in E(S)$ such that $\gamma_v \leq \gamma_u$ (or at least $\gamma_v \leq \gamma_u + \epsilon$, $\epsilon > 0$)? This would be useful in order to prove that $\widetilde{\gamma}(S) = \gamma(S)$. Let $\mathcal{L} \subset PSH(\mathbf{C}^n)$ denote the class of plurisubharmonic functions of minimal growth (i.e. $\gamma_\rho \leq 1$ for $\rho \in \mathcal{L}$). One has to associate to u a non-empty family $\mathcal{F}_u \subset \gamma_u\mathcal{L}$ such that every $\rho \in \mathcal{F}_u$ has logarithmic poles in S. The natural way to insure $\mathcal{F}_u \neq \emptyset$ would be to impose $u \in \mathcal{F}_u$. Let $v(z) = \sup\{\rho(z) : \ \rho \in \mathcal{F}_u\}$. If $\{v < +\infty\}$ is not pluripolar, then it is a standard result that the upper-semicontinuous regularization $v^\star \in \gamma_u\mathcal{L}$. As $u \leq v^\star$ we have $\gamma_{v^\star} = \gamma_u$. Moreover $v^\star \in \widetilde{E}(S)$. Indeed, for $p \in S$ fix a ball $B(p,r)$ and M such that $v^\star \leq M$ on $B(p,r)$. Then every $\rho \in \mathcal{F}_u$ satisfies $\rho(z) \leq \log\frac{\|z-p\|}{r} + M$ on $B(p,r)$, hence the same holds for $v^\star$. Since $u \leq v^\star$ we get $v^\star(z) = \log\|z-p\| + O(1)$ as $z \to p \in S$. However in general $v^\star \notin E(S)$:

Lemma 3.9 *Let* $u \in \widetilde{E}(S)$ *be such that* $\gamma_u = \delta_u > |S|^{1/n}$*, and let* $v \in E(S)$*. Then* $\liminf_{\|z\|\to\infty}\frac{v(z)}{u(z)} < 1$*. In particular, if* $\rho \in PSH(\mathbf{C}^n)$ *satisfies* $u \leq \rho$*, then* $\rho \notin E(S)$*.*

Proof. We may assume $l = \liminf_{\|z\|\to\infty} \frac{v(z)}{u(z)} > 0$. By Corollary 3.3 we have

$$|S|^{1/n} \geq \delta_v = \liminf_{\|z\|\to\infty} \left(\frac{v(z)}{u(z)} \frac{u(z)}{\log \|z\|} \right) = l\gamma_u > l|S|^{1/n} \ ,$$

so $l < 1$. □

We can use however the above method to construct for a given $u \in \widetilde{E}(S)$ a function $v \in \widetilde{E}(S)$ such that $\gamma_u = \gamma_v$ and v is maximal on $\mathbf{C}^n \setminus (K \cup S)$. Here K is a suitable compact non-pluripolar set, for instance a sphere.

Proposition 3.10 *Let $u \in \widetilde{E}(S)$ and let $K = \partial B$, where B is some ball in $\mathbf{C}^n$. There exists $v \in \widetilde{E}(S)$ with the following properties:*
(i) $u \leq v$ on $\mathbf{C}^n$, $v = u$ on K, $\gamma_v = \gamma_u$.
(ii) v is maximal on $\mathbf{C}^n \setminus (K \cup S)$.

Proof. We let $\mathcal{F}_u$ be the family of functions $\rho \in \gamma_u \mathcal{L}$ which satisfy $\rho \leq u$ on K and $\rho(z) \leq \log \|z - p\| + O(1)$ as $z \to p$, for every $p \in S$. Let $v(z) = \sup\{\rho(z) : \ \rho \in \mathcal{F}_u\}$. We have $u \in \mathcal{F}_u$, $v = u$ on K, and $\{v < +\infty\} \supset K$ is not pluripolar. By the considerations preceding Lemma 3.9 it follows that $v^\star \in \widetilde{E}(S)$, $u \leq v^\star$, $\gamma_{v^\star} = \gamma_u$. We shall prove that $v^\star = u$ on K. Then $v^\star \in \mathcal{F}_u$, so $v^\star = v$. Moreover v is maximal on $\mathbf{C}^n \setminus (K \cup S)$: if G is open and relatively compact in $\mathbf{C}^n \setminus (K \cup S)$ and $\phi \in PSH(G)$ satisfies $\limsup_{\zeta \to z, \zeta \in G} \phi(\zeta) \leq v(z)$ for $z \in \partial G$, then $\rho \in \mathcal{F}_u$, where $\rho = v$ on $\mathbf{C}^n \setminus G$, $\rho = \max\{v, \phi\}$ on G; hence $\phi \leq \rho \leq v$ on G. So v has the desired properties.

In order to show that $v^\star = u$ on K we let $\{h_l\}$ be a sequence of continuous functions on $\overline{B}$, harmonic on B, such that $h_l \searrow u$ on K. By the maximum principle $\rho \leq h_l$ on $\overline{B}$, for every $\rho \in \mathcal{F}_u$, so $v^\star \leq h_l$ on B. Hence for every $z \in K$ we have $v^\star(z) = \limsup_{\zeta \to z, \zeta \in B} v^\star(\zeta) \leq h_l(z)$. Letting $l \to +\infty$ we conclude that $v^\star(z) \leq u(z)$ for $z \in K$. □

4 The case $n = 2$

We obtain in this section the values of $\gamma(S)$ and $\widetilde{\gamma}(S)$ for certain subsets S of $\mathbf{C}^2$. This is done by constructing suitable functions $u \in E(S)$. The main tools are the Bezout theorem and the following:

Theorem 4.1 *Let r be a positive integer and let P_1, P_2 be polynomials with the following properties: $S = \{z \in \mathbf{C}^2 : \ P_1(z) = P_2(z) = 0\}$, and $ord(P_1, p) \geq r$, $ord(P_2, p) \geq r$, $(P_1 \cdot P_2)_p = r^2$, for every $p \in S$. Then*

$$u = u(r, P_1, P_2) = \frac{1}{2r} \log(|P_1|^2 + |P_2|^2) \in E(S) \ . \tag{4.1}$$

Proof. Let $v(z) = \frac{1}{r}\log\|z\|$ and $f = (P_1, P_2) : \mathbf{C}^2 \to \mathbf{C}^2$. Then $(dd^c u)^2(z) = (dd^c v)^2(f(z))|\det f'(z)|^2 = 0$ on $\mathbf{C}^2 \setminus S$. By the results of [D1] it suffices to show that u has a logarithmic pole at each point of S (see (1.1)). Without loss of generality, we fix $p = 0 \in S$. Since $(P_1 \cdot P_2)_0 \geq ord(P_1, 0)\, ord(P_2, 0)$, we have $ord(P_j, 0) = r$, $P_j = h_r(P_j) + h.o.t.$, for $j = 1, 2$. For $\|z\|$ small, $|P_j(z)| \leq C\|z\|^r$ near 0, so $u(z) \leq \log\|z\| + C$. We will show that $h_r(P_1), h_r(P_2)$ have no common factors. As they are homogeneous of degree r, it follows that $U = |h_r(P_1)|^2 + |h_r(P_2)|^2$ satisfies $U(z) \geq C\|z\|^{2r}$, where $C = \min\{U(z) : \|z\| = 1\} > 0$. Moreover $|(|P_1|^2 + |P_2|^2 - U)(z)| \leq C'\|z\|^{2r+1}$, for some $C' > 0$ and for z near 0. Hence

$$u(z) \geq \frac{1}{2r}\log(C\|z\|^{2r} - C'\|z\|^{2r+1}) \geq \log\|z\| + C'' ,$$

for $\|z\|$ sufficiently small.

