Tool use in mathematics: A framework

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In the course of research into the interpretation of tools in the didactics of mathematics I found both voids and conflicts. This paper presents the results of my research and a resultant statement on tool use in mathematics education. The statement incorporates constructs from several theoretical frameworks and I consider the consistency of my statement on tool use with regard to activity theory.

Keywords: Activity theory, agency, artefacts, mediation, tools.

INTRODUCTION

In the course of work on tool use in mathematics I examined literature which I summarise in this paper. The literature sit in various theoretical frameworks and this paper, in the language of Prediger, Bikner-Ahsbahs and Arzarello's (2008), can be considered as an attempt to 'synthesise' frameworks with regard to a statement on tool use in the didactics of mathematics. This synthesis, however, does not aim at synthesising complete theories but synthesising activity theory with principles from other theories. This paper has the following structure: a definition of tools; a survey of theoretical frameworks with regard to tools; an exposition of activity theory with regard to tools; actor network theory ideas that augment an activity theoretic account of tools; an activity theoretic statement on tool use in mathematics education which incorporates ideas from outside of activity theory, and a consideration of the consistency of this statement with regard to networking theories.

TOOLS: A DEFINITION

I define a tool via four action-related distinctions, the first of which is between an artefact and a tool. An artefact is a material object which becomes a tool when it is used by an agent to do something; a compass becomes a tool when it is used to draw a circle (its intended purpose) or to stab someone. This establishes that tool use cannot be separated from the animal using the tool and the purpose of use. After being used as a tool (for whatever purpose), the compass returns to being an artefact. The materiality of an artefact is not just that open to touch. An algorithm, e.g., for adding two natural numbers, is an artefact. It is material in as much as it can be written down or programmed into a computer. My second distinction is between an artefact/tool and ways of using the artefact/tool. For example, I could use a calculator to perform 45 + 67 by typing in '45 + 67=' or I could imitate the standard written algorithm (adding the units, storing the result) and adding the tens and adding on my stored results. My third distinction is between the mental representation of a tool and material actions in tool use. This distinction comes with an interrelationship: to carry out material actions with an artefact you need some form of mental representation, which may be quite crude, of how to act with the artefact-tool, but actions with the artefact-tool will provide feedback to the user which may change the mental representation. My fourth distinction is between signs and tools. Signs, like tools, are artefacts but a sign points to something whereas a tool does something. Having said this, signs or systems of signs, can function as tools. Similarly representations can function as tools but they may also have non-tool functions, e.g., to point to an object.

Is there such a thing as a 'mathematical tool'? – only in use, a compass is a mathematical tool when it is used to draw a circle but not when it is used to stab someone. When artefacts are used for mathematical purposes they generally incorporate mathematical features, e.g., a compass encapsulates the equidistant relationship between the centre and points on the circumference of a circle.

A SURVEY OF FRAMEWORKS WITH REGARD TO TOOLS

I conduct an historical tour of theoretical frameworks employed in Western mathematics education. I select papers from the 1960s to the present which reflect dominant 'grand theories' over this time that address or ignore tools. Behaviourism regarded artefacts as a means of stimulating a response in a subject. Suppes (1969), for example, considers computers as tutorial systems that can provide:

> individualized instruction [where the] intention is to approximate the interaction a patient tutor would have with an individual student ... as soon as the student manifests a clear understanding ... he is moved on to a new concept and new exercises." (ibid, p.43).

Suppes does not consider the environment in which the tool is used. During the period when behaviourism ruled two psychologists, E. and J. Gibson, ventured on a non-behaviourist route to the theory construct of affordances (and constraints):

> The *affordances* of the environment are what it *offers* the animal, what it *provides* ... If a terrestrial surface is nearly horizontal ... nearly flat ... and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface *affords support*. (Gibson, 1979, p. 127)

There is no mention of tools in this quote but mathematics educators have learnt that the construct 'affordances' is useful in considerations of the relevance of artefacts and digital software environments to students' mathematical learning.

