Exploring univariate mixed polynomials

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Abstract

We consider mixed polynomials $P(z, \bar{z})$ of the single complex variable z with complex (or real) coefficients, of degree n in z and m in $\bar{z}$. This data is equivalent to a pair of real bivariate polynomials $f(x, y)$ and $g(x, y)$ obtained by separating real and imaginary parts of P. However specifying the degrees, here we focus on the case where m is small, allows to investigate interesting roots structures and roots counting; intermediate between complex and real algebra. Mixed polynomials naturally appear in the study of complex polynomial matrices and complex moment problems, harmonic maps, and in recent papers dealing with Milnor fibrations.

1 Introduction

An expression $P(z, \bar{z}) = \sum_{k=0..n} \sum_{j=0..m} a_{k,j} z^k \bar{z}^j$ where z and $\bar{z}$ are complex conjugated, is called a (univariate) mixed polynomial of bidegree (n, m). We will assume $m \leq n$ and concentrate on the case where m is small, in particular $m = 1$. Our aim is to study the roots in $\mathbb{C}$ of P. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and separating real and imaginary parts of P, i.e. writing $P = f(x, y) + ig(x, y)$ with $i^2 = -1$ and $z = x + iy$, we get a pair of real bivariate polynomials of degrees at most $n + m$. Conversely from a pair of bivariate polynomials $(f(x, y), g(x, y))$, letting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and $P = f + ig$, we get a univariate mixed polynomial. However, since the two representations are different, we can investigate interesting roots structures and develop algorithms, intermediate between complex and real algebra. This representation can be also used with several variables $(z - 1, .., z_l)$. It received a renewed interest with the works in Algebraic Geometry of [18], these authors investigated a new exotic sphere (à la Pham-Brieskorn), more recently Mutsuo Oka [14], thanks to mixed polynomials, answered a question of Milnor [11] on real generalizations of Milnor fibration theorem. Roots of mixed polynomials naturally appear when expressing that a complex polynomial matrix drops rank, see e.g. [1]. It also appears as Taylor expansions of non holomorphic deformations of solutions of wave or elasticity equations, see [13, 5]. They are central for the study of the complex moment problem [6]. We can also mention the study of real subvarieties of $\mathbb{C}^2$, among others by Moser and his collaborators [12]. Harmonic polynomial and rational maps are important special cases of mixed polynomials; they have been extensively studied and were applied to the study of gravitational lensing [8, 15].

Several techniques developed in Computer algebra are useful for understanding these objects. We revisit, from an algorithmic point of view, the roots study of pairs of real bivariate polynomials (f, g). The case $m = 1$ could be called "almost holomorphic", and we look for properties similar to those of "usual" univariate polynomials. Moreover after simplification it reduces to the study of $\bar{z} = r(z)$, where r is a rational rational map: we will briefly recall recent advances obtained in that field, [8, 17, 3].

One of our tool will be a variant of Vandermonde matrix that we will use to interpolate $P(z, \bar{z})$. Similarly, we will specify a set of roots in $\mathbb{C}$ and investigate the maximum number of other roots in $\mathbb{C}$ admitted by such a constrained mixed polynomial. Unfortunately, the presentation of a univariate polynomial as a product via its roots is not valid in this context. As we will see, although P of bidegree $(n, 1)$ has $2n + 2$ coefficients, it may admit more than $2n + 2$ roots in $\mathbb{C}$. We will discuss and illustrate this behavior, directly related to bounding the number of zeros of harmonic maps. Beside the case $m = 1$, the results obtained so far on harmonic polynomials, see e.g. [22, 21, 9], concentrated on m near n, while we are more attracted by small m. We will also describe, in small degrees, the partition in semi-algebraic cells of the coefficient spaces corresponding to a given number of roots: their shapes resemble to domains delimited by the generalized "swallow tails" used by R. Thom in his Catastrophes theory.

Another objects of interest are the mixed polynomials, of degrees $(n, 1)$, with given random distribution of coefficients. Experiments with the computer algebra system Maple allowed to observe interesting patterns.

The paper is organized as follows: In section 2, we give some examples and present general properties inherited from the two representations. Then, we construct generalized Vandermonde matrices and prove that they are generically invertible. In section 3, we consider the case $(n, 1)$, we prove that the only zero sets of a $(n, 1)$ mixed polynomial, having dimension one, are a circle or a line. Then, we describe the results recently obtained on zeros counting of rational harmonic maps. In section 4, we consider the case of real coefficients and discuss different representations of the input. In section 5, we investigate the effect of choosing the coefficients with several stochastic distributions.

We denote by $\bar{a}$ the complex conjugated of a complex number a, and by $\bar{P}$ the complex conjugated of a (mixed or usual) polynomial P, its coefficients are the complex conjugated of the coefficients of P.

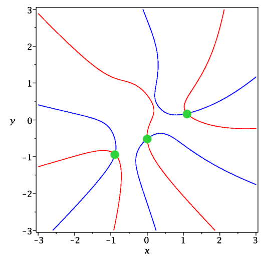

Figure 1: Example1

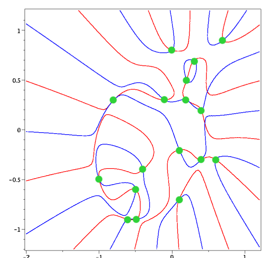

Figure 2: Example2

2 General properties

We begin with some examples of mixed polynomials and pictures of their roots.

