SYMMETRIZATION OF RATIONAL MAPS: ARITHMETIC PROPERTIES 
AND FAMILIES OF LATTÈS MAPS OF $\mathbb{P}^k$

THOMAS GAUTHIER, BENJAMIN HUTZ, AND SCOTT KASCHNER

Abstract. In this paper we study properties of endomorphisms of $\mathbb{P}^k$ using a symmetric product construction $(\mathbb{P}^1)^k/\mathbb{S}_k \cong \mathbb{P}^k$. Symmetric products have been used to produce examples of endomorphisms of $\mathbb{P}^k$ with certain characteristics, $k \geq 2$. In the present note, we discuss the use of these maps to enlighten arithmetic phenomena and stability phenomena in parameter spaces. In particular, we study notions of uniform boundedness of rational preperiodic points via good reduction information, $k$-deep postcritically finite maps, and characterize families of Lattès maps.

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Introduction

In this paper we study dynamical properties of endomorphisms of $\mathbb{P}^k$, both from the arithmetic and the complex point of view. We mainly focus on the dynamics of a symmetric product construction $(\mathbb{P}^1)^k/\mathbb{S}_k \cong \mathbb{P}^k$. Symmetric products have been used to produce examples of endomorphisms of $\mathbb{P}^k$ with certain characteristics when $k \geq 2$ and their dynamics over the field $\mathbb{C}$ of complex numbers is rather simple and completely understood (see [DS] [FS] [U], for example). In the present note, we discuss the use of these maps to enlighten arithmetic phenomena and stability phenomena in parameter spaces. In particular, we study notions of uniform boundedness of rational preperiodic points via good reduction information, $k$-deep postcritically finite maps, and characterize families of Lattès maps (in particular those containing symmetric products).

Let $d \geq 2$ be an integer. We denote by $\text{Hom}_d(\mathbb{P}^k)$ the space of holomorphic endomorphisms of $\mathbb{P}^k$ of degree $d$. The space $\text{Hom}_d(\mathbb{P}^k)$ is known to be a smooth irreducible quasi-projective variety of dimension $N_d(k) := (k + 1)(\frac{d+k}{d!k!})-1$. More precisely, it can be identified with an irreducible Zariski open set of $\mathbb{P}^{N_d(k)}$, see e.g. [BB] §1.1.

Here, we mainly investigate properties of the following construction.

Definition 1. For any $f \in \text{Hom}_d(\mathbb{P}^1)$, we define the $k$-symmetric product of $f$ as the rational mapping $F: \mathbb{P}^k \to \mathbb{P}^k$ making the following diagram commute:

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In Section 1 we prove various basic properties of the dynamics of the map $F$ in relation to $f$. Most of the properties we give seem classical. On the other hand, we did not find any reference for the arithmetic properties we need. In the remaining sections, we apply this construction to the three following problems.

- First, we investigate the arithmetic properties of symmetric products, especially the Morton-Silverman conjecture concerning the number of rational preperiodic points.
- Next, we explore properties of postcritically finite symmetric product endomorphisms of $\mathbb{P}^k$.
- Finally, we focus on deformations of Lattès maps of $\mathbb{P}^k$. We pay a particular attention to those obtained as $k$-symmetric products.

**The Morton-Silverman Conjecture for symmetric products.**

Given a map $f \in \text{Hom}_d(\mathbb{P}^k)$, we define the preperiodic points to be those with finite forward orbit. Morton and Silverman conjectured the existence of a constant $C$ bounding the number of rational preperiodic points defined over a number field of degree at most $D$ that depends only on $d$, $D$, and $k$. More precisely, they proposed the following conjecture.

**Conjecture** (Morton-Silverman [MS]). Let $d \geq 2$ be an integer and $D, k \geq 1$ be integers. There exists a constant $C$ depending on $d$, $D$, and $k$ such that for any morphism $f \in \text{Hom}_d(\mathbb{P}^k)$ the number of preperiodic points defined over a number field of degree at most $D$ is bounded by $C$.

This conjecture is open as stated; however some progress has been made under additional hypotheses, such as assuming that $f$ has good reduction at a certain prime [B, H1]. We say that $f$ has good reduction at a prime $p$ if $f$ modulo $p$ is a morphism of the same degree as $f$. In Section 2 we prove the following theorem in relation to this problem.

**Theorem 2.** Let $d \geq 2$ be an integer and $k \geq 1$ be an integer. Let $K$ be a number field and $p \in K$ be a prime. Let $f \in \text{Hom}_d(\mathbb{P}^1)$ defined over $K$ with good reduction at $p$.

1. Let $P$ be a periodic point of minimal period $n$ for $f$ defined over a Galois extension of degree $k$ of $K$. Then,

$$n \leq \sum_{i=0}^{k} (Np)^i \cdot k \cdot N(p) \cdot p^e$$

where $p = (p) \cap \mathbb{Q}$ and

$$e \leq \begin{cases} 1 + \log_2(v(p)) & p \neq 2 \\ 1 + \log_{\alpha} \left( \frac{\sqrt{5} v(2) + \sqrt{5} (v(2))^2 + 4}{2} \right) & p = 2, \end{cases}$$

where $\alpha = \frac{1 + \sqrt{5}}{2}$.

2. If we assume the existence of a uniform Morton-Silverman constant over $K$ for the $k$-symmetric product of $f$ as $C_F(K)$, then we can bound the number of rational preperiodic points over any Galois extension of degree at most $k$ as

$$C \leq k \cdot C_F(K).$$
The first part of Theorem 2 is similar to the bound from [H1], which allows $f$ to be defined over the field extension. Which bound is better depends on the properties of $p$ and value of $k$. The second part shows that one need only consider the preperiodic points defined over the field of definition for uniform boundedness in $\mathbb{P}^1$ and implies a linear growth in number of rational preperiodic points with respect to the degree of the field of definition of the points for a fixed degree of the field of definition for the map.

Additionally, this construction provides a way to determine rational preperiodic points of fields of bounded degree using the algorithm from [H2]. We give examples of a quadratic polynomial with 21 rational preperiodic points over a degree 3 number field and a quadratic polynomial with a rational 5-cycle over a degree 5 number field (see Example 1 and Example 2). Finally, we use the $k$-symmetric product to give an algorithm to compute canonical heights of number fields where all of the computations are performed over $\mathbb{Q}$.

**Postcritically finite symmetric products.**

In Section 3 we consider another problem. Recall that $F \in \text{Hom}_d(\mathbb{P}^k)$ is postcritically finite if the postcritical set of $F$

$$\mathcal{P}(F) := \bigcup_{n \geq 1} F^n(\mathcal{C}(F)),$$

i.e., the forward orbit of $\mathcal{C}(F)$, the critical locus of $F$, is a strict algebraic subvariety of $\mathbb{P}^k$.

**Definition 3.** We say that $F \in \text{Hom}_d(\mathbb{P}^k)$ is 1-deep postcritically finite if $F$ is postcritically finite and $F_{(1)} := F|_{\mathcal{C}(F)}$ is postcritically finite, meaning the orbit under iteration of $F$ of the critical locus $\mathcal{C}(F_{(1)})$ of $F_{(1)}$ is an algebraic subvariety of pure codimension 2.

We say that $F$ is $(j+1)$-deep postcritically finite if it is $j$-deep postcritically finite and $F_{(j+1)} := F|_{\mathcal{C}(F_{(j)})}$ is postcritically finite.

We say that $F$ is strongly postcritically finite if it is $(k-1)$-deep postcritically finite.

Be aware that the notion of $j$-deep postcritically finite map differs from the notion of $j$-critically finite map defined by Jonsson [J]. Indeed, in his definition Jonsson requires critical points to be non-recurrent, hence non-periodic. We don’t exclude that possibility.

We focus on the following problem.

**Question.** Is a postcritically finite endomorphism of $\mathbb{P}^k$ necessarily strongly postcritically finite?

In the case when $F$ is a symmetric product, we give a positive answer to this question. This answers a question posed during the American Institute of Mathematics workshop *Postcritically finite maps in complex and arithmetic dynamics* in the specific case of symmetric products.

Namely, we prove the following theorem.

**Theorem 4.** Let $f \in \text{Hom}_d(\mathbb{P}^1)$ and $F$ be the $k$-symmetric product of $f$ for $k \geq 2$. Then $F$ is strongly postcritically finite if and only if $F$ is postcritically finite if and only if $f$ is postcritically finite.

We also provide a one-parameter family of degree 4 rational mappings $F_a : \mathbb{P}^2 \to \mathbb{P}^2$ that are postcritically finite but *not* strongly postcritically finite.

**Families of Lattes maps of $\mathbb{P}^k$.**

Denote by $\mathcal{M}_d(\mathbb{P}^k)$ the moduli space of endomorphisms of $\mathbb{P}^k$ with algebraic degree $d$, i.e. the set of $\text{PGL}(k+1)$-conjugacy classes of the space of degree $d$ endomorphisms $\text{Hom}_d(\mathbb{P}^k)$. The space $\mathcal{M}_d(\mathbb{P}^k)$ has been proven to have good geometric properties by Levy [L].

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The canonical height application was suggested by Joseph Silverman at the ICERM semester on complex and arithmetic dynamics in 2010.
In Section 4, we examine families of Lattès maps in the moduli space $\mathcal{M}_d(\mathbb{P}^k)$ of degree $d$ endomorphisms of $\mathbb{P}^k$. Our aim here is to have a complete description of non-trivial families of Lattès maps of $\mathbb{P}^k$ when $k \geq 2$: an endomorphism $f : \mathbb{P}^k \rightarrow \mathbb{P}^k$ is a Lattès map if there exists an abelian variety $\mathcal{T}$ of dimension $k$, an affine map $D : \mathcal{T} \rightarrow \mathcal{T}$ and a finite branched cover $\Theta : \mathcal{T} \rightarrow \mathbb{P}^k$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{D} & \mathcal{T} \\
\downarrow{\Theta} & & \downarrow{\Theta} \\
\mathbb{P}^k & \xrightarrow{f} & \mathbb{P}^k
\end{array}
$$

The complex dynamics of Lattès maps has been deeply studied in several beautiful papers (see e.g. [Mi, Z, BL1, BL2, BD1, Du]). The families and the perturbations of Lattès maps of $\mathbb{P}^1$ are also quite completely understood (see [Mi, BB, BG]).

