Discrete or continuous? – A model for a technology-supported discrete approach to calculus

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In recent decades, the approach to calculus in mathematics classrooms has changed: a quite formal approach—closely linked to the teaching of calculus at university and based on the sequence concept—has been transformed to an intuitive access to the concepts of limit and derivative. The importance of sequences has so far decreased that they are sometimes no longer even taught in the calculus course. In recent years, this concept has been criticized for not developing adequate perceptions of the basic concepts of calculus and not sufficiently preparing the students for scientific courses at university. In this paper, we will present an alternative discrete step-by-step approach to the basic concepts of calculus. It shows the theoretical basis of the on-going project "ABC – A discrete Approach to the Basics of Calculus" (see Weigand, 2014).

Keywords: Calculus, discrete mathematics, limit, derivative, digital technologies.

During recent decades, teaching the concept of derivative in mathematics classrooms has changed. In the seventies and eighties of the last century, especially in continental European countries such as France, Italy, Germany, and the Eastern European countries, the limit concept was based—in close relationship to college or university mathematics—on extensive work with sequences. A formal definition of the limit of a sequence and the proving of certain theorems concerning the convergence of sequences were the basis for the definition of the derivative of real-valued functions. In the late eighties and nineties—based on the calculus courses of Emile Artin (1957) and Serge Lang (1964/1973)—the model of an *intuitive limit concept* was introduced in mathematics education, which was adopted as a concept for high schools (Blum 1975) and has since then been widely accepted in schools and is the dominant concept today (Törner et al., 2014).

Within the frame of the intuitive limit concept, no formal definition of a limit occurs. The access to the derivative starts with the discussion of real-valued continuous functions, using an intuitive limit concept such as "... coming closer and closer ..." and calculating the derivative of polynomial functions by algebraic transformations without any formal definition of the limit. Using sequences is not necessary in this concept, at least as long as one does not consider more complex functions, for example trigonometric and exponential functions. The idea of the original concept of a simplified limit concept in the 1980s has been to consider sequences "later," but in reality, for example in many new curricula such as in Germany, sequences are no longer part of calculus in mathematics lessons.

It becomes clear that this has been a turning point regarding the concept of calculus in mathematics lessons. The changes in the access to the derivative concept changed the contents and the structure of the entire calculus curriculum. A concept-oriented approach to calculus was substituted by an application-oriented approach. There is a danger that learners stay on an intuitive and technical level and that basic ideas or conceptions for a content-oriented or integrated understanding of the mathematical concepts are not given.

In the following, we ask for the *understanding* of the basic concepts of calculus, concerning:

- the present situation and (empirical) results if we look at the knowledge of students and freshmen at the university;

- the theoretical basis or the basic knowledge to be able to understand the mathematical concepts of limit and derivative;
- a constructive strategy to develop these basic concepts in the classroom.

This article gives a theoretical framework for a new or an alternative access to the basic concepts of calculus. It uses digital technologies as a calculation and drawing tool to present sequences or discrete functions and to compare the properties of different functions or sequences. It has to be seen as a basis for a follow-up empirical investigation.

CONCERNING THE UNDERSTANDING OF THE BASIC CONCEPTS OF CALCULUS

The understanding of the *concept of limit* has quite often been the subject of theoretical reflections and empirical investigations (Keene et al., 2014). It is well known that many students have problems with the formal definitions of the concepts of limit and derivative. They are either not able to use the definition properly in a given context, or they are able to solve problems on a formal level, but lack an advanced understanding of the concepts (e.g., Tall & Vinner, 1981). The main results of the investigations concerning the learning, teaching, and understanding of the limit concept in the last decades are:

a) Conceptual *understanding* of the formal limit concept is challenging for high school students as well as for some college and university students and requires explanations and visualizations using different representations (beyond the symbolic representation).

b) The understanding of the *process of the construction or calculation of limits* in the sense of *step-by-step processes on numerical and graphical levels* is essential for the understanding of the limit concept beyond a formal definition. This can be supported by computer visualizations.

To understand the *concept of derivative*, it is necessary—besides understanding limit processes—to have adequate conceptions of the *rate of change* and to understand—in relation to limit processes—the transformation from the *average* rate of change to the *local* rate of change (see Rasmussen et al., 2014).