Example 1 *A random mixed polynomial of bidegree* $(4,1)$

$$P{:=}(4{-}3i)z^4\bar{z}{+}(3{+}7i)z^4{+}(8i)z^3\bar{z}{+}(7{+}9i)z^3{+}({-}6{-}9i)z^2\bar{z}$$

$$+(6{-}3i)z^2{+}({-}5{-}6i)z\bar{z}{+}(1{-}7i)z{+}({-}5{-}9i)\bar{z}{+}4{+}2i.$$

It has 3 roots in $\mathbb{C}$ *shown in green in Figure 1. Writing* $P = f(x,y)+ig(x,y)$*, the implicit curves defined by* $f = 0$ *and* $g = 0$ *are shown in red and blue..*

Example 2 *A monic mixed polynomial of bidegree* $(8,1)$ *with 17 coefficients and 17 roots.*

Example 3 *An example of bidegree* $(4,2)$ *with real coefficients; it has 14 monomials that we do not display. Its 18 roots and the corresponding implicit curves are shown in Figure 2. Note that the non real roots appear by conjugated pairs.*

Example 4 *Examples of bidegree* $(1,1)$ *with no punctual roots.*

$$P = z\bar{z} + e$$

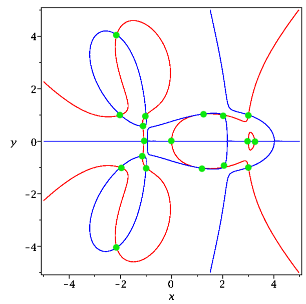

Figure 3: Example3

when $e = -1$*, the roots form a circle; while when* $e = 1$*,* P *has no root in* $\mathbb{C}$*.*

Now, we briefly review some properties of univariate mixed polynomials inherited by their representations.

2.1 Factorization

The product of two mixed polynomials $P_3 = P_1 P_2$ can be expressed by a set of algebraic conditions on their coefficients, identical to the set of conditions corresponding to "usual" bivariate polynomials with the same bidegrees. Therefore, the factorization properties and algorithms valid for bivariate polynomials, are also valid for univariate mixed polynomials.

2.2 Dimension

The real variety $V(P)$ defined in $\mathbb{C} = \mathbb{R}^2$ by $P = 0$, where P is an univariate mixed polynomial (non identically zero), can be either of dimension 1, 0 or -1 (i.e. $V(P)$ is empty).

In the first case, writing $P = f + ig$ as above, the bivariate polynomials $f(x, y)$ and $g(x, y)$ have a non constant gcd $h(x, y)$ which vanishes on a curve of $\mathbb{R}^2$. In other words this cannot happen if the gcd is constant, e.g. with probability 1 in a "random" case.

As we will see below, the third case cannot happen if the bidegree satisfy $n > m$.

So the most "common" case is the second one. If it is so, a natural question to ask is: what is the maximum number of roots for a given bidegree?

2.3 Fast Interpolation

Consider the simple case when $m = 1$, then P can be written $P = \bar{z}zP_1(z) + a\bar{z} + P_2(z)$. We first determine the coefficient a by evaluating at 0 the derivative $\frac{\partial}{\partial \bar{z}}P$, then subtracting the term $a\bar{z}$ we get a polynomial $Q =$

$P - a\bar{z} = |z|^2 P_1(z) + P_2(z)$. Now, choosing points of modulus 1 we can apply fast interpolation procedure to recover $Q_1 = P_1 + P_2$, similarly choosing points of modulus 2 we can recover $Q_2 = 4P_1 + P_2$, hence P.

Interpolation at other prescribed points will be addressed below, with generalized Vandermonde matrices.

2.4 Topological degree

Let $P = f + ig$ as above. At each of the root $z_j = (x_j, y_j)$, $j = 1..N$ of P we can attach the local topological degree of the map $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ at (x_j, y_j). Let us recall that the topological degree is defined as follows: since (f, g) is (locally) continuous and differentiable the image of a sufficiently small circle γ around (x_j, y_j) is a closed loop around $(f(x_j, y_j), g(x_j, y_j))$; the (signed) degree counts the number of turns of this loop (clockwise). For a simple root, this degree is equal to the sign of the (non vanishing) jacobian determinant of (f, g) at that root.

In particular, near a simple root z_0 of P, the local equation of P can be written $\Phi := \bar{z} - \phi(z) = 0$; by a well known formula, the jacobian of Φ is equal to $|\phi'(z)|^2 - 1$; hence this jacobian is negative if and only if $|\phi'(z)| < 1$, in other words ϕ is locally a contraction map. We will return below to this condition when we will present attractive fixed points, see 3.2.

Now, a sufficiently big circle, containing all the roots, can be viewed either as a circle "around infinity" or as a path around each root of P. So, one relates the degree "at infinity" to the sum of the local degrees at all roots of P. As observed by Oka [14], the degree at "infinity" for a univariate mixed polynomial of bidegree (n, m) is simply $n - m$. As a first consequence, if $n > m$, in particular if $m = 1, n > 1$, the zero set $V(P)$ of P cannot be empty. As a second consequence, in the "generic" case (for instance in the random case, as a claim with probability one) all roots of P are simple and the sum of the (signed) degrees is equal to $n - m$, this implies that the number of roots is $n - m + 2K$ where K is a natural integer.

2.5 Resultants

Instead of calculating the resultant of the real representation $\big(f(x, y), g(x, y)\big)$ of $P(z, \bar{z})$ to study the variety $V(P)$, we can use another resultant which respect the structure of $P(z, \bar{z})$.

As $P(z, \bar{z}) = 0$ iff $\overline{P}(\bar{z}, z) = 0$, the complex roots of $P(z, \bar{z})$ can also consider as roots of the pair of "usual" polynomials $P(z, w)$ and $\bar{P}(w, z)$, such that $w = \bar{z}$.

