# Mathematics educator transformation(s) by reflecting on students' non-standard reasoning

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*In this study, we present some results stemming from a research work exploring the way in which prospective teachers develop their interpretative knowledge and awareness by discussing students' errors and non-standard reasoning. For this purpose, we designed a particular kind of task that was administrated and discussed in our own lectures. The discussions and reflections associated with this experience allowed us, as educators, to expand our own mathematical knowledge and awareness. Based on the analysis of video-recorded lessons delivered as a part of a course based in Italy, we will argue that work grounded in discussing students' naive ideas/non-standard reasoning represents a core aspect of the mathematics teachers' education field, whereby educators are also viewed as learners.*

**Keywords**: Teacher's knowledge, educator's knowledge, interpretation, pupils' productions.

## INTRODUCTION

This work is a part of a wider research project aimed at accessing and developing the knowledge mathematic teachers require for effective instruction. Part of this work pertains to designing and implementing contexts and tasks suitable for promoting the development of such knowledge. In our previous work, the focus was mainly on prospective teachers' answers and reasoning when solving a specific task. In this paper, we expand on this early work and promote a discussion on our own reflections upon the implementation and analysis of this particular task and the way of managing it. This particular task is essentially rooted in asking the (prospective) teachers to provide sense to students' productions, some of which can be considered incomplete, containing errors, or simply being grounded on nonstandard reasoning (e.g., Jakobsen, Ribeiro, & Mellone, 2014; Ribeiro, Mellone, & Jakobsen, 2013b). Our aim is to provide reflection on how this kind of task can promote mathematical knowledge development among (prospective) teachers and teacher educators.

Several historical examples show that the growth of mathematical knowledge often occurs through a dialectic process of "proofs and refutations," where initial, and often partially incorrect, hypotheses are progressively refined through a critical analysis of their consequences (Lakatos, 1976). Moving from a phylogeny to an ontogeny level, we can argue that this is similar to the learning process experienced by an individual. Adopting this perspective, a learner that makes an error can be compared to a person that got lost on his/her journey. Thus, if (s)he had an important meeting, (s)he will likely arrive late and agitated. On the other hand, if (s)he is a tourist who is visiting new places, getting lost may be perceived as an opportunity to discover new places that (s)he wouldn't have known otherwise (Borasi, 1994).

Grounded in some of our previous work (mentioned above), and starting from these reflections, we argue that the work on errors, incomplete answers, or non-standard reasoning should represent a core aspect in and for developing mathematics teacher education. Agreeing with Tulis (2013), we claim that teachers must be sensitive to students' errors and nonstandard reasoning. We also point to the fact that, in everyday classroom, a learning climate in which errors are perceived in a positive way should be established, allowing students to learn from their mistakes. The development of positive attitudes towards errors should be pursued from the beginning of mathematics teachers' professional development (assuming that it starts in teachers' initial training). With this aim, we

worked in our own courses with prospective teachers (and in other professional development contexts) using the previously mentioned particular type of task. This task has been used in our lessons as a prompt to orchestrate a mathematical discussion (Bussi, 1996). We used mathematical discussion as main tool to develop, together with our students, mathematical knowledge and new awareness of the opportunities and richness of learning experience that can come from the work on nonstandard reasoning. Such use is thus also aimed at building a co-learning community, throughout an inquiry community perspective (e.g., Jaworski & Goodchild, 2006).

## THEORETICAL FRAMEWORK

In mathematics teacher education, only a few approaches are using mistakes and nonstandard reasoning as a resource in Mathematics Education (Tulis, 2013). From our perspective, one of the core aspects of teaching practice should focus on developing teachers' knowledge that can assist them in giving sense to students' productions and perceiving errors as a learning opportunity (e.g., Ribeiro et al., 2013b). Such knowledge would allow teachers to develop and implement ways to support students in building knowledge that is founded in their own reasoning, even when such reasoning differs from that expected by the teacher. Aiming at accessing and developing such knowledge and ability, we have been developing tasks for teacher training that require them to solve problems before trying to give sense to students' productions aimed at answering such problems. Thus far, these tasks have been used as a tool for both observing and deepening the access to prospective teachers' mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008). Driven by this prompt, we also aim to support, in our lectures, the development of prospective teachers' MKT. During this research work, we have been focusing on a particular kind of knowledge we refer to as *interpretative knowledge* (Jakobsen et al., 2014). It is essential for teachers to possess, as it entails the knowledge in and for making sense of students' solutions and helps teachers provide productive feedback to them (in the sense discussed by Bruno & Santos, 2010).