There are numerous propositions concerning the use of digital technologies and their dynamic possibilities of visualizing the approximation processes on a numerical and graphical level (e.g., Kidron & Zehavi, 2002; Martinovic & Karadag, 2012). All those suggestions have in common that they work with real-valued continuous functions and visualize—with programs such as Geogebra[1]—the limit processes dynamically with a sequence of secants converging to the tangent in a point of the graph of the function and/or the numerical process of convergence in the frame of a table (in a spreadsheet). The necessary transition from the continuous perspective to the discrete stepwise process—the discretization process—concerning the limit process, which includes selecting either a sequence of points on the graph or a sequence of numerical values converging to a selected value of the function or to a point on the graph, has to be made by the learners on their own.

The detailed (re-)construction of the limit process and the possibility of step-by-step thinking in the frame of this process has always been the strongest argument for working with sequences and their limits before starting to work with the limit of real-valued functions and their limit processes, for example the first derivative.

SEQUENCES AND DIGITAL TECHNOLOGIES

As a consequence of the increasing role of digital technologies in mathematics and mathematics education, discrete mathematics, and hence sequences, have gained importance. This was emphasized by the NCTM *Standards for School Mathematics* (1989), which included discrete mathematics as a separate standard for grades 9 to 12: “Sequences and series … should receive more attention, with a greater emphasis on their descriptions in terms of recurrence relations.”[2] Sequences are prototypes of discrete objects in mathematics.

In the *Principles and Standards for School Mathematics* (NCTM 2000), however, discrete mathematics is no longer a separate standard but is now distributed across the standards and spans the years from kindergarten through twelfth grade. *Iteration and Recursion*

1 www.geogebra.org
2 http://standards.nctm.org/Previous/CurrEvStds/9-12s12.htm

are explicitly emphasized as one of the three important areas of discrete mathematics.

Even though sequences are not explicitly defined or introduced as a separate concept in the mathematics curriculum, they are used quite often implicitly or in an intuitive way: Many real-life problems allow mathematical representations with sequences, for example growth processes or problems involving goods and their cost, or approximation algorithms such as the Heron-method for calculating irrational numbers or the Newton-method for calculating zeroes of functions are based on iteration sequences.

Nowadays, computers or digital technologies make it possible to generate sequences, to create symbolic, numerical, and graphical representations, and to switch between different representations—by just pressing of a button. In the following, digital technologies are a tool allowing a discrete access to the concept of limit and derivative as a preliminary stage working with these concepts on a continuous level.

A STEP-BY-STEP CONCEPT FOR A DISCRETE APPROACH TO CALCULUS

We will now present a concept of a discrete access to calculus, which develops the concept of the average rate of change based on a discussion of various sequences by looking at discrete functions. The advantage of this concept is not presented in the beginning, the concept of rate of change is instead developed by using discrete examples. By gradually changing the step size of the discrete actions at hand, limit processes are prepared by comprehensible step-by-step actions and are thus easier to understand. Here, the computer is used both as a tool for the representation and visualization of sequences and functions, and as well as a tool for creating recursively defined sequences in particular, which allows the user to switch between symbolic, numerical, and graphical representations (see Weigand, 2014).

Level 1: Sequences and growth processes

Sequences can be explained or defined on a formal level via an explicit mapping $a_n: \mathbb{N} \to \mathbb{R}$, or they can be defined recursively. This is widely used for the representation of growth processes, for example *linear* growth by , *exponential* growth by $a_{n+1} = A \cdot a_n$, and *limited* growth by $a_{n+1} = a_n + P \cdot (B - a_n)$, $n \in \mathbb{N}$, while all other variables are being real numbers. These sequences can easily be visualized using a spreadsheet or a computer algebra system such as Geogebra. The main goal of this first level is to become acquainted with the recursive kind of definition of sequences, to see the relationship between local aspects, between successive elements, and global aspects of the whole sequence, and to see the dependence of elements of the sequence on the initial value and the parameters. In (Thies & Weigand, 2003) and (Weigand, 2004) it is shown that high school students (grade 11) can solve problems in the frame of growth processes while working experimentally with digital representations of recursively defined sequences.

