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THE FOURIER EXPANSION OF $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$

CHRISTIAN KASSEL AND CHRISTOPHE REUTENAUER

ABSTRACT. We compute the Fourier coefficients of the weight one modular form $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ in terms of the number of representations of an integer as a sum of two squares. We deduce a relation between this modular form and translates of the modular form $\eta(z)^4/\eta(2z)^2$.

1. Introduction

In this note we consider the $\eta$-product

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \prod_{n \geq 1} \frac{(1 - q^n)^2}{1 - q^n + q^{2n}},$$

where $q = e^{2\pi iz}$. Recall that $\eta(z)$ is Dedekind’s eta function

$$\eta(z) = e^{\pi iz/12} \prod_{n \geq 1} (1 - q^n).$$

The $\eta$-product $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ is a modular form of weight 1 and level 6. Since it is invariant under the transformation $z \mapsto z + 1$, it has a Fourier expansion of the form

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \sum_{n \geq 0} a_6(n) q^n,$$

where the Fourier coefficients $a_6(n)$ are integers. For general information on $\eta$-products, see [4, Sect. 2.1].

Our first result expresses $a_6(n)$ in terms of the number $r(n)$ of representations of $n$ as the sum of two squares, i.e. the number of elements $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 = n$. Observe that $r(n)$ is divisible by 4 for all $n \geq 1$ (for $n = 0$ we have $r(0) = 1$). The sequence $r(n)$ appears as Sequence A004018 in [5].

Theorem 1.1. For all non-negative integers $m$ we have

$$a_6(3m) = (-1)^m r(3m),$$

$$a_6(3m + 1) = (-1)^{m+1} \frac{r(3m + 1)}{4},$$

$$a_6(3m + 2) = (-1)^{m+1} \frac{r(3m + 2)}{2}.$$
Theorem 1.2. Set $j = e^{2\pi i/3}$. We have the following linear relation between weight one modular forms:

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \frac{1}{4} \frac{\eta(z)^4}{\eta(2z)^2} + \frac{1 - j}{4} \frac{\eta(z + 1/3)^4}{\eta(2z + 3/2)^2} + \frac{1 - j^2}{4} \frac{\eta(z + 2/3)^4}{\eta(2z + 1/2)^2}.$$ 

Both modular forms $\eta(z)\eta(2z)/\eta(6z)$ and $\eta(z)^4/\eta(2z)^2$ came up naturally in [3], where we computed the number $C_n(q)$ of ideals of codimension $n$ of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality $q$. Equivalently, $C_n(q)$ is the number of $\mathbb{F}_q$-points of the Hilbert scheme of $n$ points on a two-dimensional torus. We proved that $C_n(q)$ is the value at $q$ of a palindromic one-variable polynomial $C_n(x) \in \mathbb{Z}[x]$ with integer coefficients, which we computed completely (see [3] Th. 1.3).

We also showed (see [3] Cor. 6.2) that the generating function of the polynomials $C_n(x)$ can be expressed as the following infinite product:

$$(1.3) \quad 1 + \sum_{n \geq 1} \frac{C_n(x)}{x^n} q^n = \prod_{n \geq 1} \frac{(1 - q^n)^2}{1 - (x + x^{-1})q^n + q^{2n}}.$$ 

It follows from the previous equality that $C_n(1) = 0$. Actually, we proved (see [3] Th. 1.3 and 1.4) that there exists a polynomial $P_n(x) \in \mathbb{Z}[x]$ such that $C_n(x) = (x - 1)^2 P_n(x)$. Moreover, $P_n(x)$ is palindromic, has non-negative coefficients and its value at $x = 1$ is equal to the sum of divisors of $n$: $P_n(1) = \sum_{d \mid n} d$.

When $x = e^{2\pi i/k}$ with $k = 2, 3, 4$, or 6, then $x + x^{-1} = 2 \cos(2\pi/k)$ is an integer. For such an integer $k$, we define the sequence $a_k(n)$ by

$$(1.4) \quad \sum_{n \geq 0} a_k(n) q^n = \prod_{n \geq 1} \frac{(1 - q^n)^2}{1 - 2 \cos(2\pi/k)q^n + q^{2n}}.$$ 

Since $2 \cos(2\pi/k)$ is an integer, so is each $a_k(n)$. It follows from (1.3) that these integers are related to the polynomials $C_n(x)$ by

$$C_n(e^{2\pi i/k}) = a_k(n) e^{2\pi i n/k}.$$ 

In [3] we computed $a_2(n), a_3(n),$ and $a_4(n)$ explicitly in terms of well-known arithmetical functions. In particular, we established the equality

$$(1.5) \quad a_2(n) = (-1)^n r(n),$$ 

where $r(n)$ is the number of representations of $n$ as the sum of two squares.