The elimination of a variable in

$$\begin{cases} P(z, w) = 0 \\ \overline{P}(w, z) = 0. \end{cases} \tag{1}$$

leads to a "biprojectif" resultant of degree n^2+m^2, a consequence of multi-projective Bézout theorem, see [20]. The number of solutions $z_j=(x_j,y_j)$ in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ is n^2+m^2.

Notice that the elimination of one variable in the system of polynomials equations $f(x,y)=0$ and $g(x,y)=0$ leads to $(n+m)^2$ solutions (x_j,y_j) in $\mathbb{P}^2(\mathbb{C})$. With this representation, the size of the Sylvester matrix is $2(n+m)$ whereas in the case of the resultant in (z,w) the size is $n+m$.

2.6 Vandermonde matrices

In this section, we consider the interpolation problem for finding the $N=(n+1)(m+1)$ coefficients of a univariate mixed polynomial $P(z,\bar{z})$ of bidegree (n,m), knowing its values at N points $w_j,\ j=1..N$ of $\mathbb{C}$. Writing the corresponding linear constraints, we obtain a square (N,N) complex matrix which is a generalization of the classical Vandermonde matrix. Its determinant Δ is a (not identically zero) multivariate mixed polynomial but, in this setting, it is not true that when the N points w_j are pairwise distinct, Δ does not vanish.

Since we are more interested by characterizing P by its roots, we will consider variants of that problem. First, we fix to 1 the constant coefficient (we could similarly have fixed the highest bidegree coefficient) to get rid of the trivial solution. Then for the case of the simple roots problem, we force P to vanish on $N-1$ points $w_j,\ j=1..N-1$. While for the case of simple and double roots problem, we force P to have a simple root at $N-1-2K$ points and a double root (with a specified direction) at K other pairs of points and directions (w_j,θ_j) $j=1..K$, with $w_j\in\mathbb{C}$ and $\theta\in[0,\pi[$.

To ease the presentation, we consider separately the two cases and skip the study of the interpolation problem which is very similar to the first case. With the same approach, we can consider mixed polynomials with real coefficients.

2.6.1 Simple roots

Given $N-1$ distinct points $w_l,\ l=1..N-1$, or equivalently a point $W\in\mathbb{C}^{N-1}$, and sorting the pairs $(k,j),\ k=1..N-1, j=1..N-1$ lexicographically, we construct the $(N-1,N-1)$ square matrix A whose l-th row is formed by the evaluation at w_l of the monomial $z^k\bar{z}^j$. Let us denote by $\Delta(W)$ its determinant.

Proposition 1 *$\Delta(W)$ is a non identically zero mixed polynomial (in several variables).*

Proof: It is clear that $\Delta(W)$ is a mixed polynomial. We will show that it admits a higher derivative in $(z,\bar{z})$ non identically equal to zero.

Since $\Delta(W)$ is a determinant, its derivatives are linear function of its rows. For a fixed l, observe that the derivative, with respect to $(w_l, \bar{w}_l)$, of a row where the variables $(w_l, \bar{w}_l)$ do not appear, is just a zero row. While the highest order derivative, with respect to $(w_l, \bar{w}_l)$, of a row where the only appearing variables are $(w_l, \bar{w}_l)$, is a row with a non zero constant entry and all the other entries are zero.

We proceed by induction. We first consider the higher derivative $\Delta_1(W)$ of order n in w_1 and order m in $\bar{w}_1$ of Δ. By the previous observations, it is a determinant of a matrix similar to A but where the first row has been replaced by $n!m!$ times the unit row $(1, 0, ..., 0)$: a 1 followed by $N-2$ zeros. Hence $\Delta_1(W)$ does not depend on $(w_1, \bar{w}_1)$ and is equal to $n!m!$ times the determinant of the first principal $(N-2, N-2)$ sub matrix of A.

The argument can be repeated, and the proposition is proved by induction. •

When $n = m = 1$, three points w_1, w_2, w_3 in $\mathbb{C} \setminus 0$ determine a unique circle or a unique line (if they are aligned).

When $n = 2, m = 1$, five points w_1, w_2, w_3, w_4, w_5 in $\mathbb{C} \setminus 0$ on a circle centered at the origin of radius R, satisfy $w_l \bar{w}_l = R^2$, hence $w_l^2 \bar{w}_l = R^2 w_l$, for all $l = 1..5$. In other words, with the previous notations, the first and the fourth column of the determinant Δ are proportional; hence $\Delta = 0$. This indicates that, unlike the usual polynomial case, the zero locus of Δ can be rather complicated. Here the "bad" points W corresponded to a factorization of a mixed polynomial P of bidegree $(2, 1)$ into a mixed polynomial of bidegree $(1, 1)$ $(z\bar{z}/R^2 - 1)$ and a polynomial of bidegree $(1, 0)$.

2.6.2 Simple and double roots

We consider the case when we impose a double root at w in the direction $u = e^{i\theta}$. Infinitesimally, this amounts to consider the limit situation where P vanishes at w and at $w + \epsilon u$ when ϵ tends to zero. We keep the first row $L_1 = (w^k \bar{w}^j)_{k,j}$ like in the previous matrix A, but we replace the following row $L_2 = [(w + \epsilon u)^k (\bar{w} + \epsilon \bar{u})^j)_{k,j}]$ by the limit L_2' of the linear combination $(L_2 - L_1)/\epsilon$ when ϵ tends to zero. More precisely we have $L_2' = [(kw^{k-1}u\bar{w}^j + jw^k \bar{u}\bar{w}^{j-1})_{k,j}]$. Factoring out $\bar{u} = e^{-i\theta}$, we obtain

$$L'(w, \bar{w}, \theta) = e^{-i\theta}[kw^{k-1}e^{2i\theta}\bar{w}^j + jw^k\bar{w}^{j-1}]$$

for $k = 1..N-1, j = 1..N-1$.