The demise of behaviourism in mathematics education saw the rise of cognitive studies and Piaget was the Guru. The interesting thing about Piaget's extensive output with regard to tool use is that he says nothing at all about the role of tools in cognitive development. Piaget's work inspired several 'local theories' in mathematics education: Brousseau's theory of didactical situations (TDS), constructionism and constructivism. TDS was developed over decades starting in the 1960s. The influence of Piaget in Brousseau's work is explicit. An important construct of TDS came to be called the 'milieu' which includes the teacher, the materials and the designed learning strategies. I know of no explicit consideration of mathematical tools in 20th century TDS but tools are a part of the milieu. Papert, who spent several years with Piaget, experimented with children using the computer language *Logo*. Constructionism views that learning occurs through the construction of meaningful products. *Logo* is integral to constructionism but, despite statements that these languages equip students with tools to think with, there is no clear statement as to what a tool is in Papert (1980) and a clearer constructionist view of tools did not emerge until Noss and Hoyles (1996) – by which time constructionism had relinquished its Piagetian roots and embraced socio-cultural viewpoints.

Piaget's ideas inspired constructivism, which focused on the ontogenic development of the individual child but developed to include a focus on microgenetic (child-environment) development (social constructivism). Yackel and Cobb (1996) is a developed form of social constructivism which examines teacher-student discussions and argumentation in a classroom context. This paper introduced the construct 'sociomathematical norms'. The classroom considered in the paper had various resources (centicubes and an overhead projector) but the paper does not mention tools. This neglect has been noticed by others, e.g., Hershkowitz and Schwarz (1999, p. 149) "... socio-mathematical norms do not arise from verbal actions only, but also from computer manipulations as communicative non-verbal actions."

In summary, 20th century mathematics educator frameworks influenced by Piagetian ideas had little to say on tools in learning and teaching but outside of mathematics education, deep ideas, published in the 1970s, on tools were in circulation.

Wartofsky (1979) includes an essay on perception, "an *historically* evolved faculty ... based on the development of historical human practice" (ibid, p.189). Practice is "the fundamental activity of producing and reproducing the conditions of species existence ... human beings do this by means of the creation of artefacts ... the 'tool' may be *any* artefact created for the purpose" (ibid, p. 200). Wartofsky extends the concepts of artefacts to the skills required to use artefacts as tools:

> Primary artefacts are those directly used in this production; secondary artifacts are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artefacts are therefore representations of such modes of action (ibid, p. 202)

Vygotsky (1978), published posthumously, was to have a profound influence on mathematics education. Vygotsky was interested in language, signs and mediation. His interest in tools was in their mediating qualities, "the basic analogy between sign and tool rests on their mediating function that characterizes each of them" (1978, p. 54). The difference between signs and tools rests on:

> The tool's function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to a change in objects … The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is *internally* oriented. (ibid, p. 55)

The reader, however, may note the influence of both Wartofsky and Vygotsky in my definition of tools above. I now move on to the work of B. Latour and P. Rabadel[1]. Latour is a sociologist and, around 1980 and with others, established what is now called actor network theory (ANT); Latour (2005) is a fairly recent exposition. ANT is a theory about how to study social phenomena – by following the actors, where an actor is "*any thing* that does modify a state of affairs by making a difference" (ibid., 71). ANT symmetrically views both society and nature as being in a state of flux and looks to the performance of the actors in situations. Objects (artefacts/tools) can make difference in performance and so can be actors, exerting agency, in the playing out of social situations. Pickering (1995), who is 'almost ANT' in my opinion, examines practices of 20th century elementary particles physics. He accepts ANT's human and material agencies and adds 'disciplinary agency' (in our discipline $a+a=2a$ regardless of what we might want it to be). He proposes a 'dance of agency' where, in the performance of scientific inquiry, human, material and disciplinary agencies "emerge in the temporality of practice and are definitional of and sustain one another" (ibid., p.21). I see this in 'dance' in techno-mathematics lessons – a myriad of influences between students, teachers, computers and mathematics.

Rabadel introduced the 'instrumental approach' which distinguishes between an artefact, as a material object, and an instrument as a psychological construct. An instrument is an emergent entity that begins its existence when a person appropriates an artefact to do something; this has influenced my distinction, above, between an artefact and a tool. The instrumental approach has been well known in mathematics education since Guin and Trouche (1999). This views an instrument as a composite entity composed of the artefact and knowledge (knowledge of the artefact and of the task constructed in using the artefact). Artefact and the agent(s) are interrelated: the artefact shapes the actions of the agent, *instrumentation*; the user shapes the use of the artefact, *instrumentalisation*. The process of turning an artefact into an instrument is called 'instrumental genesis'. The agent brings her/his knowledge and the artefact brings its potentialities and constraints to the artefact agent interaction.