To complete the proof we assume that $h_r(P_1), h_r(P_2)$ have a common factor, say $z_2 - \alpha z_1$, and we show that $(P_1 \cdot P_2)_0 \geq r^2 + 1$. By a change of coordinates $z_1' = z_1, z_2' = z_2 - \alpha z_1$ we may assume that $h_r(P_1), h_r(P_2)$ are divisible by z_2. Making another change of coordinates $z_1 = z_1' + \lambda z_2', z_2 = z_2'$, with a suitable λ, we may assume that

$$h_r(P_1) = z_2^r + \sum_{j=1}^{r-1} a_j z_1^{r-j} z_2^j \ , \ h_r(P_2) = z_2^r + \sum_{j=1}^{r-1} b_j z_1^{r-j} z_2^j \ .$$

As $P_1(0, z_2) = z_2^r + h.o.t.$ we have $P_1 = AQ$, where $A(0) \neq 0$ and $Q = z_2^r + \sum_{j=0}^{r-1} f_j(z_1) z_2^j$ is a Weierstrass polynomial. If $Q = h_m(Q) + h.o.t.$ it follows that $m = r$, $h_r(P_1) = A(0)h_r(Q)$, so $A(0) = 1$, $h_r(P_1) = h_r(Q)$. Hence $f_0(z_1) = a_0 z_1^{r+1} + h.o.t.$ and $f_j(z_1) = a_j z_1^{r-j} + h.o.t.$, $j = 1, \ldots, r-1$.

We write $Q = BQ_1 \ldots Q_l$, where $B(0) \neq 0$, Q_i are irreducible Weierstrass polynomials of degree n_i in z_2, and $n_1 + \cdots + n_l = r$. If $ord(Q_i, 0) = m_i$ then $m_i \leq n_i$ and $m_1 + \cdots + m_l = ord(Q, 0) = r$. So $m_i = n_i$, $B(0) = 1$, and $h_r(Q) = \prod_{i=1}^{l} h_{n_i}(Q_i)$. Since $z_2 | h_r(Q)$ we assume without loss of generality that $z_2 | h_{n_1}(Q_1)$. We conclude as above that

$$h_{n_1}(Q_1) = z_2^{n_1} + \sum_{j=1}^{n_1-1} c_j z_1^{n_1-j} z_2^j \ , \ Q_1 = z_2^{n_1} + \sum_{j=0}^{n_1-1} g_j(z_1) z_2^j \ ,$$
$$g_0(z_1) = c_0 z_1^{n_1+1} + h.o.t. \ , \ g_j(z_1) = c_j z_1^{n_1-j} + h.o.t. \ , \ j = 1, \ldots, n_1 - 1 \ .$$

There exists a local normalization of $\{Q_1 = 0\}$ near 0, $t \to (t^{n_1}, \phi(t))$, defined for $t \in \mathbf{C}$ in a neighborhood of 0. Moreover, $ord(\phi, 0) = ord(g_0, 0) \geq n_1 + 1$ (see e.g. [G], Theorem 5.7, and p.85). As $(Q_i \cdot P_2)_0 \geq ord(Q_i, 0)\, ord(P_2, 0) = n_i r$, $i = 2, \ldots, l$, we have

$$(P_1 \cdot P_2)_0 = \sum_{i=1}^{l} (Q_i \cdot P_2)_0 \geq (Q_1 \cdot P_2)_0 + \sum_{i=2}^{l} n_i r \ .$$

By definition $(Q_1 \cdot P_2)_0 = ord(P_2(t^{n_1}, \phi(t)), 0)$. Since $P_2 = z_2^r + \sum_{j=1}^{r-1} b_j z_1^{r-j} z_2^j + h.o.t.$ and since $ord(t^{n_1(r-j)}\phi(t)^j, 0) \geq n_1(r-j) + (n_1+1)j \geq n_1 r + 1$, it follows that $(Q_1 \cdot P_2)_0 \geq n_1 r + 1$. Thus $(P_1 \cdot P_2)_0 \geq \sum_{i=1}^{l} n_i r + 1 = r^2 + 1$, a contradiction. $\square$

Let u be of form (4.1) with $r = 1$ and let $f = (P_1, \widetilde{P}_2)$, where $\widetilde{P}_2 = P_2 + QP_1$ for an arbitrary polynomial Q. For $p \in S$ we have $\nabla \widetilde{P}_2(p) = \nabla P_2(p) + Q(p)\nabla P_1(p)$, so $\nabla \widetilde{P}_2(p), \nabla P_1(p)$ are linearly independent and $(P_1 \cdot \widetilde{P}_2)_p = 1$. Theorem 4.1 implies that $\log \|f\| \in E(S)$.

Recall from Theorem 3.6 that for a finite set $S \subset \mathbf{C}^2$ we have $\sqrt{|S|} \leq \gamma(S) \leq |S|$. Moreover $\gamma(S) = \widetilde{\gamma}(S) = |S|$ if and only if S is contained in a complex line. If $|S| = d^2$ and S is the complete intersection of two algebraic curves of degree d then $\gamma(S) = \widetilde{\gamma}(S) = \sqrt{|S|}$ (see [Co]). We now turn our attention to the computation of $\gamma(S)$ and $\widetilde{\gamma}(S)$ in some special cases.

Theorem 4.2 *Let $S \subset \mathbf{C}^2$ be such that $m_1(S) > |S|/2$. Then $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$.*

Proof. If we set $|S| = k$ and $m_1(S) = k - l$, then $m_1(S) > |S|/2$ is equivalent to $k \geq 2l + 1$. Let $S = \{p_1, \dots, p_k\}$ and assume that $p_{l+1}, \dots, p_k$ lie on a line L_0 and $p_1, \dots, p_l$ are not on L_0. We can choose parallel complex lines $L_1, \dots, L_l$ such that $L_j \cap S = \{p_j\}$, $j = 1, \dots, l$, and we denote by p_0 their intersection point on the line at infinity of $\mathbf{P}^2$. Moreover $L_1, \dots, L_l$ are chosen such that $p_0 \notin L_0$. Let $P_1 = L_0 L_1 \dots L_l$, $\deg P_1 = l + 1$. We will construct a polynomial P_2 of degree $\leq k - l$ such that $S = \{z \in \mathbf{C}^2 : P_1(z) = P_2(z) = 0\}$ and $(P_1 \cdot P_2)_p = 1$ for all $p \in S$. Then the function $u = u(1, P_1, P_2)$ given by (4.1) is in $E(S)$, $\gamma_u \leq k - l$, and the theorem follows.