Recall also that a family of degree $d$ endomorphisms of $\mathbb{P}^k$ is a morphism $f : \Lambda \times \mathbb{P}^k \rightarrow \mathbb{P}^k$, where $\Lambda$ is a quasiprojective variety of dimension $m \geq 1$ and for all $\lambda \in \Lambda$, the map $f_\lambda := f(\lambda, \cdot) : \mathbb{P}^k \rightarrow \mathbb{P}^k$ is an endomorphism of algebraic degree $d$. Equivalently, the map $\lambda \in \Lambda \mapsto f_\lambda \in \text{Hom}_d(\mathbb{P}^k)$ is a morphism of quasiprojective varieties. We also say that $(f_\lambda)_{\lambda \in \Lambda}$ is trivial if there exists a holomorphic map $\lambda \in \Lambda \mapsto m_\lambda \in \text{PGL}(k + 1, \mathbb{C})$ and $\lambda_0 \in \Lambda$ such that

$$f_\lambda = m_\lambda^{-1} \circ f_{\lambda_0} \circ m_\lambda, \quad \lambda \in \Lambda.$$

Denote by $\Pi : \text{Hom}_d(\mathbb{P}^k) \rightarrow \mathcal{M}_d(\mathbb{P}^k)$ be the canonical projection.

**Definition 5.** When $(f_\lambda)_{\lambda \in \Lambda}$ is a family of endomorphisms of $\mathbb{P}^k$, we say that it has dimension $q$ in moduli if the set $\{\Pi(f_\lambda) ; \lambda \in \Lambda\}$ is an analytic set of $\mathcal{M}_d(\mathbb{P}^k)$ of dimension $q$. We then set $\dim_{\mathcal{M}}(f_\lambda, \Lambda) := q$.

We focus on maximal families of Lattès maps containing a specific Lattès map $f$.

**Definition 6.** A family $(f_\lambda)_{\lambda \in \Lambda}$ of Lattès maps of $\mathbb{P}^k$ containing $f \in \mathcal{M}_d(\mathbb{P}^k)$ is maximal if for any family $(f_t)_{t \in X}$ of Lattès maps containing also $f$, we have $\dim_{\mathcal{M}}(f_\lambda, \Lambda) \geq \dim_{\mathcal{M}}(f_t, X)$.

When $k = 1$, the families of Lattès maps have been completely classified by Milnor [Mi]. He proves that a family $(f_t)$ of Lattès maps of $\mathbb{P}^1$ has positive dimension in moduli, i.e. is non-trivial, if and only if $d = a^2$ is the square of an integer $a \geq 2$ and the affine map inducing any map $f_{t_0}$ is of the form $z \mapsto az + b$. We aim here at giving a generalization of that precise statement.

Our main result on this problem is the following.

**Theorem 7.** Let $f \in \mathcal{M}_d(\mathbb{P}^k)$ be a Lattès map and let $D$ be an isogeny inducing $f$. Then, the dimension in moduli of any family $(f_\lambda)_{\lambda \in \Lambda}$ of Lattès maps containing $f$ satisfies $\dim_{\mathcal{M}}(f_\lambda, \Lambda) \leq k(k + 1)/2$. More precisely, if $(f_\lambda)_{\lambda \in \Lambda}$ is a maximal family containing $f$, then

1. either the family is trivial, i.e. $\dim_{\mathcal{M}}(f_\lambda, \Lambda) = 0$, in which case the eigenspaces of the linear part of $D$ associated with $\sqrt{d}$ and $-\sqrt{d}$ are trivial,

2. or $\dim_{\mathcal{M}}(f_\lambda, \Lambda) = q > 0$, in which case $\sqrt{d}$ is an integer and the linear part of $D$ has eigenspaces associated with $\sqrt{d}$ and $-\sqrt{d}$ of respective dimensions $q_+$ and $q_-$ with $(q_+ + q_-) \cdot \frac{q_+ + q_-}{2} = q$ for any $\lambda \in \Lambda$.

Beware that this result provides examples of families which are stable in the sense of Berteloot-Bianchi-Dupont [BB, BD], since the function $f \mapsto L(f)$, where $L(f)$ is the sum of Lyapunov exponents of $f$ with respect to its maximal entropy measure, is constant equal to $\frac{k}{2} \log d$, hence pluriharmonic on any such family of Lattès maps.
In Section 5, we come back to symmetric products and apply Theorem 7 to symmetric product Lattès maps. We give a precise description of maximal families of Lattès maps on \( \mathbb{P}^k \) containing the \( k \)-symmetric products of a rational map. More precisely, we prove the following consequence of Theorem 7.

**Theorem 8.** Let \( k \geq 2 \) be an integer and let \( d \geq 2 \) be an integer. Let \( F \in \Hom_d(\mathbb{P}^k) \) be the \( k \)-symmetric product of \( f \in \Hom_d(\mathbb{P}^1) \). Then

1. \( f \) is rigid if and only if \( F \) is rigid,
2. otherwise, \( F \) belongs to a family of Lattès maps which has dimension \( k(k+1)/2 \) in moduli.

In particular, \( F \) can be approximated by Lattès maps which are not symmetric products.

We give a more precise result in dimension 2, relying on the work of Rong \( \text{[R]} \). A few words are in order to describe this result. Rong’s result says that (up to taking an appropriate iterate) Lattès maps on \( \mathbb{P}^2 \) are either symmetric product or preserve an algebraic web associated with a cubic curve (see \( \text{[5]} \) for more details). In turn, our Theorem 8 says that any symmetric product Lattès map is either rigid, or contained in a family of Lattès maps of dimension 3 in moduli which symmetric product locus is a strict subfamily of dimension 1 in moduli.

Finally, we give examples of rigid Lattès maps of \( \mathbb{P}^2 \) and \( \mathbb{P}^3 \). We also provide an example of a non-rigid family of Lattès maps of \( \mathbb{P}^2 \). All three examples rely on Milnor’s famous examples of Lattès rational maps of degree 2 and 4 (see \( \text{[M]} \)).

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## 1. Dynamics of symmetric products

This section is devoted to a description of the dynamics of symmetric products. Some of the statements of the present section are known and classical, but many of the arithmetic properties are not in the literature.

Let \( \eta_k : (\mathbb{P}^1)^k \rightarrow \mathbb{P}^k \) be the holomorphic map defined as the quotient map of the action of symmetric group \( \mathfrak{S}_k \) on \( (\mathbb{P}^1)^k \) by permutting the terms of the product. In fact, one can write \( \eta_k = [\eta_{k,0} : \cdots : \eta_{k,k}] \) where \( \eta_{k,j} \) is the homogeneous degree \( k \) symmetric function given by the elementary homogeneous symmetric polynomials

\[
\eta_{k,j}([z_1 : t_1], \ldots, [z_k : t_k]) := \sum_{\ell \in \{0,1\}^k \atop \sum_{\ell_k = k-j} \ell_k} \prod_{\ell=1}^k z_{\ell_1}^{t_1} \cdots z_{\ell_k}^{t_k}.
\]

We collect some basic properties of \( \eta_k \) in the following proposition.

**Proposition 1.1.** Let \( \eta_k : (\mathbb{P}^1)^k \rightarrow \mathbb{P}^k \) be the \( k \)-th symmetric product map.

1. The map \( \eta_k \) is a finite branched cover of algebraic degree \( k! \) and topological degree \( k! \).
2. For any number field \( K \), the map \( \eta_k : M_d(\mathbb{P}^1)(K) \rightarrow M_d(\mathbb{P}^k)(K) \) is an embedding.

*Proof.*

1. see e.g. \( \text{[Ma]} \) for more details about symmetric products.
2. We need to show that a conjugate of \( f \) goes to a conjugate of \( F = \eta_k(f) \). Let \( \alpha \in \PGL(2) \). We denote the conjugate as \( f^\alpha = \alpha \circ f \circ \alpha^{-1} \). Choose \( k+2 \) points \( \{P_i\} \) in \( \mathbb{P}^1(K) \) so that no \( k+1 \) of \( \{\eta_k(P_i)\} \) are co-hyperplanar and no \( k+1 \) of \( \{\eta_k(f^\alpha)(P_i)\} \) are co-hyperplanar. Since these are both closed conditions, we can find such points. Define \( \beta \in \PGL(k+2) \) as the map that sends \( \{P_i\} \) to \( \{\eta_k(f^\alpha)(P_i)\} \). Then by construction we have

\[
\eta_k \circ f^\alpha = F^\beta \circ \eta_k.
\]
In particular, we have the following diagram with all squares commuting.

\[
\begin{array}{c}
P^1 \xrightarrow{f} P^1 \\
\downarrow \alpha \downarrow \alpha \quad \downarrow \eta_k \downarrow \eta_k \\
P^1 \xrightarrow{f} P^1 \\
\downarrow \eta_k \downarrow \eta_k \\
P^k \xrightarrow{F} P^k \\
\downarrow \beta \downarrow \beta \\
P^k \xrightarrow{F^\beta} P^k
\end{array}
\]

To show injectivity, assume there are two distinct \( f, g \) such that \( \eta_k(f) = \eta_k(g) \). Since the maps are distinct, there must be some point \( P \in \mathbb{P}^1 \) such that \( f(P) \neq g(P) \) but that \( \eta_k(f)(\eta_k(P)) = \eta_k(g)(\eta_k(P)) \). This is a contradiction, since the diagram commutes we must have \( \eta_k(f)(\eta_k(P)) = \eta_k(g(P)) \neq \eta_k(g)(\eta_k(P)) \).

\[\square\]

One can actually show that the \( k \)-symmetric product of \( f \) is holomorphic and enjoys good dynamical properties with respect to the ones of \( f \).

**Proposition 1.2.** For any \( f \in \operatorname{Hom}_d(\mathbb{P}^1) \), the \( k \)-symmetric product \( F \) of \( f \) is well-defined. Moreover, it satisfies the following properties.

1. \( F \) is an endomorphism of algebraic degree \( d \) of \( \mathbb{P}^k \).
2. If \( f \) is a degree \( d \) polynomial mapping of \( \mathbb{P}^1 \), then \( F \) is a degree \( d \) polynomial mapping of \( \mathbb{P}^k \), i.e., there is a totally invariant hyperplane.
3. If \( K \) is the field of definition of \( f \), then \( F \) is also defined over \( K \).
4. If \( K \) is the field of moduli of the equivalence class \([f] \in \mathcal{M}_d(\mathbb{P}^1)\), then \( K \) is the field of moduli of the equivalence class \([F] \in \mathcal{M}_d(\mathbb{P}^k)\).

**Proof.**

1. Since \( f \) can be written \( f([z : w]) = [P(z, w) : Q(z, w)] \) in homogeneous coordinates with \( P, Q \) homogeneous polynomials, the map \( \eta_k \circ (f, \ldots, f) : (\mathbb{P}^1)^k \to \mathbb{P}^k \) is invariant under the diagonal action of \( \mathfrak{S}_k \). In particular, there exists homogeneous degree \( d \) polynomials \( P_0, \ldots, P_k \) such that

\[\eta_k \circ (f, \ldots, f) = [P_0 : \cdots : P_k] \circ \eta_k\]

and the map \( F \) exists as a rational mapping of \( \mathbb{P}^k \). Now, remark that

\[\bigcap_{j=0}^k \{ \eta_{k,j} \circ (f, \ldots, f) = 0 \} = \emptyset\]

by construction and if \( x \in \mathbb{P}^k \) is an indeterminacy point of \( F \), it means that \( \eta_k^{-1}\{x\} \in \{ \eta_{k,j} \circ (f, \ldots, f) = 0 \} \) for any \( 0 \leq j \leq k \), which is impossible. In particular, the map \( F \) has no indeterminacy points.