I leave my historical tour at this point with the observation that a lot of the frameworks used in mathematics education pay scant regard to the nature of the tools used in doing mathematics but frameworks initiated by Wartofsky, Vygotsky, Latour, Pickering and Rabardel provide interesting, though diverse, insights into the role of tools in activity. I now turn to the focus framework of this paper, activity theory.

ACTIVITY THEORETIC CONSIDERATION OF TOOLS

I briefly outline activity theory (AT), trace its genesis into mathematics education research (MER) and consider differences in approaches.

AT is an approach to the study of human practices. It sees constant change (flux) in practice. Activity became a focus for Vygotsky in his conviction that consciousness originated in socially meaningful activity. In AT 'object orientated activity' is the *unit of analysis*, that which preserves the essence of concrete practice. 'Object' here refers to *raison d'etre* of the activity. Educators employing AT must take care that they do not merely employ the word 'activity' without considering the object and the unit of analysis. Vygotsky's AT is often presented via a triangle with 'subject', 'object' and 'mediating artefacts' at its vertices. Leont'ev (1978) developed Vygotsky's work by considering individual and collective *actions* (usually with tools) and *operations* (things to be performed or modes of using tools) involved in socially organized *activity*. Engeström (1987) extends Vygotsky's and Leont'ev's frameworks to 'activity systems' and extends the focus on mediation through signs and tools to multiple forms of mediation including the community and social rules underlying activity. Activity systems research often examines interactive activity systems with a focus on the objects of activity in the two systems; the place of tools in such research usually emphasises tool use in the context of the whole system. I now turn to the influence of activity theory in MER.

I was curious of AT's introduction into Western MER literature and I traced its introduction into the journal *Educational Studies in Mathematics* (ESM). Two AT papers appeared in ESM in 1996. Crawford (1996) is an exposition of Vygotskian AT and asks "What difference does the use of tools such as computers and calculators make to the quality of human activity?" (ibid, p. 47) but does not explore the nature of tools further. Bartolini Bussi (1996) reports on a teaching experiment on geometric perspective. The word 'tool' has two uses in the paper: Leont'ev's theory as a tool for analysis; 'semiotic tools', which are defined via examples. In 1998 two ESM AT papers considered tool use in different ways to Bartolini Bussi (1996). Chassapis (1998) focuses on the processes by which children develop a formal mathematical concept of the circle by using various instruments to draw circles: by hand; using circle tracers and templates; and using a compass. "The process of learning to use a tool ... involves the construction of an experiential reality that is consensual with that of others who know how to use [the tool]" (ibid, p. 276). Pozzi, Noss and Hoyles (1998) focuses on nursing and ask "how do resources enter into professional situations, and how do they mediate the relationship between mathematical tools and professional know–how?" (ibid, p. 110) The paper states that AT provides evidence that "acts of problem solving are contingent upon structuring resources, including a range of artefacts such as notational systems, physical and computational tools" (ibid, p. 105). Radford (2000) focuses on early algebraic thinking "considered as a sign-mediated cognitive *praxis*" (ibid, p. 237):

> to accomplish actions as required by the contextual activities ... The sign-tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification (ibid, p. 241).

The first mention of Engeström in ESM is in Jaworski (2003, p. 249). This outlines "*insider* and *outsider* research and *co-learning* between teachers and educators in promoting classroom inquiry" and is not concerned with tool use in mathematics.

Thus, although AT is quite an old theory, it is a fairly recent theory in terms of Western MER and there is wide variation with regard to the meaning of tools in ESM AT papers from 1996 to 2003. After 2003 a considerable number of ESM papers used AT as a theoretical papers but I do not have room to summarise. To get a handle on contemporary AT conceptions of tools in MER I go to a special edition of *The International Journal for Technology in Mathematics Education* devoted to AT approaches to mathematics classroom practices with technology. For reasons of space I focus on three (of 11) papers which illustrate a range of approaches.