We consider the vector space V of polynomials P in z_1, z_2 of degree at most $k - l$ which verify

(i) $P(p) = 0$, for all $p \in S$.

(ii) $\partial^\alpha P(p_0) = 0$ for $|\alpha| \leq k - l - 2$.

Here (ii) is to be interpreted for the homogeneous polynomial $\widetilde{P}$ associated to P, and in local coordinates around p_0. Since (i), (ii) impose $k + (k-l-1)(k-l)/2$ linear conditions on P we have

$$\dim V \geq \frac{(k-l+1)(k-l+2)}{2} - k - \frac{(k-l-1)(k-l)}{2} = k - 2l + 1 \,.$$

We note that for $P \in V$ we have $(\widetilde{P} \cdot \widetilde{L}_j)_{p_0} \geq k - l - 1$, $j = 1, \dots, l$, so $(\widetilde{P} \cdot \widetilde{P}_1)_{p_0} \geq l(k - l - 1)$. So if $P_2 \in V$ is such that none of the L_j divides P_2, $j = 0, \dots, l$, then by the Bezout theorem

$$(k-l)(l+1) \geq (\deg \widetilde{P}_2)(\deg \widetilde{P}_1) \geq \sum_{p \in S} (P_1 \cdot P_2)_p + (\widetilde{P}_1 \cdot \widetilde{P}_2)_{p_0} \geq k + l(k - l - 1) \,.$$

Hence $\deg \widetilde{P}_2 = k - l$, $S \subset \mathbf{C}^2$ is the set of common zeros of P_1, P_2, and $(P_1 \cdot P_2)_p = 1$ for all $p \in S$.

To prove the existence of such $P_2 \in V$ we consider the subset $W \subseteq V$ consisting of those polynomials $P \in V$ such that $L_j|P$ for some $j = 0, \dots, l$. We will show that W is a vector space of dimension $k-2l$. Since $\dim V \geq k-2l+1$ we can find $P_2 \in V \setminus W$.

We can assume without loss of generality $p_0 = [1:0:0]$, and we let $P \in W$. If $L_0|P$ then $P = L_0 R$, $\deg R \leq k-l-1$, and $(\widetilde{R} \cdot \widetilde{L}_j)_{p_0} = (\widetilde{P} \cdot \widetilde{L}_j)_{p_0} \geq k-l-1$, $j = 1, \dots, l$. As $R(p_j) = 0$ it follows by Bezout that $L_j|P$, $j = 1, \dots, l$, so $P_1|P$. If $L_1|P$ then $P = L_1 R$, $\deg R \leq k-l-1$ and $R(p_j) = 0$, $j = l+1, \dots, k$, so by Bezout $L_0|R$. Hence $P = L_0 L_1 R_1$, $\deg R_1 \leq k-l-2$, and $(\widetilde{R}_1 \cdot \widetilde{L}_2)_{p_0} = (\widetilde{P} \cdot \widetilde{L}_2)_{p_0} - (\widetilde{L}_1 \cdot \widetilde{L}_2)_{p_0} \geq k-l-2$. Since $R_1(p_2) = 0$ we conclude $L_2|R_1$, and similarly $L_j|R_1$ for $j \geq 3$. So we have shown that if $P \in W$ then $P = QP_1$ for some polynomial Q. By condition (ii) in the definition of V we have in coordinates (z_2, t) near $p_0 = (0,0)$:

$$\widetilde{P}(1, z_2, t) = h_{k-l-1}(\widetilde{P}) + h_{k-l}(\widetilde{P}) \ , \ \widetilde{P}_1(1, z_2, t) = h_l(\widetilde{P}_1) + h_{l+1}(\widetilde{P}_1) \ ,$$

so $\widetilde{Q}(1, z_2, t) = h_{k-2l-1}(\widetilde{Q}) + h.o.t.$. Here $h_j(\widetilde{P})$ denotes the homogeneous part of degree j of $\widetilde{P}(1, z_2, t)$. As $\deg \widetilde{P}(1, z_2, t) \leq k-l$, $\deg \widetilde{P}_1(1, z_2, t) = l+1$, we see that $\deg \widetilde{Q}(1, z_2, t) \leq k-2l-1$, so $\widetilde{Q}(1, z_2, t)$ is homogeneous of degree $k-2l-1$. It follows that

$$W = \{QP_1 : \ \widetilde{Q}(1, z_2, t) \text{ is homogeneous of degree } k-2l-1\} \ ,$$

hence W is a vector space of dimension $k-2l$. $\square$

Theorem 4.3 *Let $S \subset \mathbf{C}^2$ with $|S| \geq 4$ be contained in an irreducible conic C. Then $\gamma(S) = \widetilde{\gamma}(S) = |S|/2$.*

Proof. Let $S = \{p_1, \dots, p_k\}$. By Theorem 3.4 applied to C we have $\widetilde{\gamma}(S) \geq k/2$. Let $P_1 = C^2$. We show that there exists a polynomial P_2 of degree k such that $S = \{z \in \mathbf{C}^2 : \ P_1(z) = P_2(z) = 0\}$, and $ord(P_2, p) \geq 2$, $(P_1 \cdot P_2)_p = 4$ for all $p \in S$. By Theorem 4.1 we have $u = u(2, P_1, P_2) \in E(S)$ and $\gamma_u \leq k/2$, so $\gamma(S) \leq k/2$.