We denote by B this second generalization of the Vandermonde matrix corresponding to the case where we force P to have a simple root at $N-1-2K$ points $w_{2K+l}, l = 1..N-2K-1$, and a double root at K other pairs of points and directions $(w_l, u_l = e^{i\theta_l})$ $l = 1..K$. B is also a square $(N-1, N-1)$ matrix, its first $2K$ rows $L_1, L_1', L_2, L_2', ...L_K'$ are modeled as described in the previous paragraph. Dividing out each L_l' by the corresponding $e^{-i\theta_l}$ we obtain a determinant that we denote by $\Delta(W, \Theta)$. To prove that it is not identically zero, it is sufficient to exhibit one of its higher derivative which is not identically zero. We follow roughly the same argument than in the previous subsection. Here we first perform the K differentiations with respect to $\theta_1, .., \theta_K$ and divide out by the factors $2ie^{2i\theta_l}, l = 1..K$; which amounts to get another determinant $\Delta'(W)$ where the K rows L'_l have been replaced by the K rows $(kw_l^{k-1}\bar{w_l}^j)_{k,j}$. Hence we got rid of the variables θ_l.

Now, we perform on Δ' the maximum higher differentiation with respect to w_1 and $\bar{w_1}$ i.e. $2n-1$ times with respect to w_1 and $2m$ times with respect to $\bar{w_1}$; we obtain a determinant which first row is $n!m!$ times the unit row and the other rows are unchanged. Again, we perform the maximum higher differentiation with respect to w_1 and $\bar{w_1}$ so, we obtain a constant times a sub-principal minor of Δ', of two orders less, which does not contain neither w_1 nor $\bar{w_1}$.

We can iterate the argument, and the generalization of the previous proposition to the case of double roots is proved by induction. •

2.6.3 Number of roots

As a consequence of the previous subsections, for $n > 1$ and for a "generic" (i.e. in the complement of an hypersurface) set W of $mn + m + n$ simple or double points (counted with multiplicities), there exist a unique mixed polynomial P of bidegree (n, m), with a punctual zero set, that vanishes on W. However, Example 3 shows that P can have more than $mn + n + m$ roots. By biprojective Bezout theorem, applied to $(P, \bar{P})$, we know that the number of solutions is less or equal to $n^2 + m^2$.

This raises the question: What is the maximal number $M(n, m)$ of roots of a mixed polynomial of bidegree (n, m) having a punctual zero set ?

In the sequel of this article, we will only consider mixed polynomials of bidegree $(n, 1)$, i.e. we assume $m = 1$.

3 Rational harmonic map

In this section, we consider the case of mixed polynomials of bidegree $(n, 1)$: $P(z, \bar{z}) := \bar{z}q(z) - p(z)$ with $deg(q) = n,\ deg(p) \leq n$; after a translation on z also called Tchirnhausen transform, we can assume $deg(p) \leq n - 1$. In the first subsection, we consider the case with dimension $V(P)$ equal one, then in the sequel of the paper, we will assume $n > 1$ and $gcd(p, q) = 1$ which implies that the dimension of $V(P)$ is zero.

3.1 Dimension of $V(P)$

We already observed that the mixed equation of a circle of center $a \in \mathbb{C}$ and radius R is the mixed polynomial of bidegree $(1, 1)$, $Q_{a,R} := z\bar{z} - \bar{a}z - a\bar{z} + |a|^2 - R^2 = 0$. In that case the imaginary part $g(x, y)$ of $Q_{a,R}$ is identically zero. Similarly, the mixed equation of a general line in $\mathbb{C} \setminus 0$ is the mixed polynomial of bidegree $(1,1)$, $L_a := az + \bar{a}\bar{z} - 1 = 0$.

Multiplying one of these two equations by a "usual" polynomial $p(z)$ of degree $n-1$, we get a mixed polynomial $P(z,\bar{z})$ of bidegree $(n,1)$ such that its zero set has real dimension one. Indeed its zero set $V(P)$ contains a circle or a line.

Proposition 2 *The only possible curve contained in the zero set $V(P)$ of a mixed polynomial $P(z,\bar{z})$ of bidegree $(n,1)$ is either a circle or a line.*

Proof: 1) Clearly, $P(z,\bar{z})$ of bidegree $(1,1)$ can be equal to the real polynomial $f(x,y)$ of degree 1 if and only if f is the equation of a line.

2) $P(z,\bar{z})$ of bidegree $(1,1)$ can be equal to the real polynomial $f(x,y)$ of degree 2 if and only if f is the equation of a circle.

Indeed, the highest total degree form of P can be factorized over $\mathbb{C}$ as $F := (a_1z - a_2\bar{z})(b_1z - b_2\bar{z})$, but since its degree in $\bar{z}$ must be 1, b_1 or b_2 must be zero, say $b_2 = 0$. F is equal to the highest degree form of $f(x,y)$ then should be real; since it is equal to $a_1b_1(x^2 - y^2 + 2ixy) + a_2b_1(x^2 + y^2)$, then $a_1b_1 + a_2b_1$, $-a_1b_1 + a_2b_1$ and $i(a_1b_1)$ should be real. This is only possible if $a_1b_1 = 0$, hence $a_1 = 0$. Therefore the highest total degree form of P is $(a_2b_1\bar{z}z)$, which implies that f is the equation of a circle.

3) We similarly consider a polynomial $f(x,y)$ of degree $K+1 > 2$, equal to $f = P(z,\bar{z})$ of bidegree $(n,1)$.