Chiappini (2012) focuses on the learning and teaching of algebra with software with a visual 'algebraic line' and conventional algebraic notation, to draw students' attention to the culture of mathematics. Chiappini is interested in 'cultural affordances', which, "allow students to master the meanings, values and principles of the cultural domain" (ibid, p. 138). With regard to tools, Chiappini's focus is the evaluation of software designed to exploit visuo-spatial and deictic affordances and allow teachers to consolidate student learning. Ladel and Kortenkamp (2013) focuses on the design and use of a multi-touch-digital-table to engage young children in meaningful work with whole number operations, "We want to restrict the students' externalizing actions to support the internalization of specific properties ... mediation through the artefact is characterized by restriction and focussing." Artefacts are the focus of attention and the word 'tool' is not mentioned in the paper. They hold that "the artefact itself does not have agency and is only mediating ... [but] the artefact changes the way children act drastically and in non-obvious ways" (ibid, p. 3). Mariotti and Maracci (2012) outline the Theory of Semiotic Mediation (TSM) with regard to "the use of artefacts to enhance mathematics learning and teaching, with a particular focus on technological artefacts" (ibid, p. 21); like Ladel and Kortenkamp (2013) above, the word 'artefact' is favoured over the word 'tool'. This paper continues the work of Bartolini Bussi (1996) considered in the previous section and is critical of research where "the mediating function of the artefact is often limited to the study of its role in relation to the accomplishment of tasks" (ibid). TSM views that "teaching-learning ... originates from an intricate interplay of signs... mathematical meanings can be crystallized, embedded in artefacts and signs" (ibid) The paper presents a rather strange (to me) take on mediation, "The mediator is not the artefact itself but it is the person who takes the initiative and the responsibility for the use of the artefact to mediate a specific content" (ibid, p. 22). To mediate the learning of mathematics the teacher has to design specific circumstances, a didactical cycle, aimed at fostering specific semiotic mediation processes.

Differences in the papers outlined above include the unit of analysis, cognition, the words used, mediation and agency. Some papers explicitly state the unit of analysis, e.g., Chassapis (1998), but many do not. Chassapis' unit of analysis is 'quite small' compared to Engeström's, the activity system itself. I think the 'size' of the unit of analysis impacts on the extent to which the AT analysis permits a study of microgenetic learner development with tools (i.e., Chassapis' unit of analysis allows a focus on cognition and tool use but details of cognitive development are easily 'lost' when the focus is on activity systems). With regard to the words used it is clear that some scholars use 'artefact' for what I refer to as a tool. This seems unimportant but the difference between sign and tool is important and the fact that this difference is sometimes blurred does not downplay this importance; some of the papers do not consider signs vs tools. With regard to mediation the biggest difference is between Ladel and Kortenkamp, where artefact mediation is central, and Maracci and Mariotti, which holds that people and not artefacts mediates. My final consideration concerns agency. Only Ladel and Kortenkamp comment on this, to claim that artefacts do not have agency. The differences noted above show that AT in MER is a collection of approaches in which there are many ways to view tools within AT.

ANT IDEAS THAT AUGMENT AN AT ACCOUNT OF TOOLS

I am drawn to AT as a framework as it mirrors my view that tools are important but tool use is not an activity in itself though tool use and activity are interrelated. But I detect an anthropocentric position in AT – even though AT recognises that people think through/with tools, people are at the centre, they appear as 'the' agents. This anthropocentrism is explicit in Maracci and Mariotti's view that artefacts are not mediators and Ladel and Kortenkamp's statement that artefacts do not have agency. I think tools can be powerful things and I am drawn to an ANT view on material agency, but can ANT ideas be brought into AT? I first look at a potential major obstacle to networking these theories and a difference between Latour and Pickering.