We also observed in [3], (1.8)] that

$$(1.6) \quad \sum_{n \geq 0} a_2(n) q^n = \frac{\eta(z)^4}{\eta(2z)^2} \quad \text{and} \quad \sum_{n \geq 0} a_6(n) q^n = \frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)}.$$ 

The question of finding an explicit expression for $a_6(n)$ had been left open in [3]. This is now solved with Theorem 1.1 of this note. In view of this theorem, of (1.5), and of (1.6), for all $m \geq 0$ we obtain

$$a_6(3m) = a_2(3m),$$

$$a_6(3m + 1) = a_2(3m + 1)/4,$$

$$a_6(3m + 2) = -a_2(3m + 2)/2.$$
Lemma 2.1. Let \( n \) be a positive integer which is not divisible by \( E \). We had experimentally observed (see [2, Note 7]) that \( a_6(n) = 0 \) whenever \( a_2(n) = 0 \). As a consequence of (1.7) we can now state that \( a_6(n) = 0 \) if and only if \( a_2(n) = 0 \), i.e. if and only \( n \) is not the sum of two squares.

Theorems 1.1 and 1.2 will be proved in the next two sections.

Remarks 1.3. (a) The sequence \( a_6(n) \) is Sequence A258210 in [5]. The sequence \( a_6(3n + 1) \) is probably the opposite of Sequence A258277 in loc. cit.

(b) It can be seen from Table 1 that \( a_6(n) \) is not a multiplicative function. Indeed, \( a_6(10) \neq a_6(2)a_6(5) \) or \( a_6(18) \neq a_6(2)a_6(9) \) or \( a_6(20) \neq a_6(4)a_6(5) \).

Table 1. First values of \( a_6(n) \)

| \( n \) | \( 1 \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| \( a_6(n) \) | -1 | -2 | 0 | 2 | 0 | -2 | -4 | 2 | 0 | 0 | -2 | 0 | 0 | 1 | 4 | 4 | 0 | -4 |

2. Proof of Theorem 1.1

2.1. For any odd integer \( m \) we set \( \xi(m) = -2 \sin(m\pi/6) \). Because of the well-known properties of the sine function, \( \xi(m) \) depends only on the class of \( m \) modulo 12 and we have the following equalities for all odd \( m \):

\[
\xi(-m) = -\xi(m) \quad \text{and} \quad \xi(m + 6) = -\xi(m),
\]

which is equivalent to \( \xi(-m) = -\xi(m) \) and \( \xi(6 - m) = \xi(m) \).

We have

\[
\xi(m) = \begin{cases} 
-1 & \text{if } m \equiv 1, 5 \pmod{12}, \\
-2 & \text{if } m \equiv 3 \pmod{12}, \\
1 & \text{if } m \equiv 7, 11 \pmod{12}, \\
2 & \text{if } m \equiv 9 \pmod{12}.
\end{cases}
\]

Next consider the excess function \( E_1(n; 4) \) defined by

\[
E_1(n; 4) = \sum_{d|n, d \equiv 1 \pmod{4}} 1 - \sum_{d|n, d \equiv -1 \pmod{4}} 1.
\]

It is a multiplicative function, i.e., \( E_1(mn; 4) = E_1(m; 4) E_1(n; 4) \) whenever \( m \) and \( n \) are coprime. It is well known that the excess function can be computed in terms of the prime decomposition of \( n \). Write \( n = 2^r p_1^{a_1} p_2^{a_2} \cdots q_1^{b_1} q_2^{b_2} \cdots \), where \( p_i, q_i \)'s are distinct prime numbers such that \( p_i \equiv 1 \pmod{4} \) et \( q_i \equiv 3 \pmod{4} \). Then \( E_1(n; 4) = 0 \) if and only if one of the exponents \( b_i \) is odd. If all \( b_i \)'s are even, then

\[
E_1(n; 4) = (1 + a_1)(1 + a_2) \cdots.
\]

In the sequel we will need the following result.