The highest total degree form of P can be factorized over $\mathbb{C}$ as $F := (a_1z - b_1\bar{z})(a_2z - b_2\bar{z})..(a_{K+1}z - b_{K+1}\bar{z})$. As above, only one factor in $\bar{z}$ is allowed, so we can assume $b_2 = .. = b_{K+1} = 0$. Up to the multiplication by a constant, we get $F = (c+id)z - \bar{z})z^K$, that we expand in (x,y). We write $z^K = A(x,y) + iB(x,y)$ and $(c+id)z - \bar{z} = (cx - dy - x) + i(cy + dx + y)$.

We denote by e the values -1 or 1.

If K is even, then $A = x^K + ... + ey^K$ and $B = Kx^{K-1}y + ... + eKxy^{K-1}$ so x and y divide B but neither divides A.

Expanding the expression we must get a real form, in other words the imaginary part vanishes: $(cx - dy + x)B + (dx + cy - y)A = 0$. Then, the divisibility implies $c = 1, d = -1$, so $yB - xA = 0$ which is false, since we assumed $K > 1$.

If K is odd, a similar argument holds. Indeed, $A = x^K + ... + eKxy^{K-1}$ and $B = Kx^{K-1}y + ... + ey^K$, so x divides A but not B, and y divides B but not A. The divisibility implies $c = 1, d = 0$, so $yB - xA = 0$ which is false, since we assumed $K > 1$.

4) Now, suppose that P can be factorized into irreducible mixed polynomials: $P = P_1..P_r$ then $(n,1) = (n_1,1) + (n_2,0) + ..(n_r,0)$. Hence P_2, .. P_r should be "usual" polynomials and have only punctual roots. This implies that a potential zero set of dimension 1 corresponds to an irreducible mixed polynomial. In other words, the ones we just analyzed.

Therefore, the only possible zero sets of dimension 1 are a circle or a line. •

The corresponding property for $\bar{z} = r(z)$ was proved (differently) in [4].

From now on, we will only consider the case where $V(P) = 0$ and assume $n > 1$.

3.2 Counting roots of $\bar{z} = r(z)$

We assume $n > 1$, $deg(p) \leq n-1$, $gcd(p, q) = 1$, and $gcd(q, q') = 1$; $deg(q) = n$. The roots of $P(z, \bar{z}) = \bar{z}q(z) - p(z)$ are the roots of $\bar{z} = r(z)$, with $r(z) := \frac{p(z)}{q(z)}$. We will also write $r(z) := \sum_{j=1}^{n} \frac{\mu_j}{z - z_j}$, where z_j denotes the (distinct) roots of $q(z)$ and $\mu_j \in \mathbb{C}$. Counting the roots of $\bar{z} = r(z)$ has been an active field of research due to its interpretation in gravitational lensing, see e.g. [15] and important progresses have been achieved.

Theorem 1 ([8]) *The number $N(r, n)$ of roots of $\bar{z} = r(z)$ is bounded by $5n - 5$.*

Theorem 2 ([17]) *There exists a family of rational fractions r_n, $n > 1$ such that $N(r_n, n) = 5n - 5$.*

Theorem 3 ([3]) *There exists a family of rational fractions $r_{n,k}$, $n > 1, k = 0, .., 2n-2$, such that $N(r_{n,k}, n) = n - 1 + 2k$.*

Let us briefly comment these results. We already observed that $N(r, n) \leq n^2 + 1$ and that $N(r, n) = n - 1 + 2k$, by the count of topological degrees (see section 2). Let z_0 be a (simple) root of $P = f + ig$, hence of $z = \overline{r(z)}$. Then a straightforward computation shows that the topological degree at z_0 of (f, g) is 1, resp. -1, iff $|r'(z_0)| > 1$, resp. $|r'(z_0)| < 1$; moreover, z_0 is called sense preserving, resp. reversing, and z_0 is a repelling, resp. attractive, fixed point of the discrete dynamics $z_{l+1} := \overline{r(z_l)}$, $l \in \mathbb{N}$. Denoting by N_+ and N_- the numbers of attractive and repelling fixed points, we have $N_- = N_+ + (n-1)$, then $N(r, n) = N_+ + N_- = 2N_+ + n - 1$. Therefore, the first result reduces to prove that $N_+ \leq 2(n-1)$. The strategy, developed in [8], is to show that each of the N_+ attractive fixed point, also attracts at least $n + 1$ critical points of the rational fraction $Q(z) := \overline{r(\overline{r(z)})}$, which has $2(n^2 - 1)$ critical points.

Rhies'examples [17] are invariant under rotations centered at the origin of angle $\frac{2\pi}{n}$, in particular this makes the number of roots easy to count. They have a physical interpretation since they correspond to a configuration of equal masses ($\mu_j = \mu > 0$) equally spaced on a circle centered at the origin and another smaller mass at the origin.

The construction of generalizations of this configuration, in [3], proceeds for a fixed k, by induction on n, by adding a small enough mass which produces the expected effect but does not destroy the previous count, moving a little bit (almost infinitesimally) the previous roots.

The remaining question is: What happens far from these regular configurations and their small perturbations?

4 Real coefficients

In this section, we consider the case of mixed polynomials P with real coefficients. They are rather general while their investigation is easier.

4.1 Vandermonde

We first consider the case where we force a real mixed polynomial P of bidegree $(n, 1)$ to vanish at K distinct pairs of conjugated complex numbers $w_1, \bar{w_1}, ..., w_K, \bar{w_K}$ and $N-1-2K$ real numbers w_{2K+k}, with $N = 2n+1$. We also let the coefficient of $z^n \bar{z}$ be equal to 1.