We consider the vector space

$$V = \{P \in \mathcal{P}_k : \ ord(P, p) \geq 2 \ \forall p \in S\} \ .$$

Then $\dim V \geq \frac{(k+1)(k+2)}{2} - 3k = \frac{(k-1)(k-2)}{2}$. Let W be the set of $P \in V$ such that P, P_1 have common factors. As C is irreducible we have

$$W = \{CQ : \ Q \in \mathcal{P}_{k-2}, \ Q(p) = 0 \ \forall p \in S\} \ .$$

Thus W is a vector space isomorphic to $\ker E_{k-2}$, where $E_{k-2} : \mathcal{P}_{k-2} \to \mathbf{C}^k$, $E_{k-2}(F) = (F(p_1), \dots, F(P_k))$. We claim that E_{k-2} is surjective. Indeed, if $k = 2l+1$ we consider the complex lines L_j determined by p_{2j-1} and p_{2j}, $j = 1, \dots, l$. Then $L_j \cap C = \{p_{2j-1}, p_{2j}\}$, since C is irreducible. So $F(p_{2l+1}) \neq 0$, where $F = L_1 \dots L_l \in \mathcal{P}_{k-2}$, hence $(0,\dots,0,1) \in E_{k-2}(\mathcal{P}_{k-2})$. Arguing similarly with p_j in place of p_{2l+1} we conclude that $E_{k-2}(\mathcal{P}_{k-2}) = \mathbf{C}^k$. If $k = 2l$ we let L_j be the complex line through p_{2j-1} and p_{2j}, $j = 1,\dots,l-1$, and L_l be a complex line through p_{2l-1} such that $p_{2l} \notin L_l$. Then $F = L_1 \dots L_l \in \mathcal{P}_{k-2}$ and $F(p_{2l}) \neq 0$, so $E_{k-2}(\mathcal{P}_{k-2}) = \mathbf{C}^k$.

It follows that

$$\dim W = \dim \mathcal{P}_{k-2} - k = \frac{(k-1)(k-2)}{2} - 1 < \dim V \ .$$

So we can find $P_2 \in V \setminus W$. We have by Bezout

$$4k \le \sum_{p \in S} (P_1 \cdot P_2)_p \le \deg P_1 \, \deg P_2 \le 4k \ ,$$

so $S = \{P_1 = P_2 = 0\}$ and $(P_1 \cdot P_2)_p = 4$ for all $p \in S$. $\Box$

Corollary 4.4 *If $S \subset \mathbf{C}^2$ with $|S| \ge 4$ is contained in a conic C then*

$$\gamma(S) = \widetilde{\gamma}(S) = m(S) = \max\{m_1(S), |S|/2\} \ .$$

Proof. If C is irreducible then $m_1(S) = 2$ and the corollary follows directly from the above theorem. Assume now $C = L_1 L_2$, where L_1, L_2 are complex lines, and let $S = \{p_1,\dots,p_k\}$. If $k = 2l+1$ one of the lines L_1, L_2 contains at least $l+1$ of the points of S, so $\gamma(S) = \widetilde{\gamma}(S) = m_1(S) > |S|/2$ by Theorem 4.2.

If $k = 2l$ and say $|S \cap L_1| \ge l+1$ then the corollary follows again from Theorem 4.2. Otherwise $|S \cap L_1| = |S \cap L_2| = l$, so the intersection point of L_1, L_2 is not in S. Let $S \cap L_1 = \{p_1, p_3, \dots, p_{2l-1}\}$, $S \cap L_2 = \{p_2, p_4, \dots, p_{2l}\}$. We consider the lines E_j joining p_{2j-1}, p_{2j}, $j = 1,\dots,l$. If $P = E_1 \dots E_l$ and $u = u(1, P, C)$, then $u \in E(S)$ and $\gamma_u \le l$, so $\gamma(S) = \widetilde{\gamma}(S) = l$. $\Box$

Using the above results we can describe these invariants for any set $S \subset \mathbf{C}^2$ with $|S| \le 5$:

$|S| \in \{1,2,3\}$: $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$.

$|S| = 4$: $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$, if $m_1(S) \in \{3,4\}$. Otherwise S is the complete intersection of two conics and $\gamma(S) = \widetilde{\gamma}(S) = 2$.

$|S| = 5$: $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$, if $m_1(S) \in \{3,4,5\}$. Otherwise S is contained in an irreducible conic and $\gamma(S) = \widetilde{\gamma}(S) = 5/2$.

Theorem 4.5 *For $S \subset \mathbf{C}^2$ with $|S| = 6$ we have one of the following possibilities:*

(i) If $m_1(S) \in \{3,4,5,6\}$ then $\gamma(S) = \widetilde{\gamma}(S) = m_1(S)$.

(ii) If $m_1(S) = 2$ and S is contained in a conic then $\gamma(S) = \widetilde{\gamma}(S) = 3$.

(iii) If $m_1(S) = 2$ and S is not contained in a conic then $\gamma(S) = \widetilde{\gamma}(S) = 5/2$.

Proof. Let $S = \{p_1,\dots,p_6\}$ and let $L_\infty = \mathbf{P}^2 \setminus \mathbf{C}^2$ denote the line at infinity.

(i) If $m_1(S) \ge 4$ we apply Theorem 4.2. We assume $m_1(S) = 3$. Then $3 \le \widetilde{\gamma}(S) \le \gamma(S)$. We construct two polynomials P_1, P_2 of degree 3 such that $S = \{z \in \mathbf{C}^2 :$

$P_1(z) = P_2(z) = 0\}$ and $(P_1 \cdot P_2)_p = 1$ for all $p \in S$. Then $u = u(1, P_1, P_2) \in E(S)$ has $\gamma_u \leq 3$, so $\gamma(S) \leq 3$.

We consider the finite set $\mathcal{L}$ of complex lines which pass through at least two points of S, and we let $I_1 = \bigcup_{L\in\mathcal{L}}(L \cap L_\infty)$. Then I_1 is finite. Since no four of the points of S lie on the same complex line, it follows that, given any 5 points of S, there exists a unique conic containing them. We denote by $\mathcal{C}$ the set of conics determined by the 5 element subsets of S, $|\mathcal{C}| = 6$. Then the set $I_2 = \bigcup_{C\in\mathcal{C}}(C \cap L_\infty)$ is finite.

We fix $p_0 \in L_\infty \setminus (I_1 \cup I_2)$. There exists a non-trivial polynomial $P_1 \in \mathcal{P}_3$ such that $P_1(p_j) = 0$, $j = 1, \dots, 6$, and $ord(\widetilde{P}_1, p_0) \geq 2$. Indeed, we are imposing 9 linear conditions on polynomials in $\mathcal{P}_3$, and $\dim \mathcal{P}_3 = 10$. We claim that P_1 is irreducible. Assuming the contrary, we have either $P_1 = LC$, for a complex line L and an irreducible conic C, or $P_1 = L_1L_2L_3$, where L_j are complex lines. In the first case, both L and C are smooth curves, so $ord(\widetilde{P}_1, p_0) \geq 2$ implies $p_0 \in L \cap C$. By the choice of p_0 we have $|L \cap S| \leq 1$ and $|C \cap S| \leq 4$, which is impossible. In the second case, p_0 belongs to 2 of the lines L_j, say L_1, L_2. Then $|L_1 \cap S| \leq 1$, $|L_2 \cap S| \leq 1$, so $|L_3 \cap S| \geq 4$, a contradiction.