2. If \( f \) is a degree \( d \) polynomial, one can write \( f([z : t]) = [P(z, t) : t^d] \), where \( P \) is homogeneous of degree \( d \). One then can write \( \eta_k \circ (f, \ldots, f)([z_1 : t_1], \ldots, [z_k : t_k]) \) as follows:

\[
\prod_{\ell=1}^k P(z_\ell, t_\ell) : \sum_{j=1}^k t_j^d \prod_{\ell=1}^k P(z_\ell, t_\ell) : \cdots : \prod_{\ell=1}^k t_\ell^d
\]
Since \( \eta_{k,k}([z_1 : t_1], \ldots, [z_k : t_k]) = \prod_j t_j \), this gives \( P_k(x_0 : \cdots : x_k) = x_k^d \). Hence, the map \( F \) can be written in homogeneous coordinates \( F([x_0 : \cdots : x_k]) = [P_0(x_0, \ldots, x_k) : \cdots : P_{k-1}(x_0, \ldots, x_k) : x_k^d] \). Hence \( F^{-1}\{x_k = 0\} \subset \{x_k = 0\} \), i.e., the map \( F \) is a polynomial endomorphism of \( \mathbb{P}^k \).

(3) This statement is obvious since the map \( \eta_k \) is defined over \( \mathbb{Q} \).

(4) We define the field of moduli of \( g \in \text{Hom}_d(\mathbb{P}^k) \) is the smallest field \( K \) such that \( [g] \in \mathcal{M}_d(\mathbb{P}^k)(K) \). Equivalently, the field of moduli is the fixed field of the set

\[
G([g]) = \{ \tau \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : g^\tau = g^\alpha \text{ for some } \alpha \in \text{PGL}(k + 1, \overline{\mathbb{Q}}) \}.
\]

Since we know from Proposition 1.3 that \( \eta_k \) is an embedding of the moduli space for each element of \( G([f]) \), there is a corresponding element in \( G([F]) \) and the fields of moduli must be the same.

\[ \square \]

Let \( F \in \text{Hom}_d(\mathbb{P}^k) \) be defined over the field of complex numbers. Let \( \mu_F \) be the maximal entropy measure of \( F : \mathbb{P}^k \to \mathbb{P}^k \). Let also \( \pi_j : (\mathbb{P}^1)^k \to \mathbb{P}^1 \) be the projection onto the \( i \)-th coordinate. Recall that the sum of Lyapunov exponents of \( F \) with respect to \( \mu_F \) is the real number

\[
L(F) := \int_{\mathbb{P}^k} \log |DF| \mu_F,
\]

where \(| \cdot |\) denotes any hermitian metric on \( \mathbb{P}^k \). One of the many consequences of the work of Briend and Duval [BD2] is that for any \( F \in \text{Hom}_d(\mathbb{P}^k) \), we have \( L(F) \geq k \log \sqrt{d} \).

An easy result is the following. We include a proof for the sake of completeness.

**Lemma 1.3.** Let \( F \in \text{Hom}_d(\mathbb{P}^k) \) be the \( k \)-symmetric product of \( f \in \text{Hom}_d(\mathbb{P}^1) \). Then

\[
L(F) = k \cdot L(f).
\]

**Proof.** Let \( \tilde{f} := (f, \ldots, f) \). First, one easily sees that the probability measure

\[
\nu_{f,k} := \bigwedge_{j=1}^k (\pi_j)^* \mu_f
\]

is \( \tilde{f} \)-invariant and has constant jacobian \( d^k \). As a consequence, \( (\eta_k)_* (\nu_{f,k}) \) is invariant and has constant jacobian \( d^k \). Since the only probability measure having these properties of the maximal entropy measure of \( F \), we get

\[
\mu_F = (\eta_k)_* (\nu_{f,k}).
\]

Hence

\[
L(F) = \int_{\mathbb{P}^k} \log |\det DF| \mu_F = \int_{\mathbb{P}^k} \log |\det DF| \cdot (\eta_k)_* (\nu_{f,k})
\]

\[
= \int_{(\mathbb{P}^1)^k} ((\eta_k)^* \log |\det DF|) \cdot \nu_{f,k}.
\]

By construction of \( F \), the chain rule gives

\[
(\eta_k)^* \log |\det DF| = (\tilde{f})^* \log |\det D\eta_k| + \log |\det D\tilde{f}| - \log |\det D\eta_k|.
\]
As \( \nu_{f,k} \) is the intersection of closed positive \((1,1)\)-currents with continuous potentials on \((\mathbb{P}^1)^k\), it does not give mass to pluripolar sets. In particular, one has \( \log |\det D\eta| \leq L^1(\nu_{f,k}) \)

\[
L(F) = \int_{(\mathbb{P}^1)^k} \log |\det D\tilde{f}| \cdot \nu_{f,k} + \int_{(\mathbb{P}^1)^k} \left( (\tilde{f})^* \log |\det D\eta| - \log |\det D\eta| \right) \cdot \nu_{f,k}
\]

\[
= \sum_{\ell=1}^k \int_{(\mathbb{P}^1)^k} |f' \circ \pi_{\ell}| \cdot \nu_{f,k} + \int_{(\mathbb{P}^1)^k} \log |\det D\eta_k| \cdot (\tilde{f})_* (\nu_{f,k}) - \nu_{f,k}) .
\]

As \( \nu_{f,k} = \bigwedge_{j=1}^k (\pi_j)^* \mu_f \) is \((\tilde{f})_*\)-invariant, we have \((\tilde{f})_* (\nu_{f,k}) - \nu_{f,k} = 0 \) and Fubini gives

\[
L(F) = \sum_{\ell=1}^k \int_{(\mathbb{P}^1)^k} \log |f' \circ \pi_{\ell}| \cdot \nu_{f,k}.
\]

\[
= \sum_{\ell=1}^k \left( \prod_{j=1}^k \int_{(\mathbb{P}^1)^k} (\pi_j)^* \mu_f \right) \cdot \int_{(\mathbb{P}^1)^k} (\pi_{\ell})^* (\log |f'| \cdot \mu_f)
\]

\[
= \sum_{\ell=1}^k \left( \int_{\mathbb{P}^1} \mu_f \right)^{k-1} \cdot \int_{(\mathbb{P}^1)^k} (\pi_{\ell})^* (\log |f'| \cdot \mu_f)
\]

\[
= \sum_{\ell=1}^k \int_{\mathbb{P}^1} \log |f'| \cdot \mu_f = k \cdot L(f) ,
\]

which ends the proof. \( \square \)

2. Preperiodic points and canonical heights via symmetric products

In this section we modify slightly \( \eta_k \). Instead of taking the symmetric product of \( k \) copies \((f, \ldots, f)\) or \((P, \ldots, P)\), we work over a Galois field and take the symmetric product of the \( k \)-Galois conjugates. To preserve the embedding \( \mathcal{M}_d(\mathbb{P}^1) \to \mathcal{M}_d(\mathbb{P}^k) \) proven in Proposition 1.1, we need \( f \) to be defined over the ground field. This is because the elementary multi-symmetric polynomials have relations among them and if we need to consider \( \eta_k \) applied where both the coefficients of the map and the point have non-trivial conjugates, the resulting \( F \) must use multi-symmetric polynomials and is not a morphism. To avoid confusion, we specify this as \( \tilde{\eta}_k(P) = \eta_k(P_1, \ldots, P_k) \) where \( P_i \) are the Galois conjugates of \( P \). Note that requiring Galois fields is not overly restrictive as we can always extend to the Galois closure.

Remark. Note that \( \tilde{\eta}_k \) is a cover of topological degree \( k \) which sends the \( k \) Galois conjugates to a single point as opposed to \( \eta_k \) which is a cover of topological degree \( k! \).

**Lemma 2.1.** Let \( L \) be an extension of the number field \( K \). Let \( S \subset L \) be the set of primes of bad reduction for \( f : \mathbb{P}^1(L) \to \mathbb{P}^1(L) \) with \( f \) defined over \( K \). Then the primes of bad reduction for \( F = \tilde{\eta}_k(f) \) is the set \( \{ \mathfrak{p} \cap K : \mathfrak{p} \in S \} \).

**Proof.** Assume that \( \mathfrak{p} \) is a prime of bad reduction of \( f \). Then (modulo \( \mathfrak{p} \)) there is a point \( P = (x, y) \) such that \( f(x, y) = (0, 0) \) with not both \( x, y = 0 \). So, we also have

\[
F(\tilde{\eta}_k(P)) \equiv (0, \ldots, 0) \pmod{\mathfrak{p}\mathcal{O}_L \cap K}.
\]

Since one of \( x, y \neq 0 \), we have at least one coordinate of \( \tilde{\eta}_k(P) \neq 0 \) and, thus, \( F \) has bad reduction.

Now assume that \( F \) has bad reduction at a prime \( \mathfrak{p} \in K \). Then there is a point

\[
P = (\beta_0, \ldots, \beta_N)
\]
such that \( F(P) = (0, \ldots, 0) \) with not all \( \beta_i = 0 \). Thus, the polynomial defining the Galois conjugates \( \beta_i \) is not identically 0:

\[
\prod_{i=1}^{N}(y_iX - x_iY) = \beta_0X^N - \beta_1X^{N-1}Y + \cdots + (-1)^N\beta_NY^N.
\]

So there is at least one non-identically zero pair \((x_i, y_i)\) for which \( f(x_i, y_i) = (0, 0) \). Thus, \( f \) has bad reduction at a prime of \( L \) dividing \( p \).

**Proposition 2.2.** Let \( L \) be a Galois field of degree \( k \) over \( K \) and let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be defined over \( K \). The rational preperiodic points of \( F = \eta_k(f) \) come from \( \eta_k(P_1, \ldots, P_k) \), where \( P_i \) are preperiodic points for \( f \). Furthermore, if each \( P_i \) is periodic of exact period \( n_i \), then \( \eta_k(P_1, \ldots, P_k) \in \mathbb{P}^k \) is periodic of exact period dividing \( \text{lcm}(n_i) \) for \( F \).

**Proof.** Since \( \eta_k \circ f^n = F^n \circ \eta_k \), it is clear that preperiodic points go to preperiodic points and that periodic points of exact period \( n \) for \( f \) are periodic points of exact period dividing \( n \). □

Note that this allows one to find all the rational periodic points defined over a Galois field \( L \) with \([L : K] = k\) for a map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) defined over \( K \). In particular, this method searches over all possible degree \( k \) extensions simultaneously.