In particular, we recognize the peculiar and specific nature of this knowledge and thus consider it a part of the Specialized Content Knowledge (SCK) domain of the MKT model, while also recognizing its links to the Pedagogical Content Knowledge (PCK). Findings of our previous studies indicated that a poor Common Content Knowledge (CCK) compromised the prospective teachers' ability to give sense to students' solutions that differed from their own. Indeed, we found evidence in support of our hypothesis that a lack of common knowledge on a particular mathematical topic hinders the prospective teachers in forming a flexible perception of that topic, making it difficult to move to different visions and their potential use in teaching (Ribeiro et al., 2013b; Jakobsen et al., 2014).

At this time, our research work is moving to another aspect we consider intrinsically involved in the kind of mathematical activity developed in teacher training—our growth and development as mathematics teacher educators. For this reason, we are complementing the previous focus on MKT and on the interpretative knowledge with the Inquiry community perspective (see, for example, Jaworski & Goodchild, 2006). According to this approach,

> Didacticians have designed activity to create opportunity to work with teachers, to ask questions and to see common purposes in using inquiry approaches that bring both groups closer in thinking about and improving mathematics teaching and learning . . . . This design process is generative and transformative. (Jaworski & Goodchild, 2006, p. 354)

The principal aim of this approach is to work with teachers as co-learning professionals, with didacticians and teachers each contributing with their specialist knowledge in order to collaboratively develop new knowledge in practice (e.g., Schön, 1987; Wagner, 1997). By using this type of tasks in our lectures, and paying attention to the response we receive from the attendees, we, as educators, are living a transformative experience, derived from participating in what we consider a co-learning community with our students. The idea of Inquiry Community is rooted in the Activity Theory Framework (Vygotsky, 1978), comprising of a subject, an object, and a mediation between them. In our case, the subjects involved are the community of prospective teachers and teacher educators. The mediation corresponds to the task, whereby the prospective teachers are asked to give sense and feedback to students' solutions. Finally, the outcome is the MKT development of both prospective teachers and mathematics teacher educators. The considered perspective for professional

development of teacher educators is grounded, from one side, in reflecting and discussing upon our own practices (e.g., Avalos, 2011). On other side, complementarily, we also assume that being a teacher educator involves much more than applying the skills of school teaching to a different context (e.g., Loughran, 2014). In our view, it requires a specialized type of (complementary) knowledge that the teachers need (e.g., Superfine & Li, 2014) in order to expand teacher trainers' vision of teaching. At this stage, working with students must be different from working with teachers, and should focus on deepening the "hows," the "whys" and the connections among and within topics.

## METHOD

The context of this study is the mathematics courses in teacher education, in particular some courses where we (the authors) were the lectures. In this paper, for brevity, we limit our focus to the data pertaining to two classes of the course held in Italy. This course is taught in the third year of a five-year program of the master degree in education, and can be taken upon passing the Foundation of Mathematics exam.

In the course, different tasks were explored, focusing on problem solving aimed at exploring particular mathematics education issues in depth (e.g., arithmetical, algebraic, and geometrical thinking) and the students' MKT pertinent to such issues. Complementarily, the course also focused on more general mathematics education approaches (e.g., sociocultural nature of learning, semiotic mediation process), among other goals. The entire course was conceptualized assuming the Inquiry Community Perspective, with the aim of promoting the development of a co-learning community. Around 100 prospective primary teachers participated in these classes. In addition, one educator/researcher was responsible for delivering the course, while one researcher took responsibility for the data collection (gathering the prospective teachers' written answers, as well as audio and video recording the classes). This was the second course in which this task was implemented and pertinent data gathered.