If the corresponding generalized Vandermonde matrix is invertible, the unique solution P will have real coefficients, the reason is that $P(w_l, \bar{w_l}) = 0$ implies $\bar{P}(\bar{w_l}, w_l) = 0$ hence, $\bar{P}$ satisfies the same equations $\bar{P}(w_l, \bar{w_l}) = 0$ for $l = 1..N-1$, then by unicity $\bar{P} = P$.

The first $2K$ rows of the generalized Vandermonde determinant are made of K pairs of conjugated rows, hence can be replaced by rows formed by their real and imaginary parts. It turns out that we can find adapted higher differentiations to generalize the argument of the section 2.6.

Requiring only $r < 2n+1$ such linear independent conditions, we expect to obtain an affine space of dimension $n+1-r$.

4.2 Bounds and Points at infinity

The result of subsection 2.5 can be refined, as follows.

Proposition 3 *Let $P(z, \bar{z}) = f(x, y) + ig(x, y)$ be a mixed polynomial of bidegree (n, m) with real coefficients; where f and g are real bivariate polynomials. Let also $Y = y^2$. Then y is a factor of g, writing $g = y\hat{g}$, f and $\hat{g}$ are polynomials in (x, Y), that we denote by $\tilde{f}$ and $\tilde{g}$. Moreover, the number of solutions in $\mathbb{C}^2$ of $\tilde{f}(x, Y) = 0,\ \tilde{g}(x, Y) = 0$ is bounded by $\frac{n(n-1)}{2} + \frac{m(m-1)}{2}$.*

Proof:

1) Expanding any monomial $z^k \bar{z}^j = (x+iy)^k (x-iy)^j$, the real part is an even polynomial in y and the imaginary part is an odd polynomial in y. As a consequence, y is a factor of g, writing $g = y\hat{g}$, f and $\hat{g}$ are indeed polynomials in (x, Y).

2) Since P has real coefficient $\bar{P} = P$ and, as noticed in section 2, $P(z, \bar{z}) = 0$ is equivalent to the system $P(z, w) = 0,\ P(w, z) = 0,\ z = \bar{w}$. Moreover $P(z, w) = 0,\ P(w, z) = 0$ has at most $m^2 + n^2$ solutions where z

and w are complex, among them the $m+n$ complex solutions of $P(z,z)=0$. Substituting with z and w complex, $x=\frac{z+w}{2}$ and $y=\frac{z-w}{2i}$, hence $Y=y^2=\frac{(z-w)^2}{-4}$, in $\tilde{f}(x,Y)+iy\tilde{g}(x,Y)$, we get $P(z,w)=\tilde{f}(\frac{z+w}{2},\frac{(z-w)^2}{-4})+(z-w)\tilde{g}(\frac{z+w}{2},\frac{(z-w)^2}{-4})$. Similarly in $P(w,z)$. The two expression differ only by the sign of $z-w$.
Subtracting and adding $P(z,w)=0$, $P(w,z)=0$, these equations are equivalent to $\tilde{f}(x,Y)=0$, $y\tilde{g}(x,Y)=0$ and $Y=y^2$. Hence has the same number of solutions. The claim follows by an easy count. •

Consequences:

1. The resultant of $\tilde{f}(x,Y),\tilde{g}(x,Y)$ with respect to x or to Y has at most degree $\frac{n(n-1)}{2}+\frac{m(m-1)}{2}$.
2. The number of points at infinity of the complex variety defined by $(f=0,g=0)$, counted with multiplicity, is at least $2nm$.
3. The only point at infinity is (generically) $(0,1,0)$, in homogeneous coordinates (x,y,T), it has a multiplicity $2nm$.

4.3 Exploration tools

Several techniques developed in Computer algebra are useful for finding examples and investigating roots sets.

4.3.1 With (f,g)

We write, a mixed polynomial $P:=\bar{z}z^n+\sum a_jz^j+\bar{z}\sum b_jz^j$ then, $P=f(x,y)+ig(x,y)$. Since the $2n$ coefficients a_j and b_j are real, y is a factor of g. Moreover f and g/y are polynomials in (x,Y) with $Y=y^2$. We call $R(Y)$ the discriminant with respect to x of f and g/y. It is a polynomial in Y; its coefficients are polynomials in the coefficients a_j and b_j. We also consider $F(x)=f(x,0)$, which is a real polynomial of degree $n+1$. The roots of P in $\mathbb{C}$ are: first, the real roots of F and second, the real pairs (x,y) with $Y=y^2$ such that Y is a non negative root of R, then, generically, x can be written as a polynomial in Y, defined by the system $(f,g/y)$ (e.g. thanks to a Groebner basis, or with minors of the Sylvester matrix).

We consider a probabilist exploration tool as follows.
We first require that $x=0$ is a root of F if the degree of F is odd. We (repeatedly) choose randomly real values $x_1,x_2,..,x_{n-1}$; and real positive values $Y_1,Y_2,..,Y_{n-1}$, and we require that P vanishes at the $2(n-1)$ pairs of conjugated roots $(x_k+i\sqrt{Y},x_k-i\sqrt{Y})$, $k=1..n-1$. These constraints are independent with probability one. They define an affine subspace of

dimension 2 that we parameterize with two coefficients, say a_1 and b_0.

We denote by F_1 the evaluation of F (divided by x, if the degree of F is odd) and by R_1 the evaluation of R divided by $(Y - Y_1)..(Y - Y_{n-1})$.
We compute the discriminant A_1 of F_1, viewed as as a polynomial in x. Its graph is the image (by a change of coordinates) of a section of a "swallow-tail", studied in Catastrophe theory, it delimits portions of the plane (a_1, b_0) where $F1$ admits a fixed number of real roots.