Lemma 2.1. Let \( n \) be a positive integer which is not divisible by 3. We have

\[
\sum_{d|n, d \text{ odd}} \xi(d) = -E_1(n; 4) \quad \text{and} \quad \sum_{d|n, d \text{ odd}} \xi(3d) = -2E_1(n; 4).
\]
We have \( a \) becomes
\[
\sum_{d \mid n, \ d \text{ odd}} \xi(d) = \sum_{d \mid n, \ d \equiv 3 \pmod{4}} 1 - \sum_{d \mid n, \ d \equiv 1 \pmod{4}} 1 = -E_1(n; 4).
\]

Similarly, \( \xi(3d) = \xi(3) = -2 \) if \( d \equiv 1, 5 \) and \( \xi(3d) = \xi(9) = 2 \) if \( d \equiv 7, 11 \pmod{12} \). Therefore,
\[
\sum_{d \mid n, \ d \text{ odd}} \xi(3d) = \sum_{d \mid n, \ d \equiv 3 \pmod{4}} 2 - \sum_{d \mid n, \ d \equiv 1 \pmod{4}} 2 = -2E_1(n; 4).
\]

\( \square \)

2.2. We now express \( a_6(n) \) in terms of the function \( \xi \) introduced above.

\textbf{Proposition 2.2.} We have
\[
(2.4) \quad a_6(n) = \sum_{d \mid n, \ d \text{ odd}} \xi \left( \frac{2n}{d} - d \right).
\]

Note that \( 2n/d - d \) is an odd divisor of \( n \).

\textbf{Proof.} Set \( u = \pi/k \) and \( \omega = d \) in Formula (9.3) of [2, p. 10]. It becomes
\[
(2.5) \quad \sum_{n \geq 0} a_k(n) q^n = 1 - 4 \sin(\pi/k) \sum_{n \geq 1} \left( \sum_{d \mid n, \ d \text{ odd}} \sin \left( \frac{2n}{d} - d \pi/k \right) \right) q^n.
\]

Consider the special case \( k = 6 \) of (2.5). Since \( \sin(\pi/6) = 1/2 \), Equality (2.5) becomes
\[
\sum_{n \geq 0} a_6(n) q^n = 1 - 2 \sum_{n \geq 1} \left( \sum_{d \mid n, \ d \text{ odd}} \sin \left( \frac{2n}{d} - d \pi/6 \right) \right) q^n
\]
\[
= 1 + \sum_{n \geq 1} \left( \sum_{d \mid n, \ d \text{ odd}} \xi \left( \frac{2n}{d} - d \right) \right) q^n.
\]

The formula for \( a_6(n) \) follows.

\( \square \)

\textbf{Proof of Theorem 1.1.} Let us first mention the following well-known fact (see [1, § 51, Th. 65]): the number \( r(n) \) of representations of \( n \) as a sum of two squares is related to the excess function \( E_1(n; 4) \) by
\[
(2.6) \quad r(n) = 4 E_1(n; 4)
\]
for all \( n \geq 0 \). It follows from this fact and from (1.5) that
\[
(2.7) \quad a_2(n) = (-1)^n 4 E_1(n; 4).
\]

We now distinguish three cases according to the residue of \( n \) modulo 3.

(a) We start with the case \( n \equiv 1 \pmod{3} \). We have \( n = 3\ell + 1 \) for some non-negative integer \( \ell \). Since the odd divisors \( d \) of \( n \) are not divisible by 3, they must
satisfy \( d \equiv 1, 5, 7 \text{ or } 11 \pmod{12} \). Such divisors are invertible \( \pmod{12} \) et we have \( d^2 \equiv 1 \pmod{12} \). Consequently,

\[
\frac{2n}{d} - d = \frac{2nd^2}{d} - d = 2nd - d \pmod{12}.
\]

Hence,

\[
\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6dl + d) = ((-1)^d)^d \xi(d) = (-1)^d \xi(d)
\]

in view of (2.1). Therefore, by Proposition 2.2,

\[
a_6(n) = (-1)^f \sum_{d|n, d \text{ odd}} \xi(d).
\]

Together with Lemma 2.1 and (2.7), this implies

\[
a_6(n) = (-1)^f \cdot 1 E_1(n; 4) = (-1)^{n + \ell + 1} a_2(n)/4.
\]

Finally observe that \( n \) is odd (resp. even) if \( \ell \) is even (resp. odd). Therefore, \( a_6(n) = a_2(n)/4 \).