During the first lesson of the activity, a questionnaire containing the task was given to prospective teachers, asking them to answer it individually within one hour. The task required them to solve a very "simple" problem. They were given the following instructions: *Teacher Maria wants to explore with her students some notions concerning fractions. For this purpose, she has prepared a sequence of tasks involving five chocolate bars. Let us look at one of them: What amount of chocolate would each child get if we divide five chocolate bars equally among six children?*

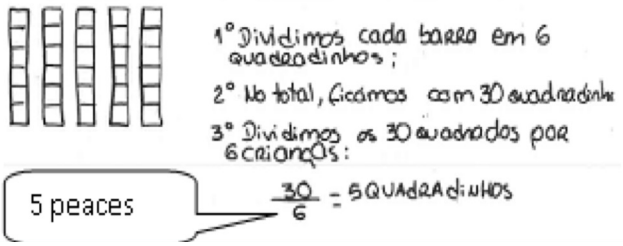


**Figure 1:** Ricardo's solution "1º We divide each bar in six squares; 2º In total, we have 30 squares; 3º We divide 30 squares among the 6 children. 30/6=5 squares"

After completing the task (i.e., providing their own answer to the aforementioned problem), we asked the participants to consider some students' solutions to the same problem. Some responses contained errors while others were incomplete, and some involved a nonstandard approach to problem-solving. Figure 1 and 2 present two of those responses, while a broader discussion of the productions included in the questionnaire is given in Ribeiro, Mellone, and Jakobsen (2013a). In particular, we asked prospective teachers to give sense to a set of pupils' productions, while following specific requirements: (i) For each pupil's production, decide if you consider it mathematically correct (adequate) or not, and justify the (in)adequacy of the mathematical rationality shown; (ii) Give a constructive feedback to pupils, especially to students whose answers you consider inadequate, and create a set of possible questions in order to promote their mathematical knowledge development.Data from the questionnaire was analyzed in terms of prospective teachers' own solutions to the problem, taking into consideration the different types of answers given,

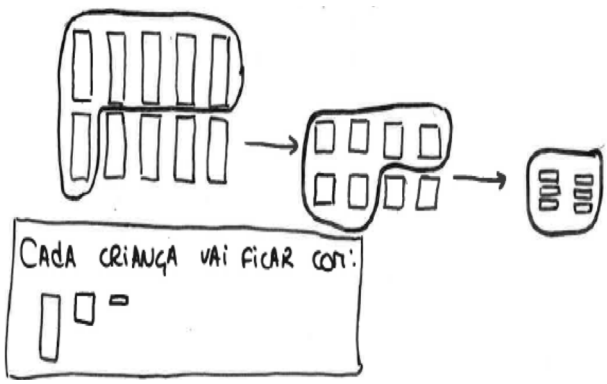


**Figure 2:** Mariana's solution "Each child will get"

the number of representations used, and their evaluations of the pupils' solutions (for further details, see Ribeiro et al., 2013b). The subsequent lesson, grounded in the collective mathematical discussion (Bussi, 1996) of the task was video recorded and analyzed in terms of prospective teachers' and educators' own reflections, discoveries, and points of turn in the MKT development. The fact that we, as educators, are considered an integral part of the inquiry community, in particular community of learners, is the key point of this paper. Nonetheless, this same fact represents a sensitive point of the method we chose for our analysis, since we analyzed "ourselves." However, in this case, we analyze the Italian experience. Thus, at the beginning of the lesson, the Italian researcher was chosen to provide her point of view on some crucial points of the mathematical discussion that took place. In the next stage, we conducted a joint analysis, followed by joint discussion and reflection.

## SOME EVIDENCE OF THE TRANSFORMATIVE EXPERIENCE

The transformative experience is a complex process and there are several considerations to take into account. Here, we will focus on two examples of the kind of reflections that are driving our experience, as well as our reflections and discussions. These examples were chosen because they are perceived as constituting crucial and critical points of the mathematical discussion undertaken. In particular, we focus on some of the mathematical discussion that took place when commenting on Mariana's solution (Figure 2). It should be noted that prospective teachers' answers to the questionnaire revealed that some found interpretation of this solution particularly difficult. This finding is in line with the reports of Norwegian (Jakobsen et al., 2014) and Portuguese experiences. Among the prospective teachers' comments to this solution, the following answers were particularly interesting:

> Mariana's solution is not understandable, so the first question would be: what does this representation mean? After I have listened to her answer, I will try to show her my own representation and we will reach the solution together"; or
> "She does not understand fractions – she is just dividing the pieces.