Assume now without loss of generality that $p_0 = [1 : 0 : 0]$ and let $(a, b) \in \mathbf{C}^2$ be such that $(a, b) \neq (0, 0)$ and $h_2(\widetilde{P}_1)(a, b) = 0$, where $h_2(\widetilde{P}_1)$ is the homogeneous part of degree two of $\widetilde{P}_1(1, z_2, t)$ at $(0, 0)$. We consider the vector space V of polynomials $P \in \mathcal{P}_3$ such that $P(p_j) = 0$, $j = 1, \dots, 6$, $\widetilde{P}(p_0) = 0$, and $a\frac{\partial \widetilde{P}}{\partial z_2}(1, 0, 0) + b\frac{\partial \widetilde{P}}{\partial t}(1, 0, 0) = 0$ (i.e. (a, b) is tangent to $\{\widetilde{P}(1, z_2, t) = 0\}$ at $(0, 0)$). So $(\widetilde{P}_1 \cdot \widetilde{P})_{p_0} \geq 3$ for $P \in V$. Since $\dim V \geq 2$ we can choose $P_2 \in V$ linearly independent from P_1. As P_1 is irreducible we have by Bezout $3 \times 3 \geq \sum_{j=1}^{6}(P_1 \cdot P_2)_{p_j} + (\widetilde{P}_1 \cdot \widetilde{P}_2)_{p_0} \geq 6 + 3$. So P_1, P_2 have the desired properties.

(ii) If S is contained in a conic, this conic has to be irreducible since otherwise $m_1(S) \geq 3$. So Theorem 4.3 applies.

(iii) In view of Theorem 3.4 applied to a conic passing through 5 points of S we have $5/2 \leq \widetilde{\gamma}(S) \leq \gamma(S)$. We construct two polynomials P_1, P_2 of degree 5 which satisfy the hypotheses of Theorem 4.1 with $r = 2$. Then $u = u(2, P_1, P_2) \in E(S)$ has $\gamma_u \leq 5/2$, so $\gamma(S) \leq 5/2$.

We denote by L_{ij} the complex line joining p_i and p_j, and by C_j the unique conic containing $S \setminus \{p_j\}$. Then $p_j \notin C_j$, and C_j is irreducible, hence smooth, since $m_1(S) = 2$. Let $\mathcal{C} = \{C_1, \dots, C_6\}$, $\mathcal{L} = \{L_{ij}\}$, $|\mathcal{L}| = 15$. We claim that there exists $L \in \mathcal{L}$ such that $L \cap L_\infty \cap C = \emptyset$ for all $C \in \mathcal{C}$. Indeed, let $\mathcal{L}_j = \{L \in \mathcal{L} : \ L \cap L_\infty \cap C_j \neq \emptyset\}$, $j = 1, \dots, 6$. We show $|\mathcal{L}_j| \leq 2$, so $|\bigcup_{j=1}^{6} \mathcal{L}_j| \leq 12$ hence $\mathcal{L} \setminus \bigcup_{j=1}^{6} \mathcal{L}_j \neq \emptyset$. To see $|\mathcal{L}_1| \leq 2$, we note that $\mathcal{L}_1 \subset \{L_{12}, \dots, L_{16}\}$, since $L_{ij} \cap C_1 = \{p_i, p_j\}$ if $i \neq 1 \neq j$. We have $|C_1 \cap L_\infty| \leq 2$. If $q \in C_1 \cap L_\infty$ then q lies at most on one of the lines L_{1j}: otherwise, if say $q \in L_{12} \cap L_{13}$ then $L_{12} = L_{13}$ is the line through p_1, q, so p_1, p_2, p_3 are colinear. Consequently $|\mathcal{L}_1| \leq 2$.

Hence we may assume that $L = L_{12}$ satisfies $L_{12} \cap L_\infty \cap C = \emptyset$ for all $C \in \mathcal{C}$. So $p_0 \notin C$, where $\{p_0\} = L_{12} \cap L_\infty$.

Let $P_1 = C_1C_2L_{12}$. Then $\deg P_1 = 5$, $ord(P_1, p) = 2$ for $p \in S$, $\widetilde{P}_1(p_0) = 0$. Let $P_2 = C_6Q$, where $\deg Q \leq 3$ and $\widetilde{Q}(p_0) = 0$, $Q(p_j) = 0$, $j = 1, \ldots, 5$, $ord(Q, p_6) \geq 2$. Since we are imposing 9 linear conditions on polynomials in $\mathcal{P}_3$, we can find such non-trivial Q. Then $\deg P_2 \leq 5$, $ord(P_2, p) \geq 2$ for $p \in S$, $\widetilde{P}_2(p_0) = 0$. We will show that P_1, P_2 have no common factors. Then by Bezout $25 \geq \sum_{j=1}^{6}(P_1 \cdot P_2)_{p_j} + (\widetilde{P}_1 \cdot \widetilde{P}_2)_{p_0} \geq 6 \times 4 + 1$, so P_1, P_2 have the desired properties.

We assume $C_1|Q$. Then $Q = C_1L'$, $\deg L' \leq 1$. Since $ord(C_1, p_j) = 1$, $j = 2, \ldots, 6$, and $C_1(p_1) \neq 0 \neq \widetilde{C}_1(p_0)$, it follows that $L'(p_1) = 0$, $\widetilde{L}'(p_0) = 0$, and $L'(p_6) = 0$ as $ord(Q, p_6) \geq 2$. So $L' = L_{12}$ is the line joining p_0, p_1, hence p_1, p_2, p_6 are colinear, a contradiction. A similar argument shows that C_2 does not divide Q. Finally, we assume $L_{12}|Q$, so $Q = L_{12}C$, $\deg C \leq 2$. Then $C(p_j) = 0$ for $j = 3, 4, 5$, and $ord(C, p_6) \geq 2$ as $L_{12}(p_6) \neq 0$. Thus C is a reducible conic, $C = L'L''$, and $p_6 \in L' \cap L''$. It follows that one of the lines L', L'' contains at least three of the points p_3, p_4, p_5, p_6, a contradiction. We conclude that P_1, P_2 have no common factors. □

Remark. In view of these results we have $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = m(S)$ for sets S with $|S| \leq 6$.

We now consider sets $S = \{p_1, \ldots, p_7\} \subset \mathbf{C}^2$ such that $m_1(S) = 2$, $m_2(S) = 5$. This happens for generic sets S with $|S| = 7$.