**Example 1.** Consider the map

\[
f : \mathbb{P}^1 \to \mathbb{P}^1
\]

\[
(x, y) \mapsto (x^2 - \frac{29}{16}y^2, y^2).
\]

The 3-symmetric product is

\[
F : \mathbb{P}^3 \to \mathbb{P}^3
\]

\[
(x_0, x_1, x_2, x_3) \mapsto (4096x_0^2 - 8192x_1x_3 - 22272x_3^2,
-14848x_0^2 + 4096x_1^2 - 8192x_0x_2 + 29696x_1x_3 + 40368x_3^2,
13456x_2^2 - 7424x_1^2 + 14848x_0x_2 + 4096x_2^2 - 26912x_1x_3 - 24389x_3^2,
4096x_3^2).
\]

We find the \( \mathbb{Q} \) rational preperiodic points of \( F \) and construct the corresponding number field preperiodic points for \( f \). Every preperiodic point defined over any number field of degree 3 is among these points. We find that over \( \mathbb{Q}(\alpha) \) where the minimal polynomial of \( \alpha \) is \( 64x^3 + 16x^2 - 16x + 23 \), there are 21 \( K \)-rational preperiodic points for \( f \).

Over \( \mathbb{Q} \) it is conjectured that there are at most 9 rational preperiodic points \([\mathbb{P}], \) and for quadratic fields it is conjectured there are at most 15 rational preperiodic points \([\mathbb{D}F\mathbb{K}] \).

**Example 2.** Consider \( f_c : \mathbb{P}^1 \to \mathbb{P}^1 \) a quadratic polynomial defined by \( f(x, y) = (x^2 + cy^2, y^2) \).

For \( c \in \{-2, -169, -64/9\} \) there is a rational 5-cycle defined over a degree 5 number field. The authors are not aware of any examples of 5-cycles for lower degree fields.

**Proposition 2.3.** Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be defined over \( K \) with periodic points \( P_1, \ldots, P_k \). We determine the characteristic polynomial of the multiplier matrices of \( \eta_k(f) \) for \( \eta_k(P_1, \ldots, P_k) \). We will denote the multiplier matrix for \( f \) at \( P \) as \( df_P \).

1. If a point with multiplier \( \lambda \) occurs \( m \)-times in \( (P_1, \ldots, P_k) \), then the characteristic polynomial of \( dF_{\eta_k(P_1,\ldots,P_k)} \) will contain

\[
(x - \lambda)(x - \lambda^2) \cdots (x - \lambda^m).
\]
(2) If \( \eta_k(P_1, \ldots, P_k) \) causes a \( nt \)-cycle to collapse to a \( t \)-cycle (this can occur when say the whole cycle is contained in \( \{P_1, \ldots, P_k\} \)), then the characteristic polynomial will contain \( (x^m - \lambda) \).

Proof. We first reduce to the case where the points in \( \{P_1, \ldots, P_k\} \) all have the same multiplier.

Let \( P = (P_1, \ldots, P_m, P_{m+1}, \ldots, P_k) \) where the \( P_i \) for \( 1 \leq i \leq m \) all have the same multiplier. Then we change coordinates so that the collection of points becomes

\[
P = (P_1, \ldots, P_m, (0:1), \ldots, (0:1)).
\]

The multiplier is a local object, so we can just consider the multiplier of \( \tilde{\eta} \). Denote \( \tilde{\eta} \) the restriction to the first \( m \) coordinates of \( F \). With \( P \) we have \( (\eta_k, \ldots, \eta_k, k, \ldots, k)(P) = (\eta_k, \ldots, \eta_k, 0, \ldots, 0) \). So the upper left \( k \times k \) block of the multiplier matrix of \( F \) is the same as the multiplier of \( \tilde{\eta} \). Since we can always rearrange coordinates, we are restricted to the case where all the multipliers are the same.

Now assume \( P = (P_1, \ldots, P_m) \) all have the same multiplier \( \lambda \) (and are fixed by \( f \)). We can change coordinates so that locally we have

\[
f = [\lambda x_1 + O(x_1^2), \ldots, \lambda x_m + O(x_m^2)].
\]

Note that locally the \( \eta_{m,i} \) are the elementary symmetric polynomials \( \sigma_i \). If we evaluate the \( \sigma_i \) at \( f \), it is clear that the leading terms are

\[
\eta_k(f) = [\lambda \sigma_1 + O(\sigma_1^2), \lambda^2 \sigma_2 + O(\sigma_2^2), \ldots, \lambda^m \sigma_m + O(\sigma_m^2)].
\]

Now assume \( P = (P_1, \ldots, P_m, \ldots, P_k) \) where \( P_1, \ldots, P_m \) form an \( m \)-cycle with multiplier \( \lambda \) for \( f \). In particular, \( \eta_k \) collapses this \( m \)-cycle to a fixed point. Then we use the simple fact that if we write \( f \) as a local power series

\[
f(x) = \lambda x + O(x^2)
\]

then

\[
f^m(x) = \lambda^m x + O(x^2).
\]

So that \( (f^m)'(P_1) = \cdots = (f^m)'(P_m) = \prod_{i=1}^m f'(P_i) = \lambda \). Since the \( P_i \) are distinct and \( f' \) is non-constant, we must have \( f'(P_i) \neq f'(P_j) \) for \( i \neq j \), so that \( \prod_{i=1}^m (x - f'(P_i)) = x^m - \lambda \). \( \square \)

We can now prove Theorem \( \text{[2]} \) about the boundedness of rational preperiodic points over an extension of the base field.

Proof of Theorem \( \text{[2]} \). We consider \( F = \tilde{\eta}_k(f) \in \text{Hom}_d(\mathbb{P}^k) \) defined over \( K \). Additionally, any preperiodic point \( P \) defined over a Galois extension \( L/K \) of degree at most \( k \) gets sent by \( \tilde{\eta}_k \) to a preperiodic point defined over \( K \).

(1) We first prove the bound on the length of a rational periodic cycle.

We know from \( \text{[3]} \) that the minimal period of \( \tilde{\eta}_k(P) \) is decomposed as

\[
n = m r e^g
\]

where \( m \) is the period modulo \( p \) for \( \tilde{\eta}_k(P) \) by \( F \), \( r \) is the multiplicative order of the multiplier of \( \tilde{\eta}_k(P) \) by \( F \), \( p = \mathfrak{p} \cap \mathbb{Q} \), and \( e \) is explicitly bounded as

\[
e \leq \begin{cases} 
1 + \log_2(v(p)) & p \neq 2 \\
1 + \log_\alpha \left( \frac{\sqrt{5}v(2) + \sqrt{(5v(2))^2 + 1}}{2} \right) & p = 2,
\end{cases}
\]

where \( \alpha = \frac{1 + \sqrt{5}}{2} \).
We have
\[ m \leq \sum_{i=0}^{k} (Np)^i \]
\[ r \leq k \cdot N(p). \]

The bound on \( m \) is simply the number of points modulo \( p \) in \( \mathbb{P}^k \). The bound on \( r \) is more subtle. From Proposition 2.3 we have a relation between the multipliers of \( P \) and \( \tilde{\eta}_k(P) \). The multiplicative order of the multiplier of \( P \) is either the same as the multiplicative order of the multiplier of \( \tilde{\eta}_k(P) \) or the cycle collapses and the order increases. Since the cycle can at most collapse by a factor of \( k \), we have the given maximal possible multiplicative order of the multiplier of \( \tilde{\eta}_k(P) \).

(2) We now prove the bound on the total number of preperiodic points. We are constructing a bound \( C_f(k,K) \) on the number of \( L \)-rational preperiodic points where \( L \) is a Galois extension of \( K \) of degree at most \( k \). If \( C_F(K) \) is a bound on the number of preperiodic points for \( F \) over \( K \), then since \( \tilde{\eta}_K \) is a cover of degree \( k \)
\[ C_f(k,K) \leq k \cdot C_F(K). \]

\[ \Box \]

2.1. Canonical heights. To apply this construction to computing canonical heights over number fields, we need to determine the relation between \( \hat{h}_f(P) \) and \( \hat{h}_F(\tilde{\eta}_k(P)) \). We know that \( \deg(f) = \deg(F) \) and that \( \tilde{\eta}_k(f^n(P)) = F^n(\tilde{\eta}_k(P)) \). So all we need to determine is the relation between the heights
\[ H(P) \quad \text{and} \quad H(\tilde{\eta}_k(P)). \]

There is a standard relation between the height of the coefficients of a polynomial and the height of its roots

Lemma 2.4. \([S2, \text{Theorem VIII.5.9}]\) Let
\[ f(X,Y) = e_0X^k - e_1X^{k-1}Y + \cdots + (-1)^ke_kY^k \]
\[ = (\alpha_{01}X - \alpha_{00}Y) \cdots (\alpha_{k1}X - \alpha_{k0}Y). \]

Then
\[ 2^{-k} \prod_{j=1}^{k} H(\alpha_{ij}) \leq H([e_0, \ldots, e_k]) \leq 2^{k-1} \prod_{j=1}^{k} H(\alpha_{ij}). \]

Theorem 9. Let \([L : \mathbb{Q}] = k \) be a Galois field. We have
\[ \hat{h}_f(P) = \frac{1}{k} \hat{h}_F(\tilde{\eta}_k(P)). \]

Proof. By Lemma 2.4 we have
\[ h(\tilde{\eta}_k(P)) = kh(P) + O(1), \]
and by construction we have
\[ \tilde{\eta}_k(f^n(P)) = F^n(\tilde{\eta}_k(P)). \]

Thus,
\[ \hat{h}_F(\tilde{\eta}_k(P)) = \lim_{n \to \infty} \frac{h(F^n(\tilde{\eta}_k(P)))}{\deg(F)^n} = \lim_{n \to \infty} \frac{kh(f^n(P)) + O(1)}{\deg(f)^n} = k\hat{h}_f(P). \]

\[ \Box \]
We address the practical problem of computing the $\hat{h}_f(\sigma(\alpha))$ by computing the local Green’s functions with the “flip-trick” as in [S1 Exercise 5.29]. Recall that we only need to compute the Green’s functions for the primes of bad reduction, and the primes of bad reduction for $f$ and $F$ are the same (Lemma 2.1).