Miettinen (1999) considers ANT and AT as approaches to studying innovations and locates the main division between these approaches as ANT's generalised principle of symmetry which states that the same "vocabulary must be used in the description and explanation of the natural and the social ... no change of register is permissible when we move from the technical to the social aspects of the problem studied" (ibid, pp. 172–173). This is a problem for AT because the object (of activity) is generated from human needs. OK, humans do generate the object but once the object is established the agency which follows in the activity can be distributed. Indeed, Latour (2005) states that he abandoned most of the symmetry metaphor because what he had in mind was a "joint dissolution of both collectors" (ibid, p. 76). Pickering (1999, p. 15) also considers the generalised principle of symmetry to be problematic, "As agents, we humans seem to be importantly different from nonhuman agents". With the generalised principle of symmetry 'put in to perspective' I now look to two commonalities in principles between Latour and Pickering: focus on performance; don't restrict agency to animals (humans) alone.

Latour (2005) mentions performance with to regard groups, social aggregates. Classical sociologists are accused of making *ostensive* definitions of groups – there's a group of teachers – and focusing on stability but, from an ANT point of view, "the rule is performance and what has to be explained, the troubling exceptions, are any type of stability over the long term [and this cannot be explained] without looking for vehicles, tools, instruments, and materials able to provide such a stability" (ibid, p. 35). This focus on performance is akin to flux in AT. A sketch of a performative view of science is presented early in Pickering (1995, p. 6), instead of a world where scientists only generate knowledge from facts, he sees a world filled with agency:

> The world ... is continually doing things, things that bear upon us not as observation statements upon disembodied intellects but as forces upon material beings ... Much of everyday life ... character of coping with material agency, agency that comes at us from outside the human realm and that cannot be reduced to anything within that realm.

Later, in Pickering (1995), 'disciplinary agency' and the 'dance of agency', as described above, are introduced. Neither Latour nor Pickering are concerned with mathematics education but their 'multi-agent' stance resonates with my experience of mathematics classrooms. When a teacher uses a tool in a mathematics class, then s/he is only one of the agents in the activity, other potential agents are: other teachers; students; the curriculum; the institution; other available artefacts; and the tool itself.

I now consider mediation and what mediates: language, signs, artefacts or people? I think the problem here can be viewed via the ostensive-performative distinction. Scholars have different interests and tend to point to something and say "that (those) is (are) the mediator(s)" whereas the mediator in a specific situation exists in relation to what is actually done (the activity/performance). I am, for instance, interested in artefact/tool-mediation but two learners may be involved in ostensibly similar activities with a mathematical tool but one learner may be heavily reliant on the tool whereas the use of this tool to the other learner may be peripheral; mediation by the tool comes down to the actual use of the tool. Similarly Mariotti and Maracci (2012) may expect the mediator to be the teacher but I doubt if this is always the case. Latour (2005, p. 39) appears to present a similar idea in distinguishing between mediators and intermediaries, "An intermediary ... is what transports meaning or force without transformation ... Mediators transform, translate, distort, and modify".

A STATEMENT ON TOOL USE IN MATHEMATICS EDUCATION

The considerations above, together with those in the previous three sections, provide a basis for the following statement (in italics) on tool use in mathematics education.

AT provides a framework to interpret tool use in practice but the level of detail on tool use will depend on the 'size' of the unit of analysis. An AT account of tools would benefit from being augmented by constructs from instrumentation theory and the theory of affordances. Activity is mediated by human and non-human mediators but this mediation cannot be stipulated in advance of the performance of the activity. Human and non-human agents impact the activity; as with mediation, the impact of these agents cannot be stipulated in advance of the performance of the activity.

I now state my networking argument. The theories of affordances and of instrumentation have few assumptions and a lot of application. Recognition of the relationship between learners and their environments is important in AT as is the process by which an artefact becomes a tool for learners. Both theories can be used in MER to shed light on the action and operation aspects of AT without compromising any tenets of AT. With regard to taking ideas from Latour and Pickering I focus on the two principles outlined above. The 'focus on performance' principle is entirely consistent with the concept of flux in AT. AT focuses on describing practice and tools (and, I add, other things) are used as they are used (or not) – there is no pre-ordained plan. As for not restricting agency to humans alone, well, this is a problem for many activity theorists because the object of an activity is generated by humans. But if the principle of non-human agency is weakened to restrict non-humans from initiating activity, then I don't think there is a problem.

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ENDNOTE

1. I would have liked to have considered tool-focused work within the Anthropological Theory of Didactics as well but this was not possible in the length restrictions for this paper.