Level 2: Difference sequences

The aim of this second level is the introduction of the concept of difference sequences $(\Delta a_n)_{\mathbb{N}}$, $\Delta a_n := a_{n+1} - a_n$, of a given sequence $(a_n)_{\mathbb{N}}$. The concept may be introduced in connection with real-life problems, for example the average air temperature in one year during the last 100 years, which may be presented in a table and a graph. The given relations in examples like this are not based on algebraic formulas. This would encourage students to not immediately working on a formal level and foster the understanding of the relationship between the sequence and the difference sequence by operating s step-by-step.

Level 3: The concept Z-functions and their difference functions

3.1 Quadratic Z-functions

Starting with sequences or functions defined on the domain $\mathbb{N}$, we gradually extend the concept of sequence to functions defined on $\mathbb{Z}$, $f: \mathbb{Z} \to \mathbb{R}$, and advance to more subdivided discrete domains. We call functions $f: \mathbb{Z} \to \mathbb{R}$ "Z-functions." These functions f with $y = f(z)$ are "extended sequences," defined on integers $z \in \mathbb{Z}$,

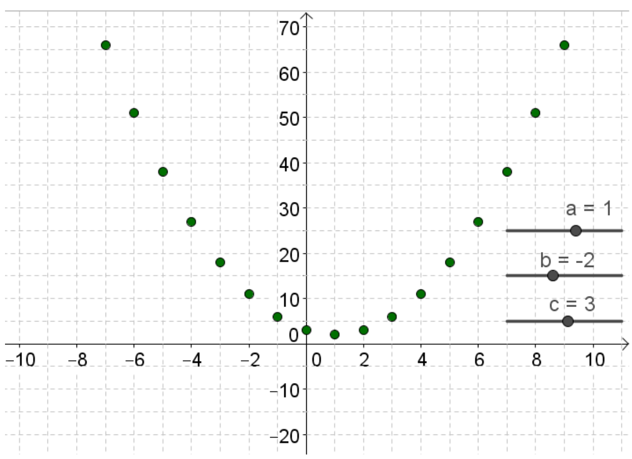

Figure 1: $f(z) = z^2 - 2z + 3$

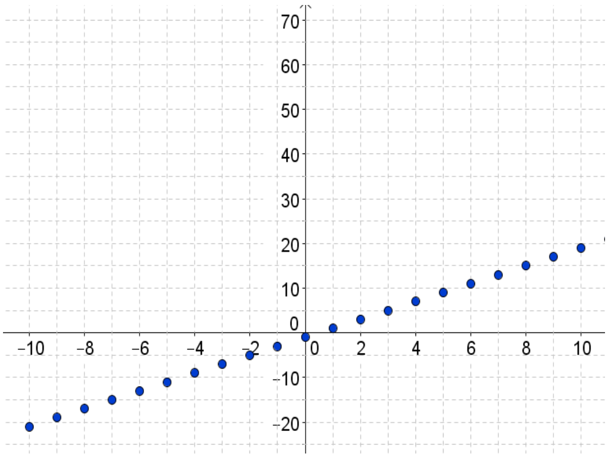

Figure 2: $D_f(z) = f(z + 1) - f(z)$

for example $f(z) = z^2 - 2z + 3$. We will now look at these Z-functions in relation to their difference-Z-functions D_f: $D_f(z) = f(z + 1) - f(z)$.

The values of D_f can be interpreted as the slope of a right-angled triangle with two legs of lengths $|f(z + 1) - f(z)|$ and $|\Delta z| = 1$. $D_f(z)$ is the *rate of change* of the graph between the points $(z, f(z))$ and $(z + 1, f(z + 1))$. Digital technologies are used to visualize the dependence of D_f on the used parameters of f: $f(x) = az^2 + bz + c$ graphically and to give reasons for the behavior of D_f.

3.2 Polynomial Z-functions

The concept of Z-functions can be extended to polynomial functions of a higher degree, as the respective difference functions can be obtained algebraically in an equally simple manner. For the family of Z-functions

$$f(z) = az^3 + bz^2 + c\,z + d,$$

for example, we get the (family of) difference-Z-functions D_f with

$$D_f(z) = 3az^2 + (3a + 2b)z + a + b + c.$$

We can see in particular that D_f is a quadratic function, which is also apparent in the graph.