**Example 3.** Consider $L = \mathbb{Q}(\zeta_5)$ a number field of degree 4. Let

$$f : \mathbb{P}^1(L) \to \mathbb{P}^1(L)$$

$$x \mapsto x^2 - 2.$$  

Let $P = (3,1)$ and we compute

$$\hat{h}_f(P) \approx 0.9624.$$

We apply $\eta_4$ to $f$ as

$$F : \mathbb{P}^4(\mathbb{Q}) \to \mathbb{P}^4(\mathbb{Q})$$

$$(v_0, v_1, v_2, v_3, v_4) \mapsto (v_0^2 - 2v_1^2 + 4v_0v_2 + 4v_2^2 - 8v_1v_3 - 8v_0v_4 + 16v_2v_4 + 16v_4^2,
\quad v_1^2 - 2v_0v_2 - 4v_2^2 + 8v_1v_3 + 12v_3^2 - 8v_0v_4 - 24v_2v_4 - 32v_4^2,
\quad v_2^2 - 2v_1v_3 - 6v_3^2 + 2v_0v_4 + 12v_2v_4 + 24v_4^2,
\quad v_3^2 - 2v_2v_4 - 8v_4^2,
\quad v_4^2).$$

We have

$$\eta_4(P) = (81, 108, 54, 12, 1)$$

and we compute

$$\hat{h}_F(\eta(P)) \approx 3.84969 = 4\hat{h}_f(P).$$

Applying this to a number field point $P = (\zeta_5, 1) \in \mathbb{P}^1(L)$, we compute

$$\hat{h}_f(P) = \frac{\hat{h}_F(1, -1, 1, -1, 1)}{4} = \frac{1.5536}{4} \approx 0.3884.$$

3. Postcritically finite symmetric products

In this section, we prove Theorem 4. The proof of this result relies on the following description of the critical set of $F$ in terms of critical sets of $f$ and $\eta_k$.

**Lemma 3.1.** The critical set of $F$ is given by

$$\mathcal{C}(F) = \eta_k \left( \bigcup_{s=0}^{k-1} (\mathbb{P}^1)^s \times \mathcal{C}(f) \times (\mathbb{P}^1)^{k-s-1} \cup \bigcup_{1 \leq i < j \leq k} \Delta_{i,j} \right)$$

where $\Delta_{i,j} = \{ x \in (\mathbb{P}^1)^k \mid x_i = x_j \}$.

**Proof.** First, note that the critical locus of $F$ is the projection by $\eta_k$ of the critical locus of the map $(f, \ldots, f)$ acting on $(\mathbb{P}^1)^k$ and of the critical locus of $\eta_k$. We, thus, just have to determine the critical locus of $(f, \ldots, f)$ and of $\eta_k$. The critical locus of $(f, \ldots, f)$ obviously is

$$\bigcup_{s=0}^{k-1} (\mathbb{P}^1)^s \times \mathcal{C}(f) \times (\mathbb{P}^1)^{k-s-1},$$

and the critical locus of $\eta_k$ is the set where the action of the group $\mathfrak{S}_k$ is not simply transitive, i.e. $(x_1, \ldots, x_k) \in (\mathbb{P}^1)^k$ is in the critical set of $\eta_k^i$ if and only if $x_i = x_j$ for some $1 \leq i < j \leq k$. 

□
Proof of Theorem \textup{4}. Let us first prove that $F$ is postcritically finite if and only if $f$ is postcritically finite. Remark that the postcritical set of $F$ is an algebraic variety if and only if it is the image under $\eta_k$ of an algebraic variety. According to Lemma \textup{3.1}, one, thus, sees that $F$ is postcritically finite if and only if the union of the iterates under the map $(f, \ldots, f)$ (resp. $(\Delta \subset \{\text{resp. the diagonal } \Delta \text{ of } (\mathbb{P}^1)^k\})$ is postcritically finite if and only if the restriction of $(f, f)$ is strongly postcritically finite, which concludes the proof for $k = 1$. 

We now want to give a negative answer to the question in the case when indeterminacy points are allowed. To do so, we provide an example based on a map from [KPR] of a degree 4 rational mapping of $\mathbb{P}^2$ having the wanted properties. For any $a \in \mathbb{C} \setminus \{0\}$, we let 

$$F_a : [x : y : z] \in \mathbb{P}^2 \to [-y^2 : ax^2 - axz : z^2 - x^2] \in \mathbb{P}^2.$$ 

This provides a good counter-example.

\textbf{Proposition 3.2.} The following holds:

(1) For all $a \in \mathbb{C} \setminus \{0\}$, the map $F_a^2$ is rational (i.e. $\mathcal{I}(F_a^2) \neq \emptyset$) and postcritically finite.
(2) For all but countably many \( a \in \mathbb{C} \setminus \{0\} \), the map \( F^2_a \) is not strongly postcritically finite.

**Proof.** For any \( a \neq 0 \), the point \([1 : 0 : 1]\) is clearly the unique indeterminacy point of \( F_a \). It also is an indeterminacy point of \( F^2_a \), hence \( F^2_a \) is rational. Moreover, the critical locus of \( F_a \) is
\[ C(F_a) = \{ x = 0 \} \cup \{ y = 0 \} \cup \{ z = x \} . \]
But \( F_a(\{ y = 0 \}) = \{ x = 0 \} \), \( F_a(\{ y = 0 \}) = \{ x = 0 \} \), and \( F_a(\{ x = z \}) = \{ [1 : 0 : 0] \} \) and we get
\[ F_a(C(F_a)) \subset C(F_a) . \]
Hence, \( F_a \) is postcritically finite, as is \( F^2_a \). To prove that for a good choice of \( a \neq 0 \) the map \( F^2_a \) is not strongly postcritically finite, we look at the restriction
\[ f_a := F^2_a |_{\{ y = 0 \}} : \{ y = 0 \} \cong \mathbb{P}^1 \rightarrow \{ y = 0 \} \cong \mathbb{P}^1 , \]
which is the quadratic rational map
\[ f_a([z : t]) = [-a^2 z^2 (z - t)^2 : (z^2 - t^2)^2] = [-a^2 z^2 : (z + t)^2] , \]
\([z : t] \in \mathbb{P}^1 \). Its critical points are \([0 : 1]\) and \([1 : 1]\). Moreover, \( f_a([0 : 1]) = [0 : 1] \), hence, \( f_a \) is a quadratic polynomial. It is conjugate to the map
\[ p_a : z \in \mathbb{C} \mapsto z^2 - \frac{1}{a^2} \in \mathbb{C} . \]
In particular, \( f_a \) is postcritically finite if and only if \( p_a \) is postcritically finite, i.e. if and only if \( p^n_a(0) = p_k^0(0) \) for some \( n > k \geq 0 \). But the family \( (p_a)_{a \in \mathbb{C} \setminus \{0\}} \) is a 2-to-1 cover of \( M_2(\mathbb{P}^1) \setminus \{ [z^2] \} \) so no equation of the form \( p^n_a(0) = p_k^0(0) \) is satisfied by all \( a \neq 0 \). In particular, there exists only countably many such parameters. \( \square \)

4. Families of Lattès maps in dimension at least 2

Our proof decomposes in two steps. First, we prove that the induced analytic family of complex dimension \( k \) tori up to biholomorphism has the same dimension as the family of Lattès maps. We then follow Milnor’s proof for the case \( k = 1 \) and adapt his argument.

4.1. Families of abelian varieties. This paragraph is devoted to recalling classical results concerning abelian varieties. For the material of this paragraph, we refer to [De].

The moduli space of polarized abelian varieties.

A polarization on a complex abelian variety \( \mathcal{T} \) is an entire Kähler form \( \omega \) on \( \mathcal{T} \). To \( (\mathcal{T}, \omega) \) can be associated integers \( d_1 | \cdots | d_k \) and a matrix \( \tau \in M_k(\mathbb{C}) \) with \( \tau^t = \tau \) and \( \text{Im}(\tau) > 0 \) such that \( \mathcal{T} = \mathbb{C}^k / \Gamma_\tau \), with \( \Gamma_\tau := \tau \mathbb{Z}^k \oplus \Delta \mathbb{Z}^k \) and \( \Delta = \text{diag}(d_1, \ldots, d_k) \in M_k(\mathbb{Z}) \). We say \( \mathcal{T} \) is of type \( \Delta \).

We now give the classical definition of the moduli space of type \( \Delta \) abelian varieties. Let
\[ \mathcal{H}_k := \{ \tau \in M_k(\mathbb{C}) ; \; \tau^t = \tau , \; \text{Im}(\tau) > 0 \} . \]
For any field \( K \), let \( \text{Sp}_{2k}(K) \) be the symplectic group
\[ \text{Sp}_{2k}(K) := \{ M \in \text{GL}_{2k}(K) ; \; MJM^t = J \} , \]
where \( J = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} \in M_{2k}(K) \). We also let \( \sigma_\Delta : \text{GL}_{2k}(\mathbb{Q}) \rightarrow \text{GL}_{2k}(\mathbb{Q}) \) be defined by
\[ \sigma_\Delta(M) := \begin{pmatrix} I_k & 0 \\ 0 & \Delta \end{pmatrix}^{-1} M \begin{pmatrix} I_k & 0 \\ 0 & \Delta \end{pmatrix} , \; M \in \text{GL}_{2k}(\mathbb{Q}) . \]
Finally, we let \( G_\Delta := \sigma_\Delta(M_{2k}(\mathbb{Z})) \cap \text{Sp}_{2k}(\mathbb{Q}) \).
The well-defined map

\[ M = \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tau \right) \mapsto M \cdot \tau := (A\tau + B)(C\tau + D)^{-1} \]

defines an action of the group $\text{Sp}_{2k}(\mathbb{R})$ on $\mathcal{H}_k$. Moreover, any discrete subgroup of $\text{Sp}_{2k}(\mathbb{R})$ acts properly discontinuously on $\mathcal{H}_k$. The main result we rely on is that two polarized abelian varieties $\mathbb{C}^k/\Gamma_\tau$ and $\mathbb{C}^k/\Gamma_\gamma$ of type $\Delta$ are isomorphic if and only if there exists $M \in G_\Delta$ with $M \cdot \tau = \gamma$.

**Definition 4.1.** The moduli space $\mathcal{A}_{k,\Delta}$ of polarized abelian variety of type $\Delta$ is the set of isomorphism classes of such varieties. Equivalently, this is $\mathcal{H}_k/G_\Delta$.

It is known that the moduli space $\mathcal{A}_{k,\Delta}$ is a quasi-projective variety and that it has dimension $k(k + 1)/2$.

**Dimension in moduli of families of abelian varieties.** We say that $(\mathcal{T}_t)_{t \in X}$ is a holomorphic family of abelian varieties if there exists a complex manifold $\hat{T}$ and a holomorphic map $p : \hat{T} \to X$, such that $p^{-1}\{t\} = \mathcal{T}_t$ for all $t \in X$.

We want to define properly a notion of dimension in moduli for the family $(\mathcal{T}_t)_{t \in X}$. First, notice that the polarization of $\mathcal{T}_t$ is defined by an *entire* Kähler form. As seen in the proof of [De, Proposition VI.1.3], when $t$ varies in $X$, the integers $d_1 \cdots d_k$ cannot change. In particular, the type of the polarization is constant in the family $(\mathcal{T}_t)_{t \in X}$ and there exists a holomorphic map $t \in X \mapsto \tau(t) \in \mathcal{H}_k$ with $\mathcal{T}_t = \mathbb{C}^k/(\tau(t)\mathbb{Z}^k \oplus \Delta\mathbb{Z}^k)$ for all $t \in \mathcal{H}_k$.