Theorem 4.6 *For S as above we have $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = 8/3$ and $m(S) = 5/2$.*

Proof. For $1 \leq i < j \leq 7$ let C_{ij} be the (irreducible) conic determined by $S \setminus \{p_i, p_j\}$. For each $1 \leq j \leq 7$ there exists a polynomial E_j of degree 3 such that $ord(E_j, p_j) \geq 2$ and $E_j(p) = 0$ for $p \in S$. We claim that E_j is irreducible and unique up to multiplication by constants with these vanishing properties, $ord(E_j, p_j) = 2$, $ord(E_j, p) \leq 1$ for $p \neq p_j$. Indeed, if E_1 was reducible then $E_1 = L_1L_2L_3$ or $E_1 = LC$, where L, L_1, L_2, L_3 are complex lines and C is an irreducible conic. The first case cannot occur, as $|S \cap L_l| \leq 2$, $l = 1, 2, 3$. In the second case $|S \cap L| = 2$, $|S \cap C| = 5$, $S \cap L \cap C = \emptyset$, so $ord(E_1, p_1) = 1$, a contradiction. If $ord(E_1, p_1) \geq 3$ (or $ord(E_1, p) \geq 2$ for some $p \neq p_1$) then by Bezout E_1 is divisible by the line p_1p_2 (or p_1p). Moreover, if P has degree 3 and the same vanishing properties as E_1 then Bezout's theorem implies $P = cE_1$, for some $c \neq 0$.

Let $p_0 \in L_\infty \setminus (\bigcup_{i,j} C_{ij} \cup \bigcup_j E_j)$, where L_∞ is the line at infinity of $\mathbf{P}^2$. For each $1 \leq j \leq 7$ there exists a polynomial $Q_j \neq 0$ with $\deg Q_j \leq 5$, such that $\widetilde{Q}_j(p_0) = 0$, $Q_j(p_j) = 0$, $ord(Q_j, p) \geq 2$ for $p \in S \setminus \{p_j\}$. We claim that Q_j is irreducible. Assuming that Q_1 is reducible, let Q be an irreducible factor of Q_1 of smallest degree; so $\deg Q \in \{1, 2\}$. If $\deg Q = 1$ we have $Q_1 = QR$, $\deg R \leq 4$, $ord(R, p) \geq 2$ for at least 4 points $p \in S \setminus \{p_1\}$, $ord(R, q) \geq 1$ for at least 2 other points $q \in S \setminus \{p_1\}$. Without loss of generality say $ord(R, p_j) \geq 2$ for $j = 2, 3, 4, 5$, $ord(R, p_j) \geq 1$ for $j = 6, 7$. By Bezout $C_{16}|R$, $C_{17}|R$, so up to multiplication by constants $R = C_{16}C_{17}$. It follows that the line Q passes through p_1, p_6, p_7, a contradiction. If $\deg Q = 2$ then $Q_1 = QR$, $\deg R \leq 3$. We have $ord(R, p) \geq 2$ for at least one $p \in S$, say

p_2. If $ord(R,p) \geq 2$ for some $p \in S \setminus \{p_2\}$ then R is divisible by the line p_2p, which contradicts the choice of Q. It follows that up to multiplication by constants $Q = C_{12}$, $R(p) = 0$ for $p \in S$, $ord(R,p_2) \geq 2$, so $R = E_2$. Thus we have shown that Q_1 reducible implies $Q_1 = C_{1j}E_j$, so in particular $\widetilde{Q}_1(p_0) \neq 0$ by the choice of p_0. We conclude that the Q_j's are irreducible. This implies $ord(Q_j,p_j) = 1$ (otherwise by Bezout $E_j|Q_j$), so Q_i, Q_j are coprime for $i \neq j$.

We now let $P_1 = E_1Q_1$, $P_2 = E_2Q_2$. Then $\deg P_j \leq 8$, $ord(P_j,p) \geq 3$ for $p \in S$, $\widetilde{P}_1(p_0) = \widetilde{P}_2(p_0) = 0$, and P_1, P_2 have no common factors. It follows by Bezout that P_1, P_2 satisfy the hypotheses of Theorem 4.1 with $r = 3$, so $u(3,P_1,P_2) \in E(S)$, $\gamma_u \leq 8/3$. Thus

$$\omega(S) \geq \frac{\sum_{j=1}^{7} ord(E_1,p_j)}{\deg E_1} = \frac{8}{3} \geq \gamma(S) \geq \widetilde{\gamma}(S) ,$$

and the conclusion follows by Corollary 3.5. □

Theorem 4.7 *For generic $S \subset \mathbf{C}^2$ with $|S| = 8$ we have $\gamma(S) = \widetilde{\gamma}(S) = \omega(S) = 17/6$ and $m(S) = 8/3$.*

Proof. Let $S = \{p_1, \ldots, p_8\}$. As before, by counting dimension, the following polynomials exist:

(i) F_{jl} of degree 5, $1 \leq j < l \leq 8$, such that: $ord(F_{jl},p_i) \geq 2$ for $i \neq j,l$, $F_{jl}(p_j) = F_{jl}(p_l) = 0$.

(ii) E_j of degree 6, $1 \leq j \leq 8$, such that: $ord(E_j,p_j) \geq 3$, $ord(E_j,p_i) \geq 2$ for $i \neq j$.

Moreover, for generic S we have: $ord(F_{jl},p_j) = ord(F_{jl},p_l) = 1$, $ord(F_{jl},p_i) = 2$ for $i \neq j,l$, $ord(E_j,p_j) = 3$, $ord(E_j,p_i) = 2$ for $i \neq j$, and F_{jl}, E_j are unique up to multiplication by constants with the specified degree and vanishing properties. It follows that E_i, E_j are linearly independent for $i \neq j$.

Let $p_0 \in L_\infty \setminus (\bigcup_{j,l} F_{jl} \cup \bigcup_j E_j)$. For generic S there exist polynomials Q_j of degree 11, $j = 1, \ldots, 8$, such that $\widetilde{Q}_j(p_0) = 0$, $ord(Q_j,p_j) = 3$, $ord(Q_j,p_i) = 4$ for $i \neq j$. Then Q_i, Q_j are linearly independent for $i \neq j$.

We will show that E_j, Q_j are irreducible for generic S. To this end we introduce for positive integers l the numbers

$$A_l = A_l(S) = \max\left\{\sum_{i=1}^{8} ord(P,p_i) : \ P \in \mathcal{P}_l\right\} .$$

We need the following:

Lemma. *For generic $S \subset \mathbf{C}^2$ with $|S| = 8$ we have: $A_1 = 2$, $A_2 = 5$, $A_3 = 8$, $A_4 = 11$, $A_5 = 14$, $A_6 = 17$, $A_7 = 19$, $A_8 = 22$, $A_9 = 25$, $A_{10} = 28$. The maximum $A_5 = 14$ (respectively $A_6 = 17$) is reached if and only if $P = cF_{jl}$ (respectively $P = cE_j$), for some constant $c \neq 0$.*

Proof of the Lemma. For generic S, there exists a non-trivial polynomial $P \in \mathcal{P}_l$ with $ord(P, p_i) \geq x_i$, $i = 1, \ldots, 8$, if and only if $\sum_1^8 x_i(x_i + 1) < (l + 1)(l + 2)$. Indeed, we are imposing $\sum_1^8 x_i(x_i + 1)/2$ linear conditions on polynomials in $\mathcal{P}_l$. As $\sum_1^8 x_i(x_i + 1)$ is even the above condition is equivalent to $\sum_1^8 x_i(x_i + 1) \leq l^2 + 3l$. It follows that for a fixed $l > 0$ and generic S, $A_l(S)$ is given by

$$A_l = \max \left\{ \sum_{i=1}^{8} x_i : \; x_i \in \{0, 1, \ldots, l\}, \; \sum_{i=1}^{8} x_i(x_i + 1) \leq l^2 + 3l \right\} .$$