(b) Now consider the case \( n \equiv 2 \pmod{3} \). We have \( n = 3\ell + 2 \) for some non-negative integer \( \ell \). Again the odd divisors \( d \) of \( n \) must satisfy \( d \equiv 1, 5, 7, 11 \pmod{12} \) since they are not divisible by 3. Consequently, as above,

\[
\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6dl + 3d) = (-1)^d \xi(3d).
\]

By Lemma 2.1 and (2.7), we obtain

\[
a_6(n) = (-1)^f \sum_{d|n, d \text{ odd}} \xi(3d)
\]

\[
= (-1)^{f+1} 2E_1(n; 4) = (-1)^{n + \ell + 1} a_2(n)/2.
\]

Since \( n \) and \( \ell \) are of the same parity, we have \( a_6(n) = -a_2(n)/2 \).

(c) Finally we consider the case when \( n \) is divisible by 3. We write \( n = 3^N m \), where \( m \) is coprime to 3 and \( N \geq 1 \). Any odd divisor \( d \) of \( n \) is of the form \( d = 3^r s \) for some odd divisor \( s \) of \( m \) and \( 0 \leq r \leq N \). Since \( m \) and its divisors \( s \) are not divisible by 3 and since \( s \) is odd, we again have \( s \equiv 1, 5, 7 \text{ or } 11 \pmod{12} \). Recall that for such \( s \) we have \( s^2 \equiv 1 \pmod{12} \). Thus, for \( d = 3^r s \), we obtain

\[
\frac{2n}{d} - d \equiv (2 \cdot 3^{N-r} m - 3^r) s \pmod{12}.
\]

If \( r = 0 \), then \( 2n/d - d \equiv (6 \cdot 3^{N-1}m - 1)s \pmod{12} \). Therefore,

\[
\xi\left(\frac{2n}{d} - d\right) = \xi((6 \cdot 3^{N-1}m - 1)s) = (-1)^m \xi(-s) = (-1)^{m-1} \xi(s).
\]

in view of (2.1).

If \( 0 < r < N \), then \( 2n/d - d \equiv (6 \cdot 3^{N-r}m - 3^r)s \pmod{12} \). Therefore,

\[
\xi\left(\frac{2n}{d} - d\right) = \xi((6 \cdot 3^{N-r}m - 3^r)s) = (-1)^m \xi(-3^r s) = (-1)^{m-1} \xi(3^r s).
\]

Now, \( 3^r \equiv 3 \pmod{12} \) if \( r \) is odd, and \( 3^r \equiv -3 \) if \( r > 0 \) is even. Then by (2.1),

\[
\xi\left(\frac{2n}{d} - d\right) = (-1)^{m-r} \xi(3s).
\]
Now consider the case $r = N$. If $N$ is odd, then $3^N \equiv 3 \pmod{12}$ and

$$\xi \left( \frac{2n}{d} - d \right) = \xi ((2m - 3^N)s) = \xi((2m - 3)s).$$

Now, if $m$ is odd, then $m \equiv 1, 5, 7 \text{ or } 11 \pmod{12}$. We have $2m - 3 \equiv 7 \text{ or } 11 \pmod{12}$ and the multiplication by 7 or by 11 exchanges the sets $\{1, 5\}$ and $\{7, 11\}$. Since by (2.2) $\xi$ takes opposite values on such sets, we have $\xi((2m - 3)s) = -\xi(s)$. Consequently, $\xi(2n/d - d) = -\xi(s)$ when $m$ is odd.

If $m$ is even, then $m \equiv 2, 4, 8 \text{ or } 10 \pmod{12}$. Then $2m - 3 \equiv 1 \text{ or } 5 \pmod{12}$. The multiplication by 1 or by 5 preserves each set $\{1, 5\}$ and $\{7, 11\}$, so that by (2.2) we have $\xi((2m - 3)s) = \xi(s)$. In conclusion,

$$\xi(2n/d - d) = (-1)^m \xi(s)$$

when $r = N$ is odd.