As a consequence, we can define the *type* of a holomorphic family $(\mathcal{T}_t)_{t \in X}$ of abelian varieties as the type $T_{t_0}$ of any $t_0 \in X$.

**Definition 4.2.** We say that a holomorphic family $(\mathcal{T}_t)_{t \in X}$ of abelian varieties of type $\Delta$ has dimension $q$ in moduli if the analytic set $\Pi_\Delta(\Lambda) \subset \mathcal{A}_{k,\Delta}$ has dimension $q$.

### 4.2. Lattès maps and abelian varieties.

**Families of Lattès maps.** Recall that $F \in \text{Hom}_d(\mathbb{P}^k)$ is a Lattès map if there exists a complex $k$-dimensional abelian variety $\mathcal{T}$, an affine map $I : \mathcal{T} \to \mathcal{T}$, and a Galois branched cover $\Theta : \mathcal{T} \to \mathbb{P}^k$ making the following diagram commute

\[ \begin{array}{ccc}
\mathcal{T} & \xrightarrow{I} & \mathcal{T} \\
\Theta \downarrow & & \Theta \downarrow \\
\mathbb{P}^k & \xrightarrow{F} & \mathbb{P}^k
\end{array} \]

(see e.g. [Du]). These maps are known to be postcritically finite (see e.g. [BDI]). Another way to present Lattès maps, given by Berteloot and Loeb [BL2, Théorème 1.1 & Proposition 4.1], is the following (see also [BDI]): A *complex crystallographic* group $G$ is a discrete group of affine transformations of a complex affine space $V$ such that the quotient $X = V/G$ is compact.

A degree $d$ endomorphism $F : \mathbb{P}^k \to \mathbb{P}^k$ is a Lattès map if there exists a ramified cover $\sigma : \mathbb{C}^k \to \mathbb{P}^k$, an affine map $A : \mathbb{C}^k \to \mathbb{C}^k$ whose linear part is $\sqrt{d} \cdot U$ where $U \in \text{U}(k)$ is a diagonal unitary linear map, and a complex crystallographic group $G$ such that the following diagram commutes:

\[ \begin{array}{ccc}
\mathbb{C}^k & \xrightarrow{A} & \mathbb{C}^k \\
\sigma \downarrow & & \sigma \downarrow \\
\mathbb{P}^k & \xrightarrow{F} & \mathbb{P}^k
\end{array} \]

and the group $G$ acts transitively on fibres of $\sigma$. 

Lemma 4.3. Let \((f_t)_{t \in \mathbb{X}}\) be any holomorphic family of Lattès maps of \(\mathbb{P}^k\). Then there exists a holomorphic family of complex crystallographic groups \((G_t)_{t \in \mathbb{X}}\), a holomorphic family of branched covers \((\sigma_t)_{t \in \mathbb{X}}\), and a holomorphic family of affine maps \((A_t)_{t \in \mathbb{X}}\), as above. Moreover, the linear part of \(A_t\) is independent of \(t\).

Proof. Let \((f_t)_{t \in \mathbb{X}}\) be a holomorphic family of Lattès maps and let \(G_t, A_t\) and \(\sigma_t\) be such that \(\sigma_t \circ A_t = f_t \circ \sigma_t\) on \(\mathbb{C}^k\). It is obvious that \(G_t, A_t\) and \(\sigma_t\) depend holomorphically on \(t\). Moreover, the linear part of \(A_t\) can be written \(\sqrt{d} \cdot U_t\) with \(U_t \in \mathbb{U}(k) \subset \text{GL}_k(\mathbb{C})\) and the map \(t \mapsto U_t\) is holomorphic. Hence, it has to be constant.

Equality of the dimensions in moduli. We now may prove the following key fact.

Proposition 4.4. Let \((f_{\lambda})_{\lambda \in \Lambda}\) be a holomorphic family of degree \(d\) Lattès maps of \(\mathbb{P}^k\), and let \((\mathcal{T}_{\lambda})_{\lambda \in \Lambda}\) be any induced family of abelian varieties. Then the family \((f_{\lambda})_{\lambda \in \Lambda}\) has dimension \(q\) in moduli if and only if \((\mathcal{T}_{\lambda})_{\lambda \in \Lambda}\) has dimension \(q\) in moduli.

Proof. Pick any induced holomorphic family of tori \((\mathcal{T}_{t})_{t \in \mathcal{D}}\). Let \(q\) and \(r\) stand, respectively, for the dimension in moduli of \((f_t)_{t \in \Lambda}\) and \((\mathcal{T}_t)_{t \in \mathcal{D}}\). Assume first that \(r > q\). Choose \(t_0 \in \Lambda\). By assumption, there exists a holomorphic disk \(\mathbb{D} \subset \Lambda\) centered at \(t_0\) and such that \((\mathcal{T}_t)_{t \in \mathbb{D}}\) has dimension 1 in moduli and the canonical projection \(\Pi : \mathbb{D} \mapsto \mathcal{M}_d(\mathbb{P}^k)\) is constant. In particular, there exists a holomorphic disk \((\phi_t)_{t \in \mathbb{D}}\) of \(\text{PSL}(k+1, \mathbb{C})\) such that \(f_t = \phi_t^{-1} \circ f_{t_0} \circ \phi_t\) on \(\mathbb{P}^k\). Since \((\mathcal{T}_t)_{t \in \mathbb{D}}\) is an induced family of abelian varieties, we have

\[
\begin{array}{ccc}
\mathcal{T}_t & \xrightarrow{D_t} & \mathcal{T}_t \\
\phi_t \circ \Theta_t & \downarrow & \phi_t \circ \Theta_t \\
\mathbb{P}^k & \xrightarrow{f_{t_0}} & \mathbb{P}^k
\end{array}
\]

As a consequence, \(\phi_t \circ \Theta_t : \mathcal{T}_t \mapsto \mathbb{P}^k\) and \(\Theta_{t_0} : \mathcal{T}_{t_0} \mapsto \mathbb{P}^k\) are isomorphic Galois branched covers. Hence, there exists an analytic isomorphism \(\psi_t : \mathcal{T}_t \rightarrow \mathcal{T}_{t_0}\) for any \(t \in \mathbb{D}\). Hence, \(\mathcal{T}_t\) is isomorphic to \(\mathcal{T}_{t_0}\) for any \(t \in \mathbb{D}\). This implies that \((\mathcal{T}_t)_{t \in \mathbb{D}}\) is trivial in moduli. This is a contradiction.

For the converse inequality, we also proceed by contradiction. Assume \(r < q\). Since \(\mathcal{M}_d(\mathbb{P}^k)\) is a geometric quotient, for any \(t_0 \in \Lambda\), there exists a local dimension \(q\) complex submanifold \(X_0 \subset \Lambda\) containing \(t_0\) and such that the canonical projection \(\Pi : X_0 \mapsto \mathcal{M}_d(\mathbb{P}^k)\) has discrete fibers over its image.

As above, this implies the existence of a holomorphic disk \(\mathbb{D} \subset X_0\) centered at \(t_0\) and such that the corresponding family \((\mathcal{T}_t)_{t \in \mathbb{D}}\) is trivial in moduli, i.e. \(\mathcal{T}_t \simeq \mathcal{T}_{t_0}\) as abelian varieties for any \(t \in \mathbb{D}\). We, thus, may assume that \(\mathcal{T} := \mathcal{T}_{t_0} = \mathcal{T}_t\) for all \(t \in \mathbb{D}\). We, thus, have a holomorphic family \((D_t)_{t \in \mathbb{D}}\) of degree \(\sqrt{d}\) isogenies of \(\mathcal{T}\) and a holomorphic family of Galois covers \(\Theta_t : \mathcal{T} \mapsto \mathbb{P}^k\) with

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{D_t} & \mathcal{T} \\
\Theta_t & \downarrow & \Theta_t \\
\mathbb{P}^k & \xrightarrow{f_t} & \mathbb{P}^k
\end{array}
\]

Let \(\Gamma\) be a lattice defining \(\mathcal{T}\), i.e. such that \(\mathcal{T} = \mathbb{C}^k/\Gamma\). Lifting \(D_t\) to an affine map \(\tilde{D}_t : \mathbb{C}^k \rightarrow \mathbb{C}^k\), we end up with a holomorphic map \(t \in \mathbb{D} \mapsto \tilde{D}_t \in \text{Aff}(\mathbb{C}^k)\), which satisfies \(\tilde{D}_t(\Gamma) \subset \Gamma\) and \(\tilde{D}_t = \sqrt{d} \cdot U_t + \tau_t\) with \(U_t \in \mathbb{U}_k\) and \(\tau_t \in \frac{1}{\sqrt{d}} \Gamma\). As a consequence, the map \(t \mapsto \tilde{D}_t\) is discrete, hence constant. The family \((D_t)_{t \in \mathbb{D}}\) is thus constant. Let \(D := D_t\) for all \(t\).
Finally, by assumption, there exists an isomorphism $\alpha_t : \mathcal{T}_t \to \mathcal{T}_{t_0}$ for all $t \in \mathbb{D}$, which depends analytically on $t$. Hence $\Theta_t = \Theta_{t_0} \circ \alpha_t$ for all $t \in \mathbb{D}$, and $(\Theta_t)_{t \in \mathbb{D}}$ is a holomorphic family of Galois cover of $\mathbb{P}^k$ which are isomorphic. As a consequence, there exists a holomorphic disk $t \in \mathbb{D} \mapsto \phi_t \in PSL(k+1, \mathbb{C})$ such that $\Theta_t = \phi_t \circ \Theta_{t_0}$, for all $t \in \mathbb{D}$. Hence, we get

$$\phi_t \circ (\Theta_{t_0} \circ D) = (f_t \circ \phi_t) \circ \Theta_{t_0}$$

which we may rewrite

$$f_{t_0} \circ \Theta_{t_0} = \Theta_{t_0} \circ D = (\phi_t^{-1} \circ f_t \circ \phi_t) \circ \Theta_{t_0}.$$ 

This gives $f_{t_0} = \phi_t^{-1} \circ f_t \circ \phi_t$, i.e. $\mathbb{D} \subset \Pi^{-1} \{ f_{t_0} \}$. This is a contradiction since the fibers of $\Pi$ are discrete in $X_0$.

An immediate consequence is the following.

**Corollary 4.5.** Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of degree $d$ Lattès maps of $\mathbb{P}^k$. Then the family $(f_\lambda)_{\lambda \in \Lambda}$ has dimension in moduli at most $k(k+1)/2.$

**4.3. Classification of maximal families of Lattès maps.** We now want to prove Theorem 4.6. We use the description of Berteloot and Loeb of Lattès maps [BL2]. When $U \in \mathcal{L}(\mathbb{C}^k)$, we denote by $E_\lambda := \ker (\lambda I_k - U)$ the eigenspace associated with the eigenvalue $\lambda \in \mathbb{C}$.