Then, we compute the discriminant of R_1, viewed as as a polynomial in Y. It admits generically two factors: a squared polynomial in (a_1, b_0), and a simple polynomial in (a_1, b_0), that we denote by A_2. The graph of A_2 delimits regions where R_1 admits a fixed number of real roots. But, we need regions where these real roots are non negative.

4.3.2 Physical configurations

We consider $\bar{z} = r(z)$, with $r := \frac{p}{q} = \sum \frac{\mu_j}{z - z_j}$, where $gcd(q, q') = 1,\ gcd(p, q) = 1$; in the physical interpretation μ_j are masses and should be real positive. Moreover we will assume here that p, q have real coefficients. In particular this implies that if $\mu_j > 0$ and $z_k = \bar{z_j}$ then $\mu_j = \mu_k$.

The family of examples studied by [17], and by [3], used this representation.

One can find interesting other examples fixing all the input data except two and and express graphically the constraints, which define semi-algebraic sets.

4.3.3 Attractive fixed points

In the previous section, we have seen that, over $n - 1$, the number of roots of P increases as twice the number of roots at which the topological degree is -1. Therefore, one can prescribe roots (using a generalized Vandermonde matrix), leaving two parameters free and explore regions where the corresponding jacobian are negative. The evaluation of the jacobian at a point(x_0, y_0) is a quadratic form in the coefficients. So, we are lead to deal with regions delimited by conics in the parameter plane. Notice that when P has real coefficients, the sign of the jacobian is the same at two conjugated roots. See below an example with $n = 5$.

4.4 Examples with small $n > 1$

We consider examples different from Rhies'ones, i.e. with less symmetries.

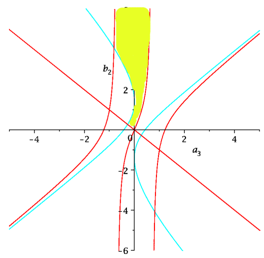

Figure 4: delimiting a region

To simplify the computation and the presentation, we set to 1 the leading coefficient and perform a translation on z, $z := z - \alpha$, (similar to the Tchirnhausen transform) in order to fix one term in the expansion of P.

4.4.1 $n = 2$

When $n = 2$, we have $2n + 1 = n^2 + 1 = 5n - 5 = 5$.

We can parameterize the coefficients of P by its roots, inverting the corresponding Vandermonde matrix. In the example of a "physical" configuration. with $z_1 = -1,\ z_2 = 2, \mu_1 = 2,\ \mu_2 = 1$, we obtain 5 solutions 3 real ones and a pair of conjugated ones.

4.4.2 $n = 3$

When $n = 3$, we have $2n + 1 = 7,\ n^2 + 1 = 5n - 5 = 10$. Looking for an example with 10 roots, we can fix 2 pairs of conjugated roots e.g. $\pm 1 + \pm i$ and a real root 0, hence 5 linear constraints on the coefficients; it remains 2 parameters, say a_3 and b_2. Figure 4 indicates the region of the plane (a_3, b_2) where the constrained P has 10 roots, it is symmetric with respect to 0, and half of it is colored in yellow.

Similarly, we constructed examples with any even number of roots from 2 to 10.

4.4.3 $n = 4$

When $n = 4$, we have $2n + 1 = 9,\ n^2 + 1 = 17,\ 5n - 5 = 15$.

We proceeded similarly choosing arbitrarily some coefficients then tunning the last two ones. We computed several examples with the maximum number of roots 15. In one of them P had 5 real roots, and 5 pairs of complex conjugated root; in another one it had 3 real roots, and 6 pairs of complex conjugated roots.

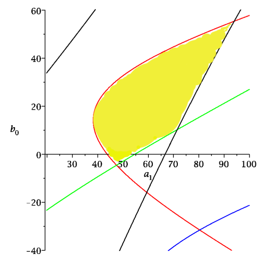

Figure 5: 4 conics

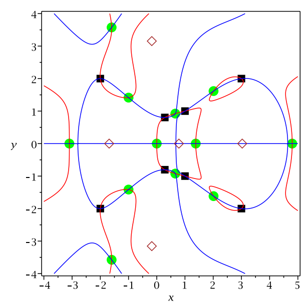

Figure 6: 20 roots

4.4.4 $n = 5$

When $n = 5$, we have $2n+1 = 11$, $n^2+1 = 26$, $5n-5 = 20$. We proceeded as explained in subsection 4.3.3: interpolating at the origin and at 4 pairs of conjugated roots, corresponding to $(x = 1, Y = 1), (x = -1, Y = 2), (x = -3, Y = 4), (x = 3, Y = 4)$, we ended with two free parameters a_0, b_0. Then, we constructed 4 conics defined by the jacobians of (f, g) at these 4 pairs. We delimited a region where 3 of the 4 jacobians were negative, see Figure 5. We chose in that region, after two trials, the value $a_0 = 70, b_0 = 40$ which corresponds to a mixed polynomial with 8 pairs of conjugated roots and 4 real roots, hence the maximum number of roots. These roots are shown in Figure 6: the graphs of real and imaginary parts are colored in red and blue, the 8 attractive fixed points of $\bar{z} = r(z)$ are indicated by black solid boxes and the 12 repelling ones by green solid discs, while the 5 poles of r are indicated by brown diamonds. Observe the distributions of the intersections points (and their color) on the different ovals.