If $r = N$ is even, then $3^N \equiv -3 \pmod{12}$ and $\xi(2n/d - d) = \xi((2m - 3^N)s) = \xi((2m + 3)s)$. A reasoning as in the odd $N$ case shows that when $N$ is even we have

$$\xi(2n/d - d) = (-1)^{m-1} \xi(s).$$

We can now compute $a_6(n)$. We start with the case of odd $N$. Putting the above information together, we obtain

$$a_6(n) = \sum_{d|n, \text{ d odd}} \xi \left( \frac{2n}{d} - d \right) = \sum_{s|m, \text{ s odd}} \sum_{r=0}^N \xi \left( \frac{2 \cdot 3^N m}{3^s} - d \right)$$

$$= \sum_{s|m, \text{ s odd}} \left( (-1)^{m-1} \xi(s) + \left( \sum_{r=1}^{N-1} (-1)^{m-r} \right) \xi(3s) + (-1)^m \xi(s) \right)$$

$$= ((-1)^{m-1} + (-1)^m) \sum_{s|m, \text{ s odd}} \xi(s) = 0.$$

On the other hand, since the power of 3 in $n$ is odd, then by (2.3) we have $a_2(n) = (-1)^n E_1(n, 4) = 0$. Therefore, $a_6(n) = a_2(n)$ in this case.

If $N$ is even, then

$$a_6(n) = \sum_{d|n, \text{ d odd}} \xi \left( \frac{2n}{d} - d \right) = \sum_{s|m, \text{ s odd}} \sum_{r=0}^N \xi \left( \frac{2n}{d} - d \right)$$

$$= \sum_{s|m, \text{ s odd}} \left( (-1)^{m-1} \xi(s) + \left( \sum_{r=1}^{N-1} (-1)^{m-r} \right) \xi(3s) + (-1)^{m-1} \xi(s) \right)$$

$$= \sum_{s|m, \text{ s odd}} \left( (-1)^{m-1} 2 \xi(s) + (-1)^{m-1} \xi(3s) \right)$$

$$= (-1)^{m-1} \left( 2 \sum_{s|m, \text{ s odd}} \xi(s) + \sum_{s|m, \text{ s odd}} \xi(3s) \right)$$

$$= (-1)^m 4 E_1(m; 4)$$

by Lemma 2.1. Now, by multiplicativity of the excess fonction,

$$E_1(n; 4) = E_1(3^N; 4) E_1(m; 4) = E_1(m; 4)$$
It follows from (1.7) that
\[ f \text{ the right-hand side is equal to } \]
\[ \Box \]
Q.e.d.

3. Proof of Theorem 1.2

Set \( f(q) = \eta(z)\eta(2z)\eta(3z)/\eta(6z) = \sum_{n \geq 0} a_6(n) q^n \) and \( g(q) = \eta(z)^4/\eta(2z)^2 = \sum_{n \geq 0} a_2(n) q^n \); see (1.6). To prove Theorem 1.2 it suffices to check that
\[ f(q) = ag(q) + bg(jq) + cg(j^2 q), \]
where \( a = 1/4, b = (1 - j)/4, \) and \( c = (1 - j^2)/4. \) Now,
\[
ag(q) + bg(jq) + cg(j^2 q) = a \sum_{n \geq 0} a_2(n) q^n + b \sum_{n \geq 0} a_2(n) j^n q^n \\
+ c \sum_{n \geq 0} a_2(n) j^{2n} q^n \\
= (a + b + c) \sum_{m \geq 0} a_2(3m) q^{3m} \\
+ (a + jb + j^2 c) \sum_{m \geq 0} a_2(3m + 1) q^{3m+1} \\
+ (a + j^2 b + jc) \sum_{m \geq 0} a_2(3m + 2) q^{3m+2}.
\]

It follows from (1.7) that
\[
ag(q) + bg(jq) + cg(j^2 q) = (a + b + c) \sum_{m \geq 0} a_6(3m) q^{3m} \\
+ 4(a + jb + j^2 c) \sum_{m \geq 0} a_6(3m + 1) q^{3m+1} \\
- 2(a + j^2 b + jc) \sum_{m \geq 0} a_6(3m + 2) q^{3m+2}.
\]

The right-hand side is equal to \( f(q) \) since \( a + b + c = 1, a + j b + j^2 c = 1/4, \) and \( a + j^2 b + jc = -1/2. \) Q.e.d.

References

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