Comments such as these reveal the difficulty these prospective teachers had in leaving their own space of solutions. In particular, when this space consists of a single element (Jakobsen et al., 2014), it seems impossible for prospective teachers to appreciate and understand different solution strategies students adopt. As a result, they are unable to exploit them to support children's deeper knowledge development on the subject. Many of the prospective teachers responded in a similar way. A particular case concerns those that provided answers using only natural numbers, similar to Ricardo's solution (Figure 1), or saying "*each will get 5/6 of each chocolate bar.*"

The collective discussion promoted some changes in reasoning and argumentation. Here, it is worth noting the difficulties in orchestrating a mathematical discussion within a community of about 100 individuals. Thus, in order to facilitate constructive discussion, Mariana's production was projected on the wall and was thus clearly visible to all participants, while the educator focused on identifying prospective teachers that wanted to comment on the solution (using a microphone). After some minutes of discussion, in which most prospective teachers expressed their difficulties in understanding Mariana's reasoning (including how she obtained 10 bars), something changed. This is evident in Miriam's contribution to the discussion.

> Miriam: She basically takes the five bars and divides them in half and so she has ten pieces, so she gives six, while four remain. Then she divides the others in half again, and then there are eight and she assigns six, so two remain. The other two are divided into three parts, so six more pieces. Then finally, she says: every child will have a half of bar, plus the half of the half of a bar and a third of a bar (she stopped)
> –voices of students who want to intervene–
> Educator: Wait a moment, give her time...
> Miriam: mmm ... it's as if ... a third of the half of the half.

It is worth pointing out that Miriam, along with the majority of the prospective teachers that took part in the discussion, did not previously understand Mariana's solution. However, by building her own reflections on those of her colleagues, she was able not only to understand something that was not clear during the individual work, but also to recognize in the pieces of Mariana's representation the particular

fractions representing parts of the unit (the chocolate bar) involved. In the next step, the educator proposed to all participants to write down the explanation Miriam provided. The aim of this task was to prompt the prospective teachers to understand and verify the equivalence between 5/6 and the sum of these particular fractions. The amazement most prospective teachers felt upon discovering the meaning of Mariana's solution was the prompt to build an interesting discussion about the fact that, just because one does not understand something, it does not mean that it is incorrect. Such prompt also facilitated exploring links between teachers' specialized and pedagogical content knowledge, as well as understanding its importance/role in practice (e.g., Ribeiro & Carrillo, 2011). One of the focal points of the task conceptualization and implementation is on exploring and developing students' awareness on the knowledge involved (nature and type) in and for elaborating constructive feedback. In taking this approach, we were deepening some aspects of Bruno and Santos's (2010) work on written feedback.

While discussing and reflecting upon Mariana's production, prospective teachers experienced some contingency moments (Rowland, Huckstep, & Thwaites, 2005), allowing us to reflect and discuss upon the work we had done thus far. Such reflection is linked to the development of a mathematical knowledge we, as mathematics teacher educators, experienced by sustaining our own professional development. The situation presented here was driven by a comment made by Francesca, another prospective teacher. Indeed, after recognizing the correctness of the subdivision of the bars presented in Mariana's solution, most prospective teachers also saw the possibility to create numerical representations of the quantities indicated within the drawings. However, another issue was raised by Francesca.

Francesca: Yes, I think it is mathematically correct, because there are six children. But, if I had seven children, for example, I don't know if this division into equal parts could work. However, it was a trial and error process and this time it succeeded; but I do not know if, with other numbers, it could work.

Educator: So . . . you are saying that this procedure does not seem to be applicable to other numbers ...

Francesca: I do not know, it amazes me, but perhaps it could not work with other numbers.