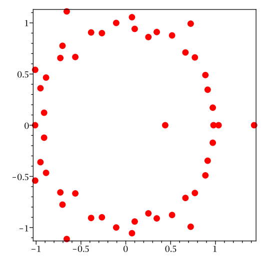

Figure 7: with uniform coefficients

5 Random Mixed polynomials

In this section, we fix a distribution law of real numbers (such as the normal Gaussian or the uniform, with mean zero). Then we study the roots of a mixed polynomial $P(z, \bar{z})$, which coefficients are chosen using this law.

A natural question is to find this expectation of the number of roots w.r.t. these choices. We made some statistics on different kinds of distributions. Recently, a related question has been studied theoretically in [16].

5.1 Uniform distribution

For $n = 20, 30, 31, 41$, we computed the roots of 100 mixed polynomials of bidegree $(n, 1)$, which coefficients are integers uniformly distributed in $-10..10$ and collected the number of roots. We obtained the following statistics:

For $n = 30$: in 67 cases we got $n - 1 = 29$ roots, in 30 cases $n + 1 = 31$ roots and in 3 cases 33 roots. The expectation is 29.7. For $n = 31$: in 69 cases we got $n - 1 = 30$ roots, in 30 cases $n + 1 = 32$ roots and in 3 cases 34 roots. The expectation is 30.8. For $n = 41$: in 68 cases we got $n - 1 = 40$ roots, in 28 cases $n + 1 = 42$ roots and in 3 cases 44 roots. The expectation is 40.7. We got similar observations with $n = 50$.
Figure 7 illustrate a typical distribution of the solutions, which roughly concentrate around the unit circle.

So, it seems that, for this uniform distribution of real coefficients, the average number of roots is about n. Moreover in two third of the cases, we get $n - 1$ i.e. the minimum number of roots respecting the lower bound provided by the topological degree (see section 2). Notice that for the "physical" case, [15] showed that the minimum number of roots is indeed $n + 1$ and not $n - 1$.

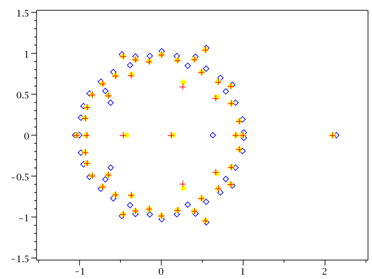

Figure 8: with equidistributed poles

5.1.1 Condensation

For "usual" univariate polynomials, when the coefficients are real and uniformly distributed, say in $-10..10$, the average number of roots is about $(2/\pi)Log(n)$, while when the size of the coefficients depends exponentially on the exponents then, as proved in [10], the number becomes a $O(n^s), 0 < s < 1$ and increase with the level of exponentiation; this phenomena was observed by Majundar and Scher in [10] who compared it to a condensation process. Therefore, one wonders if the same kind of behavior occurs for random mixed polynomials of bidegree $(n, 1)$. Our experiments indicate that it is not the case.

It would be interesting to find (random) distributions of coefficients which increase the number of attractive fixed points of the corresponding discrete dynamics; hence the number of roots of P.

5.2 Equidistribution of poles

We also considered a random case which approximate equidistribution of poles with same mass: We took a Kac type polynomial q of degree n such that its roots roughly concentrate near the circle centered at the origin of radius 1 (alternatively of radius 0.8) and its derivative p and considered the mixed polynomial $P = n\bar{z}q - p$; (alternatively we also experimented with $p = nz^{n-1}$). In all these cases we found about $n+1$ solutions near the same circle. Figure 8 shows a typical set of solutions of P in red crosses together with the n roots of q in blue diamonds and the roots of $p = \frac{\partial q}{\partial z}$ in yellow solid circles; (two more real solutions, approximately at -7 and 7, does not appear on the picture). Notice that the yellow circles and the red crosses are near: indeed most coordinates differ by less than 0.01.

On the picture, observe that the roots of such a random P are in the convex hull of the roots of q, like the roots of q' (as asserted by Gauss-Lucas theorem). Compare with the patterns described in [7].

Inverse problem: Given a random distribution of n equal masses, positioned at the roots of a polynomial $q(z)$. We observe the $n+1$ solutions Z_k of the equation $n\bar{z}q(z) - q'(z) = 0$. The problem of recovering $q(z)$ from the set $Z_k,\ k = 1..n+1$ can be addressed using approximate linear algebra.

6 Conclusion

In this paper we presented univariate mixed polynomials from a Computer algebra view point. After some general results, we concentrated on the case where the bidegree is $(n, 1)$ and the coefficients are real numbers, aiming generalizations of behaviors of "usual" univariate polynomials; the complex plane $\mathbb{C}$ included in $\mathbb{C}^2$ is viewed as a substitute of the real axis $\mathbb{R}$ included in $\mathbb{C} = \mathbb{R}^2$. The equation $\bar{z}q(z) - p(z) = 0$ is (generically) equivalent to the equation $\bar{z} = \frac{p(z)}{q(z)}$. This second equation has been extensively studied for its application in gravitational lensing, and important results were obtained, that we briefly reviewed. However from a computational point of view, much remain to be done, it would certainly be worthwhile to adapt to this setting, efficient roots isolation algorithms. Relying on resultants and generalized Vandermonde matrices, we constructed exploration tools and described some significant examples. Little is known on roots of mixed equations of bidegrees (n, m) with small m. We plan to generalize the exposed methods to address the study of the equation $\bar{z}^2 = r(z)$.

Also, the polynomial solutions of the bi-Laplacian Δ^2, called bi-harmonic functions, are of great interest since they have important applications, see e.g. [13]; they are real (and imaginary) part of mixed polynomials of the form $\bar{z}Q_1(z) + Q_2(z) + zQ_3(\bar{z}) + Q_4(\bar{z})$,

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