The assertions of the lemma now follow by direct calculations, which we omit. □

We proceed with the proof of the theorem. Assuming Q_1 is reducible, we write $Q_1 = R_1 R_2$, where $1 \leq \deg R_1 \leq \deg R_2$. Set $x_i = ord(R_1, p_i)$, $l = \deg R_1$, $y_i = ord(R_2, p_i)$, $l' = \deg R_2$. Then $\sum_1^8 x_i \leq A_l$, $\sum_1^8 y_i \leq A_{l'}$, $l + l' = 11$. Moreover $x_1 + y_1 = 3$, $x_i + y_i = 4$ for $i > 1$. We obtain $31 = \sum_{i=1}^8 (x_i + y_i) \leq A_l + A_{l'}$, $l + l' = 11$. By the previous lemma, for generic S this implies $l = 5, l' = 6$, and $R_1 = cF_{1i}, R_2 = c'E_i$ for some $i \neq 1$. By the choice of p_0 it follows $\widetilde{Q}_1(p_0) \neq 0$, a contradiction. Thus Q_j is irreducible for generic S.

A similar argument works for E_j: assuming E_j is reducible we obtain $A_l + A_{l'} \geq 17$, $l + l' = 6$, which is impossible for generic S by the above lemma.

Now let $P_1 = E_1 Q_1$, $P_2 = E_2 Q_2$. Then $\deg P_j = 17$, $\widetilde{P}_j(p_0) = 0$, $ord(P_j, p) = 6$, $p \in S$, $j = 1, 2$. For generic S, P_1, P_2 have no common factors, so by Bezout they satisfy the hypotheses of Theorem 4.1 with $r = 6$. We have $u = u(6, P_1, P_2) \in E(S)$, $\gamma_u \leq 17/6$. Thus

$$\omega(S) \geq \frac{\sum_{j=1}^8 ord(E_1, p_j)}{\deg E_1} = \frac{17}{6} \geq \gamma(S) \geq \widetilde{\gamma}(S) \geq \omega(S) \,. \quad \square$$

5 On the classes $M(S)$

We consider now the subclass $M_n(S) \subset E(S)$. A sufficient condition for $M_n(S)$ to be non-empty is given in [Co]. Here we deal with the question whether in dimension $n = 2$ this condition is also necessary. We start with a simple remark:

Lemma 5.1 *If $\widetilde{\gamma}(S) = |S|^{1/n}$ then $\Omega_0(S) = |S|^{1/n}$. In particular, this holds if $M_n(S) \neq \emptyset$.*

Proof. If $M_n(S) \neq \emptyset$ we have by Corollary 3.3 $\widetilde{\gamma}(S) = |S|^{1/n}$. By Corollary 3.5

$$|S|/\Omega_0(S) \leq \omega(S) \leq \widetilde{\gamma}(S)^{n-1} = |S|^{(n-1)/n}$$

and the lemma follows since $\Omega_0(S) \leq |S|^{1/n}$ for any set S (see [Ch]). □

¿From now on we assume $n = 2$ and write $M(S) = M_2(S)$. We have $M(S) \neq \emptyset$ provided that $|S| = d^2$ and S is the complete intersection of two algebraic curves of degree d. We will prove partial converses to this statement.

Theorem 5.2 *Assume there exist $u \in M(S)$ and a subvariety V of $\mathbf{C}^2$ of pure dimension one such that the restriction of u to $V_{reg} \setminus S$ is harmonic. Then $|S| = d^2$ for some $d \in \mathbf{N}$ and V is algebraic.*

To prove this we need the following lemma:

Lemma 5.3 *Let V be a pure one dimensional subvariety of $\mathbf{C}^2$ and let $u \in L^\infty_{loc} \cap PSH(\mathbf{C}^2)$ be harmonic along V_{reg}. Then the measure $dd^c u \wedge [V] = 0$.*

Proof. We fix first $z \in V_{reg}$ and an open neiborhood U of z such that $V \cap U$ is biholomorphic to the unit disc Δ. If $h : \Delta \to V \cap U$ is a biholomorphism and $\phi \in C^\infty_0(U)$ then

$$\langle dd^c u \wedge [V], \phi \rangle = \int_V u \, dd^c \phi = \frac{1}{2\pi} \int_\Delta (u \circ h) \, \Delta(\phi \circ h) = 0,$$

as $u \circ h$ is harmonic on Δ and $\phi \circ h \in C^\infty_0(\Delta)$. So $dd^c u \wedge [V] = 0$ on $\mathbf{C}^2 \setminus V_{sing}$. As u is locally bounded we see by the following lemma that the measure $dd^c u \wedge [V]$ has no atomic part. So $dd^c u \wedge [V] = 0$ on $\mathbf{C}^2$ since V_{sing} is discrete. □

Lemma 5.4 *If u is plurisubharmonic and bounded near $z \in \mathbf{C}^2$ and V is a pure one-dimensional variety containing z then $dd^c u \wedge [V](\{z\}) = 0$.*

Proof. We have $[V] = \sum m_j [V_j]$ near z, where V_j are the irreducible components of V at z. So we may assume that V is irreducible at z and z is a singular point of V. Then there exist a disc Δ in $\mathbf{C}$ centered at 0, an open neighborhood U of z, and a holomorphic map $h : \Delta \to V \cap U$ such that $h(\Delta) = V \cap U$, $h(0) = z$, and $h : \Delta \setminus \{0\} \to (V \setminus \{z\}) \cap U$ is biholomorphic. If $\phi \in C^\infty_0(U)$, $0 \le \phi \le 1$, and $\phi \equiv 1$ near z, then

$$\begin{aligned} dd^c u \wedge [V](\{z\}) &\le \langle dd^c u \wedge [V], \phi \rangle = \int_{V_{reg}} u \, dd^c \phi = \\ &= \frac{1}{2\pi} \int_\Delta (u \circ h) \, \Delta(\phi \circ h) = \frac{1}{2\pi} \int_\Delta (\phi \circ h) \, \Delta(u \circ h) \, . \end{aligned}$$

Since $u \circ h$ is bounded subharmonic, the measure $\Delta(u \circ h)$ has no atomic part, by the Riesz representation theorem. So $\int_\Delta (\phi \circ h) \, \Delta(u \circ h) \to 0$ as $\operatorname{supp} \phi \searrow \{z\}$. □