At that moment, the educator was not prepared for such a comment and did not prompt a discussion that would explore this issue further. However, Francesca's comment was a focus of discussion and reflection in the scope of the research group, and served as a starting point for developing a new mathematical awareness. Indeed, on one hand, according to Empson, Junk, Dominguez, and Turner (2006), Mariana's solution can be seen as a *progressive parts strategy* considering it without any anticipatory organization of the subdivision. Yet, on the other hand, such solution reflects a peculiar management of subdivisions, with the potential for being generative of a precious mathematical insight. One can argue that Ricardo's solution (Figure 1) is grounded in the understanding/assumption that, in order to equally share five chocolate bars, each bar can be partitioned into 6 parts, thus perceiving $\frac{5}{6}$ as equivalent to $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$. Linked to this strategy, in which it is possible to recognize a kind of anticipatory thinking (Empson et al., 2006), there is the view of a fraction $\frac{n}{m}$ as equivalent to the sum of $n$ unitary fractions $\frac{1}{m}$. In contrast, Mariana's progressive partitions strategy suggests an interpretation of $\frac{5}{6}$ as equivalent to $\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$. Thus, it is impressive to see how this particular representation of a fraction as the sum of unitary fractions appears naturally in her reasoning. Her approach reveals the possibility to represent uniquely any fraction as a finite sum of decreasing rational numbers, such that the first element is the integer part of the fraction and each subsequent one is the greatest unitary fraction that is contained in the remaining part, which can be represented as $\frac{n}{m}=i+\frac{1}{q_0}+\frac{1}{q_1}+\ldots+\frac{1}{q_k}$. Such a representation has the strong advantage of showing clearly a sequence of rational numbers, simpler than the one assigned, that are, in a sense, the best lower approximations of it. Thus, after developing these reflections, during the following class, the educator had the opportunity to discuss the findings with the prospective teachers. The aim was to allow them to develop complementary elements to be included in their own space of solutions (e.g., Jakobsen et al., 2014).

## SOME FINAL COMMENTS

In this paper, some insights and reflections pertaining to a particular task were presented. The discussions prompted by Mariana's production have provided the opportunity to reflect on prospective teachers' knowledge elaboration on a particular aspect of fractions, as well as on our own role in the development of such knowledge and of our own awareness. However, we want to stress that, although such reflections were grounded in a task on fractions, the mathematical topic itself and the particular mathematical fact of "discovering" the possible link between Mariana's solution and the representation of a fraction as the sum of unitary fractions, are not the focus here. Indeed, as a meta-discourse, we are arguing something more general. The experience, and the way in which we presented it, is perceived as a contribution to the discussions and the associated reflections that highlight the need for a more focused work (and research) on the interpretative knowledge needed by (prospective) teachers and their educators, and the way(s) to promote it. Moreover, assuming that teachers and teacher educators need different "ways of hearing," we argue that different aspects and different natures of professional knowledge need to be taken into account when designing and implementing teacher training. The ultimate aim of this initiative is a joint and intertwined development of all the constituents of the inquiry community (Jaworski & Goodchild, 2006).

A significant number of diverse possibilities and paths for discussion were anticipated when we discussed and prepared the task and its implementation. Fortunately, some unforeseen situations emerged, leading to improvisations, some of which corresponded to contingency moments (Rowland et al., 2005). The reflections upon these situations, and the associated discussions on our own practice, have enabled us to develop a broader perspective on the process of teaching teachers, providing us with a deeper insight into what it requires and entails. Indeed, as we listened to the prospective teachers' comments on the proposed students' solutions, we also had some difficulties in interpreting and giving sense to their reasoning. This has led to mathematical critical moments, allowing us to reflect upon them. In our view, this is an essential aspect in and for promoting professional development for teachers as well as teacher trainers. In this sense, collaboration among the authors was crucial. By reflecting upon the discussion with prospective teachers, we could appreciate the type and nature of possible connections, representations, and navigations that could be made (needed) between and within topics. This has been one triggering event, leading to the awareness of the need to help teachers develop a much deeper knowledge. The work of Superfine and Li (2014) is a good example of an effort aimed at deepening and evolving this knowledge further.

Finally, we want to call attention to inextricable links among the tasks we prepare and implement (type, nature, and focus), the role of research on (prospective) teachers' knowledge and practices, and the learning opportunities we, as facilitators, provide. In order to also bring together theory and practice, we must recognize the importance of the role teacher trainers play in developing teachers' knowledge and practices. Thus, we argue that, if we want to enable prospective teachers to give sense to students' solutions and provide constructive feedback, we, as educators, need to do the same, albeit with a different focus. We posit that the courses specifically designed for teachers must have aims differing from those classes for students are geared towards. In this sense, a practice-based approach, in which prospective teachers can experience similar situations to those we expect them to encounter with their students is essential. We also recommend that teacher training address the role and the attitude of the educator, as one of the key elements. In particular, we recognize the importance and the need for teacher educators to live/work in terms of transformative experience, being sensitive to growing opportunity it offers.

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