We may prove the following.

**Theorem 4.6.** Assume $(f_t)_{t \in X}$ is a maximal family of Lattès maps containing $f$ and let $D = \sqrt{d} \cdot U + \gamma$ be the affine map inducing $f$. Let $q_\pm := \dim \ker(U \pm I_k)$; then

$$\dim_{\mathcal{M}}(f_t, X) = (q_+ + q_-) \cdot \frac{q_+ + q_- + 1}{2}.$$ 

**Remark.** Let $(f_t)_{t \in X}$ be any holomorphic family of Lattès maps of $\mathbb{P}^k$, let $(A_t)_{t \in X}$ be the holomorphic family of affine maps given by Lemma 4.3, and write $\sqrt{d} \cdot U$ the linear part of $A_t$ with $U \in \mathcal{U}(k)$. Be aware that for any integer $m \in \mathbb{Z}$, since $U$ is unitary, we have $\dim \mathbb{C} E_m(\sqrt{d} \cdot U) = 0$ for all $m \neq \pm \sqrt{d}$ and $\dim \mathbb{C} E_m(\sqrt{d} \cdot U) = 0$ for all $m$ if $d$ is not the square on an integer.

**Proof.** Up to taking a connected subvariety of $X$, we may assume $q := \dim(f_t) = \dim_{\mathcal{M}}(f_t)$. According to Proposition 4.3, the induced family of abelian varieties $(\mathcal{T}_t)_{t \in X}$ has also dimension $q$ in moduli.

Choose $t_0 \in X$. Notice that, by Lemma 4.3, if $A_t : \mathcal{T}_t \to \mathcal{T}_t$ is the affine map inducing $f_t$, then $A_t$ for all $t$ has linear part $\sqrt{d} U \in \mathcal{U}(k)$ independent of $t$. Our assumption implies that the isogeny $A = (A_t)_t$ preserves a family of abelian varieties with dimension in moduli $q$. Set

$$q_* := \left(q_{\sqrt{d}} + q_{-\sqrt{d}}\right) \cdot \frac{q_{\sqrt{d}} + q_{-\sqrt{d}} + 1}{2}.$$ 

We now may prove it implies $q = q_*$. We prove first $q \geq q_*$. If $q_* = 0$, this is trivial. Otherwise, up to linear change of coordinates, we have

$$U = \begin{pmatrix} I_{\sqrt{d}} & 0 & 0 \\ 0 & I_{-\sqrt{d}} & 0 \\ 0 & 0 & * \end{pmatrix}$$ 

where $*$ is a $(k - q_*)$-square diagonal unitary matrix. Let $\Gamma_0 = \tau_0 \mathbb{Z}^k + \Delta \mathbb{Z}^k$ be a lattice preserved by $A_{t_0}$ with $\mathcal{T}_{t_0} \simeq \mathbb{C}^k / \Gamma_{t_0}$. Write

$$\tau_0 = \begin{pmatrix} \tau_{1,1} & \tau_{1,2} \\ \tau_{1,2} & \tau_{2,2} \end{pmatrix}$$ 

where $\tau_{1,1} \tau_{2,2} \neq 0$. Then $\tau_{1,1} + \tau_{2,2} \neq 0$, so $(\tau_{1,1}, \tau_{2,2}) \neq (0, 0)$. If $q_{\sqrt{d}} = 0$, then $q_{-\sqrt{d}} = q_*$. If $q_{-\sqrt{d}} = 0$, then $q_{\sqrt{d}} = q_*$. Thus $q_{\sqrt{d}} + q_{-\sqrt{d}} \geq q_*$. Otherwise, $q_{\sqrt{d}} = 0$ and $q_{-\sqrt{d}} = q_*$. But $q_{-\sqrt{d}} \geq q_{\sqrt{d}} = 0$, so $q_{-\sqrt{d}} = q_*$. Thus $q = q_*$. By assumption, $\mathcal{T}_{t_0} \simeq \mathbb{C}^k / \Gamma_{t_0}$. Write

$$\tau_0 = \begin{pmatrix} \tau_{1,1} & \tau_{1,2} \\ \tau_{1,2} & \tau_{2,2} \end{pmatrix}$$ 

where $\tau_{1,1} \tau_{2,2} \neq 0$. Then $\tau_{1,1} + \tau_{2,2} \neq 0$, so $(\tau_{1,1}, \tau_{2,2}) \neq (0, 0)$. If $q_{\sqrt{d}} = 0$, then $q_{-\sqrt{d}} = q_*$. If $q_{-\sqrt{d}} = 0$, then $q_{\sqrt{d}} = q_*$. Thus $q_{\sqrt{d}} + q_{-\sqrt{d}} \geq q_*$. Otherwise, $q_{\sqrt{d}} = 0$ and $q_{-\sqrt{d}} = q_*$. But $q_{-\sqrt{d}} \geq q_{\sqrt{d}} = 0$, so $q_{-\sqrt{d}} = q_*$. Thus $q = q_*$.
where \( \tau_{1,1} \in \mathcal{H}_q \), \( \tau_{2,2} \in \mathcal{H}_{k-q} \) and \( \tau_{1,2} \) is a \((q_*, k-q_*)\)-matrix. For \( \lambda \in \mathcal{H}_{q_*} \), we let

\[
\tau(\lambda) := \begin{pmatrix} \lambda & \tau_{1,2} \\ \tau_{1,2} & \tau_{2,2} \end{pmatrix} \in \mathcal{H}_k.
\]

Let \( \mathcal{T}_\lambda := \mathbb{C}^k/\Gamma_\lambda \). The family \((\mathcal{T}_\lambda)\) has dimension \(q_*\) in moduli. Remark also that

\[
\gamma = \tau_0 \cdot \underline{u} + \Delta \cdot \underline{v} \in \alpha \cdot \Gamma_0
\]

for some \( \alpha \in \mathbb{C^*} \) (resp. \( \underline{u}, \underline{v} \in \alpha \cdot \mathbb{Z}^k \)) and that this property has to be preserved in any family of Lattès maps containing \( f \). Hence, we may let

\[
\gamma(\lambda) := \tau(\lambda) \cdot \underline{u} + \Delta \cdot \underline{v} = \alpha \cdot \Gamma_0, \quad \lambda \in \Lambda,
\]

so that the map \( \gamma : \Lambda \to \mathbb{C}^k \) is holomorphic and the map \( A_\lambda := \sqrt{d} \cdot U + \gamma(\lambda) \) induces an isogeny of the abelian variety \( \mathcal{T}_\lambda \). Moreover, we have \( \mathcal{T}_{\tau_{1,1}} \simeq \mathcal{T}_{t_0} \) and \( f_{t_0} \) is induced by \( A_{\tau_0} \). Hence, \( f_{t_0} \) belongs to a family of Lattès maps which is parameterized by \( \Lambda \). By Proposition 4.4, this family has dimension at least \( q_* \) in moduli, and \( q \geq q_* \).

We now prove the converse inequality by contradiction, assuming \( q > q_* \). Assume first that \( q_* = 0 \). Since \( q > 0 \), we again can find a holomorphic disk \( D \subset X \) with \( \dim \Pi_\Delta(D) = 1 \) where \( \Delta \) is the type of the family \((\mathcal{T}_t)_{t \in D} \). By assumption, \( U \in \mathbb{U}(k) \) is diagonal and has no integer eigenvalue. Hence, \( U = \text{diag}(u_1, \ldots, u_k) \) with \( |u_i| = 1 \) and \( u_i^2 \neq 1 \). On the other hand, if \( \sigma_t : \mathbb{C}^k \to \mathbb{P}^k \) is the Galois cover making the following diagram commute

\[
\begin{array}{ccc}
\mathbb{C}^k & \xrightarrow{A_t} & \mathbb{C}^k \\
\sigma_t \downarrow & & \sigma_t \downarrow \\
\mathbb{P}^k & \xrightarrow{f_t} & \mathbb{P}^k,
\end{array}
\]

for any \( t \in D \), any \( z \in \mathbb{C}^k \), and any \( a \in \Gamma_t \), we find

\[
\sigma_t(\sqrt{d} \cdot U(z) + \gamma_t) = f_t(\sigma_t(z)) = f_t(\sigma_t(z + a)) = \sigma_t(\sqrt{d} \cdot U(z + a) + \gamma_t).
\]

This gives \( \sqrt{d} \cdot U(\Gamma_t) \subset \Gamma_t \) for all \( t \). In particular, if \((e_1, \ldots, e_k)\) is the canonical basis of \( \mathbb{C}^k \)

\[
\sqrt{d} \cdot (U \circ \tau_t) \cdot e_i = \tau_t \cdot \underline{u} + \Delta \cdot \underline{v}
\]

for some \( \underline{u}, \underline{v} \in \mathbb{Z}^k \), and \( \tau_t \) is a non-constant holomorphic map of the parameter \( t \) by assumption. Since \( \sqrt{d} \cdot U \) is diagonal, this gives

\[
\sqrt{d} \cdot u_i \sum_{j=1}^k \tau_{t,i,j} \delta_{i,j} = \sum_{j=1}^k \tau_{t,j} u_j + \Delta \cdot \underline{v}.
\]

This is impossible, since it gives that \( \tau_t \) is constant on \( D \) hence that \((\mathcal{T}_t)\) is a trivial family.

Assume finally that \( q_* > 0 \). As in the proof of Proposition 4.4 since \( \mathcal{M}_d(\mathbb{P}^k) \) is a geometric quotient, for any \( t_0 \in X \) there exists a local dimension \( q \) complex submanifold \( X_0 \subset X \) containing \( t_0 \) and such that the canonical projection \( \Pi : X_0 \to \mathcal{M}_d(\mathbb{P}^k) \) has discrete fibers over its image.