Proof of Theorem 5.2. By Lemma 5.3 we have $dd^c u \wedge [V] = 0$ on $\mathbf{C}^2 \setminus (S \cap V)$. Using this and a comparison theorem for Lelong numbers with weights of [D2] (see the proof of Theorem 3.4) we obtain

$$\int_{\mathbf{C}^2} dd^c u \wedge [V] = \sum_{p \in S \cap V} \nu(V, p) = M \in \mathbf{N} \, .$$

Proposition 3.2 implies that

$$M = \int_{\mathbf{C}^2} dd^c u \wedge [V] = \sqrt{|S|} \int_{\mathbf{C}^2} dd^c \log \|z\| \wedge [V] \ .$$

Since

$$\int_{\mathbf{C}^2} dd^c \log \|z\| \wedge [V] = \lim_{R \to \infty} \frac{\sigma_V(B(0,R))}{\pi R^2}$$

it follows that this limit is finite, hence V is algebraic and the limit is equal to $\deg V$ (see e.g. [LG]). We conclude that $M = \sqrt{|S|} \deg V$, so $|S| = d^2$ for some $d \in \mathbf{N}$. □

We assume now that $S \subset \mathbf{C}^2$ satisfies $|S| = d^2$. Recall the definitions of $e_m(S)$ and $H(S,m)$ from Section 2. We will prove that if $M(S) \neq \emptyset$ and in addition the number $H(S,m)$ satisfies certain conditions for suitable values of m then S is a complete intersection of curves of degree d.

We remark first that, by the proof of Theorem 3.4, we easily obtain the following:

Lemma 5.5 *If $|S| = d^2$ and $M(S) \neq \emptyset$ then $e_m(S) = 0$ for all $m < d$.*

Proposition 5.6 *If $|S| = d^2$, $e_d(S) \geq 2$, and $M(S) \neq \emptyset$, then S is the complete intersection of two algebraic curves of degree d.*

We note that the assumption $e_d(S) \geq 2$ is strong: it means that there exist two different curves of degree d containing S. We will use the existence of $u \in M(S)$ to prove that they cannot have a common component. We need the following:

Lemma 5.7 *Let $u \in M(S)$, $|S| = d^2$, and $V \subset \mathbf{C}^2$ be an algebraic curve of degree d which contains S. The following hold:*

(i) The restriction of u to $V \setminus S$ is harmonic and V is smooth at each point of S.

(ii) Every irreducible component of V has multiplicity one.

(iii) If $V = V_1 \cup \cdots \cup V_j$, where V_i are irreducible, and if $S_i = V_i \cap S$, then $S_i \neq \emptyset$ and $S_i \cap S_l = \emptyset$ for $i \neq l$. Hence $S_1, \ldots, S_l$ partition S.

Proof. Assertion (i) follows since

$$d^2 = \int_{\mathbf{C}^2} dd^c u \wedge [V] = \int_{\mathbf{C}^2 \setminus S} dd^c u \wedge [V] + \sum_{p \in S} \nu(V,p) \geq d^2 \ .$$

So $dd^c u \wedge [V] = 0$ on $\mathbf{C}^2 \setminus S$ and $\nu(V,p) = 1$ for $p \in S$.

To prove the remaining assertions, we write $V = m_1 V_1 + \cdots + m_j V_j$, where V_i are the irreducible components of V and $m_i \geq 1$. If $S_1 = S \cap V_1 = \emptyset$ the curve $m_2 V_2 + \cdots + m_j V_j$ contains S and has degree less than d, which contradicts Lemma 5.5. So each S_i is non-empty. If $p \in S_i$ we have $\nu(V,p) \geq m_i$, so by (i) each $m_i = 1$ and $V = V_1 + \cdots + V_j$. Moreover $S_i \cap S_l = \emptyset$, since otherwise $\nu(V,p) \geq 2$ for $p \in S_i \cap S_l$. □

Proof of Proposition 5.6. We fix $F_1, F_2 \in I_d(S)$ linearly independent and let $V_j = \{F_j = 0\}$. In order to show $S = V_1 \cap V_2$ it suffices by Bezout to prove that V_1, V_2 have no common component. Assuming the contrary, we write $F_j = FP_j$, where F is the greatest common divisor of F_1, F_2. Then P_1, P_2 are linearly independent in $\mathcal{P}_{d-\deg F}$. If $S_1 = S \cap \{F = 0\}$ and $S_2 = S \setminus S_1$, we have, by the previous lemma, that there exists $p \in S_1$ and $S_2 = S \cap \{P_1 = 0\} = S \cap \{P_2 = 0\}$, as F_j have no repeated factors. Since $P_1(p) \neq 0 \neq P_2(p)$ we choose $c \in \mathbf{C}$ such that $P_1(p) + cP_2(p) = 0$. As $P_1 + cP_2 \not\equiv 0$, we can consider the curve V defined by $(P_1 + cP_2)F$. Then $S \subset V$, $\deg V = d$ by Lemma 5.5, and $F(p) = (P_1 + cP_2)(p) = 0$. This contradicts Lemma 5.7. $\square$

Theorem 5.8 *If $|S| = d^2$, $H(S, 2d-3) < d^2$, and $M(S) \neq \emptyset$, then S is the complete intersection of two algebraic curves of degree d.*

Proof. By the result of [EP] stated in Theorem 2.1 there are two possibilities: either S is the complete intersection of two curves of degree d in $\mathbf{P}^2$ (hence in $\mathbf{C}^2$ as well), or there exists a curve V of degree $0 < m < d$ in $\mathbf{P}^2$ such that $|V \cap S| \geq m(2d - m)$. Assuming the latter, we have $|V \cap S| > md$ since $m < d$. As $M(S) \neq \emptyset$ it follows as in the proof of Theorem 3.4 that $|V \cap S| \leq d \deg(V \cap \mathbf{C}^2) \leq dm$, a contradiction. $\square$

We have by [Co] that $M(S) = \emptyset$ if $|S| \in \{2, 3, 5, 6\}$.

Proposition 5.9 *If $|S| \in \{7, 8\}$ then $M(S) = \emptyset$.*

Proof. Let p_j denote the points of S. Assume first $|S| = 7$. There exists a polynomial P of degree 3 which vanishes on S and has a zero of order ≥ 2 at p_1. Let V be the variety defined by P, so $\nu(V, p_1) \geq 2$. If $M(S) \neq \emptyset$ then, by Theorem 3.4, we obtain $8 \leq \sum_{p \in S} \nu(V, p) \leq 3\sqrt{7}$, a contradiction.

If $|S| = 8$ we can find a polynomial P of degree 6 which has zeros of order ≥ 2 at $p_2, \ldots, p_8$ and a zero of order ≥ 3 at p_1. If $V = \{P = 0\}$ and $M(S) \neq \emptyset$ we get $17 \leq \sum_{p \in S} \nu(V, p) \leq 6\sqrt{8}$, a contradiction. $\square$

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