Notice that for any \( t_0 \in X \), there exists a polydisk \( D^{t_0} \) centered at \( t_0 \) in \( X_0 \), as described above. By assumption, there exists a disk \( D \) centered at \( t_0 \) which is transverse to \( D^{t_0} \). As above, write \( \Gamma_t = \tau(t) \mathbb{Z}^k \oplus \Delta \mathbb{Z}^k \) and decompose \( \tau(t) \):

\[
\tau(t) := \begin{pmatrix} \tau_{1,1}(t) & \tau_{1,2}(t) \\ \tau_{1,2}(t)^t & \tau_{2,2}(t) \end{pmatrix}.
\]
where $\tau_{i,j}(t)$ is holomorphic on $t \in D$. Let $\Gamma'_t := \tau_{2,2}(t)Z^k + \Delta'Z^k$, where

$$\Delta' = \text{diag}(d_{k-q_1+1}, \ldots, d_k)$$

and the $d_i$'s are given by $\Delta = \text{diag}(d_1, \ldots, d_k)$ with $d_i/d_i+1$. Up to reducing $D$, we may assume $\Gamma'_t$ is not $G_{\Delta'}$-equivalent to $\Gamma'_t$ and that $\Delta'$ is the type of the family $(T'_t)_{t \in D}$ of abelian varieties defined as $T'_t := \mathbb{C}^{k-q_1}/G'_t$. We now let $U' = \text{diag}(u_{k-q_1+1}, \ldots, u_k) \in \mathbb{U}(k-q_1)$ where, again, the $u_i$'s are given by $U = \text{diag}(u_1, \ldots, u_k)$. We finally let $\gamma'_t = (\gamma_{tk-q_1+1}, \ldots, \gamma_{tk}) \in \mathbb{C}^{k-q_1}$ and $A'_t := \sqrt{d} \cdot U' + \gamma'_t$. The family $(T'_t, A'_t)$ induces a family of Lattès maps on $\mathbb{P}^{k-q_1}$ with $\dim(U' + I_{k-q_1}) = 0$. We thus have reduced to the case $q_* = 0$.

An immediate corollary is

**Corollary 4.7.** The dimension in moduli of $(f_t)_{t \in X}$ is non-zero if and only if $d$ is the square of an integer and the linear map $U$ has at least one integer eigenvalue.

We now derive Theorem 4 from Theorem 4.6.

**Proof of Theorem 4.** If $\dim_{\mathcal{M}}(f_\lambda, \Lambda) = 0$, then by Theorem 4.6, $0 = q_{\sqrt{d}} + q_{-\sqrt{d}}$. Since both $q_{\pm \sqrt{d}} \geq 0$, we must have $q_{\sqrt{d}} = q_{-\sqrt{d}} = 0$.

On the other hand, suppose $\dim_{\mathcal{M}}(f_\lambda, \Lambda) > 0$. Again by Theorem 4.6, at least one of $q_{\sqrt{d}}$ and $q_{-\sqrt{d}}$ must be positive. Recall that $U$, the linear part of $A_t$, is unitary. Thus, we have $q_m := \dim_{\mathcal{L}} E_m(\sqrt{d} \cdot U) = 0$ for all $m \neq \pm \sqrt{d}$ and $q_m = 0$ for all $m$ if $d$ is not the square on an integer. Since one or both of $q_{\pm \sqrt{d}}$ must be positive, the result follows.

5. **Symmetric product Lattès maps**

Among symmetric products, we can easily characterize Lattès ones.

**Proposition 5.1.** Let $k,d \geq 2$ and let $f \in \text{Hom}_d(\mathbb{P}^1)$, and let $F \in \text{Hom}_d(\mathbb{P}^k)$ be the $k$-symmetric product of $f$. Then $F$ is a Lattès map if and only if $f$ is a Lattès map. Moreover, if $T$ is an elliptic curve, $\mathcal{I} : T \rightarrow T$ is affine and $\Theta : T \rightarrow \mathbb{P}^1$ is the Galois cover such that

$$\begin{array}{ccc}
\mathcal{I} & \xrightarrow{\mathcal{I}} & \mathcal{I} \\
\Theta \downarrow & & \Theta \downarrow \\
\mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 \\
\end{array}$$

is a commutative diagram, then the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{T}^k & \xrightarrow{\eta_k \circ (\Theta, \ldots, \Theta)} & \mathcal{T}^k \\
\eta_k \circ (\Theta, \ldots, \Theta) \downarrow & & \eta_k \circ (\Theta, \ldots, \Theta) \downarrow \\
\mathbb{P}^k & \xrightarrow{F} & \mathbb{P}^k \\
\end{array}$$

**Proof.** Berteloot and Dupont [BD1] proved that $F$ is Lattès if and only if $L(F) = k \log \sqrt{d}$. A similar result was known by [Z] in dimension 1. Namely, $f \in \text{hom}_d(\mathbb{P}^1)$ is Lattès if and only if $L(f) = \log \sqrt{d}$. The fact that $F$ is Lattès if and only if $f$ is Lattès follows directly from Lemma [BD1]. Assume now that $f$ is a Lattès map induced by $\mathcal{I} : T \rightarrow T$ under the Galois cover $\Theta : T \rightarrow \mathbb{P}^1$. By
construction of $F$, the following diagram commutes:

\[
\begin{array}{ccc}
T^k (\Theta, \ldots, \Theta) & \xrightarrow{f} & (\Theta, \ldots, \Theta) \\
\downarrow & & \downarrow \\
(P^1)^k & \xrightarrow{f} & (P^1)^k \\
\eta_k & \xrightarrow{f} & \eta_k \\
\P^k & \xrightarrow{F} & \P^k
\end{array}
\]

which concludes the proof. \qed

We now come to the proof of Theorem S.

Proof of Theorem S. It is classical that $f$ is rigid if and only if the linear part of the induced isogeny $A$ is not an integer (see [Mi]). We now let $F$ be the $k$-symmetric product of $f$. We have the following alternatives:

1. either $q\sqrt{d} + q\sqrt{-d} = 0$,
2. or $q\sqrt{d} = k$ (or $q\sqrt{-d} = k$).

We now just have to apply Theorem 7. \qed

In dimension 2. An algebraic web is given by a reduced curve $C \subset (\P^2)^*$, where $(\P^2)^*$ is the dual projective plane consisting of lines in $\P^2$. The web is invariant for a holomorphic map $f$ on $\P^2$ if every line in $\P^2$ belonging to $C$ is mapped to another such line.

In [R, Theorems 4.2 & 4.4], Rong Feng gave the following description of Lattès maps on $\P^2$: pick a Lattès map $F \in \Hom_d(\P^2)$, then

- either $F$ or $F^2$ is the 2-symmetric product of a Lattès map $f \in \Hom_d(\P^1)$ (resp. $f \in \Hom_d^2(\P^1)$),
- or $F, F^2, F^3$ or $F^6$ is a holomorphic map preserving an algebraic web associated to a smooth cubic.

Combined with Theorem S, this description directly gives the following result.

Corollary 5.2. Let $F \in \Hom_d(\P^2)$ be a 2-symmetric product Lattès map. Then

1. either $F$ is a rigid 2-symmetric product,
2. or $F$ is a 2-symmetric product which belongs to a family of Lattès maps which has dimension 3 in moduli. In that case, $F$ can be approximated by Lattès maps admitting a suitable iterate which preserve an algebraic web associated to a smooth cubic. More precisely, any maximal subfamily of symmetric products has dimension 1 in moduli.

Proof. According to Theorem S, either $F$ is rigid or $F$ belongs to a family of Lattès maps which has dimension 3 in moduli. Now, if $F$ belongs to such a family, there is a maximal subfamily $(F_t)_t$ which consists in symmetric products which has dimension 1. The family $(F_t^m)_t$ is also a maximal family of symmetric products for all $m \geq 1$. The result follows from Rong’s classification. \qed

Explicit examples. We now give explicit examples.

Example 4 (Rigid Lattès maps of $\P^2$ and $\P^3$). In [Mi], Milnor provided an example of degree 2 Lattès map $f \in \Rat_2$:

\[
f : [z : t] \mapsto [z^2 + a^2zt : t^2 + a^2zt].
\]
As $\deg(f)$ is not the square of an integer, $f$ has to be isolated in $\mathcal{M}_2$ (see e.g. [Mi]). The 2-symmetric product of $f$ is $F: \mathbb{P}^2 \to \mathbb{P}^2$ given by

$$[z : t : w] \mapsto [F_1: F_2: F_3],$$

where $F_1(z, t, w) = z^2 + a^2zw + a^4zt$, $F_2(z, t, w) = z^2 + a^2tw + a^4zt$ and $F_3(z, t, w) = w^2 + a^2zw + a^2tw + 2(a^4 - 1)zt$. We now turn to the 3-symmetric product of $f$.

The 3-symmetric product of $f$ is $G: \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$[z : t : w : u] \mapsto [G_1: G_2: G_3: G_4],$$

where

$$
\begin{align*}
G_1(z, t, w, u) &= z^2 + a^2uz + a^4zw + a^6zt, \\
G_2(z, t, w, u) &= t^2 + a^2wt + a^4ut + a^6zt, \\
G_3(z, t, w, u) &= w^2 + a^4zw + a^2wt + 3a^6zt + a^2wu - 3a^2zt - 2tu + 2a^4ut, \\
G_4(z, t, w, u) &= u^2 + 3a^6zt + a^2wu - 3a^2zt - 2zw + a^4zw + a^4ut + a^2uz + a^4zw.
\end{align*}
$$

By Theorem 7 since $f$ is rigid, both maps $F$ and $G$ are rigid Lattès maps, respectively, in the moduli spaces $\mathcal{M}_2(\mathbb{P}^2)$ and $\mathcal{M}_3(\mathbb{P}^3)$.

**Example 5** (Flexible Lattès maps of $\mathbb{P}^2$). The one-dimensional family $(f_\lambda)_{\lambda \in \mathbb{C} \setminus \{0, 1\}}$ of degree 4 maps

$$f_\lambda : [z : t] \in \mathbb{P}^1 \mapsto \left[ (z^2 - \lambda^2)^2 : 4zt(z - t)(z - \lambda t) \right]$$

is a family of flexible Lattès maps of $\mathbb{P}^1$. The family of 2-symmetric products $F_{\lambda} : \mathbb{P}^2 \to \mathbb{P}^2$ given for $\lambda \in \mathbb{C} \setminus \{0, 1\}$ by

$$[z : t : w] \mapsto [F_{1, \lambda} : F_{2, \lambda} : F_{3, \lambda}],$$

where

$$
\begin{align*}
F_{1, \lambda}(z, t, w) &= ((z + \lambda w)^2 - \lambda^2)^2, \\
F_{2, \lambda}(z, t, w) &= (z + \lambda w)^3t + 2(\lambda + 1)(z + \lambda w)^2zw + \lambda(z + \lambda w)t^3 - 8\lambda zw(z + \lambda w) \\
&\quad - (\lambda + 1)(z^2t^2 + 3\lambda^2zw^2 + \lambda^2t^2w^2), \\
F_{3, \lambda}(z, t, w) &= 4zw(z - t + w)(z - \lambda t + \lambda^2w)
\end{align*}
$$

is a family of flexible Lattès maps of $\mathbb{P}^2$. Moreover, it is a strict subfamily of a 2-dimensional family of flexible Lattès maps.

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LAMFA, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80039 AMIENS Cedex 1, FRANCE

E-mail address: thomas.gauthier@u-picardie.fr

SAINT LOUIS UNIVERSITY, 220 N. GRAND BIVD., ST.LOUIS, MO 63103, USA

E-mail address: hutzba@slu.edu

BUTLER UNIVERSITY, 4600 SUNSET AVE., INDIANAPOLIS, IN 46208, USA

E-mail address: skaschne@butler.edu