As an example we look at the Z-function $f(z) = 0.1z^3 - z + 1$, with the corresponding difference-Z-function $D_f = 0.3z^2 + 0.3z - 0.9$ and their respective graphs.

The calculations can easily be extended to difference functions of higher order Z-functions, especially by using a CAS.

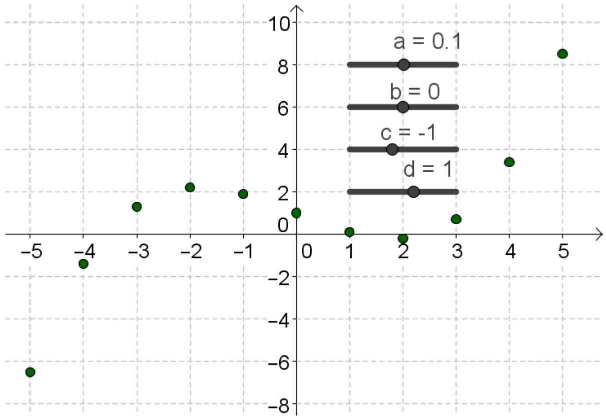

Figure 3: Z-function $f(z) = 0.1z^3 - z + 1$

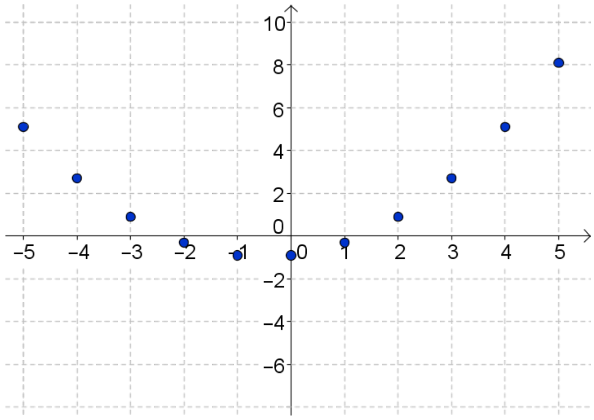

Figure 4: $D_f(z) = f(z + 1) - f(z)$

The advantage of working with these discrete functions is the possibility of obtaining the *rate of change of discrete polynomial functions* only through algebraic transformations and the possibility of *step-by-step argumentations* concerning the properties of the function, especially concerning the rate of change and the difference function.

3.3 Exponential Z-functions

Using the graphical representation of the Z-function $E(z) = a^z, a \in \mathbb{R}^+, z \in \mathbb{Z}$, and its difference-Z-function, we obtain the graphs shown in Figures 5 and 6.

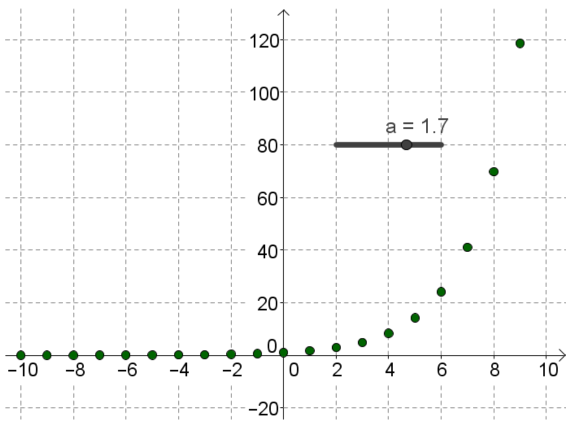

Figure 5: The Z-function $E(z) = 1.7^z$

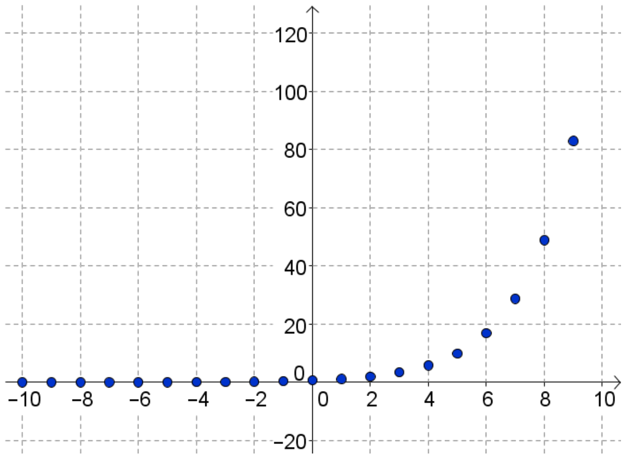

Figure 6: Difference sequence

It is striking how similar the "shapes" of the graphs of the Z-function and difference-Z-function are. A calculation on the formal level gives the following result:

$$D_e(z) = E(z+1) - E(z) = a^{z+1} - a^z = a^z(a-1) = E(z) \odot (a-1).$$

The value $D_e(z_0)$ is obtained by the difference sequence for a given value z_0 by multiplying the value $E(z_0)$ by the factor $(a-1)$. Geometrically, the graph of the Z-function is the result of an orthogonal affinity of the difference-Z-function with the z-axis as the axis of affinity. For $a = 2$, the two graphs match exactly! There is therefore a value for which the difference-Z-function equals the Z-function!

Level 4: The function of the difference quotients

The difference function D_f with $D_f(z) = f(z+1) - f(z)$ of a Z-function f can be interpreted as the slope of the Z-function f concerning the points $(z, f(z))$ and $(z+1, f(z+1))$ of the graph of f.

The next step of an expansion of the Z-function is considering a domain with non-integer values, but

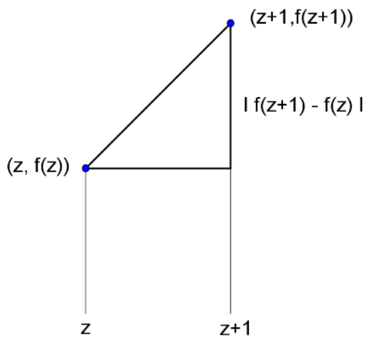

Figure 7

we will still remain within discrete domains. The idea of the difference function as well as the calculation of the slope can be used as long as the domain consists of discrete values.

In a first step the domain $\mathbb{Z}$ of the Z-function f is expanded by considering the values $z_{10} = \frac{z}{10}, z \in \mathbb{Z}$. This means $z_{10} \in \mathbb{Z}_{10} = \{..., -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, ...\}$ and we obtain the Z_{10}-function $f_{10}: \mathbb{Z}_{10} \to \mathbb{R}$. To get the rate of change of successive values, we restrict the calculation to an interval of the length , and get the *difference-quotient-Z_{10}-function*

$$D_{f10}(z_{10}) = \frac{f(z_{10} + \frac{1}{10}) - f(z_{10})}{\frac{1}{10}},$$
$$z_{10} \in \mathbb{Z}_{10} = \{..., -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, ...\}.$$

For example, the difference-quotient-Z_{10}-function belonging to the cubic Z_{10}-function $f(z_{10}) = 0.1z_{10}^{\ 3} - z_{10} + 1$ is

$$D_{f^*}(z_{10}) = \frac{f(z_{10} + \frac{1}{10}) - f(z_{10})}{\frac{1}{10}} = 0.3z_{10}^{\ 2} + 0.03z_{10} - 0.999$$

This can be generalized to an interval of the length $\frac{1}{n}, n \in \mathbb{N}$, and the difference-quotient-Z_n-function (see Weigand, 2014).

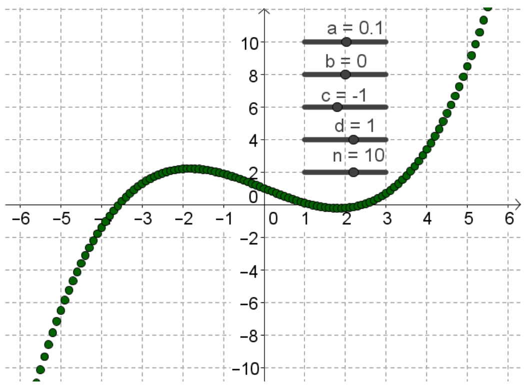

Figure 8: The Z_{10}-function $f(z_{10}) = 0.1z_{10}^{\ 3} - z_{10} + 1$

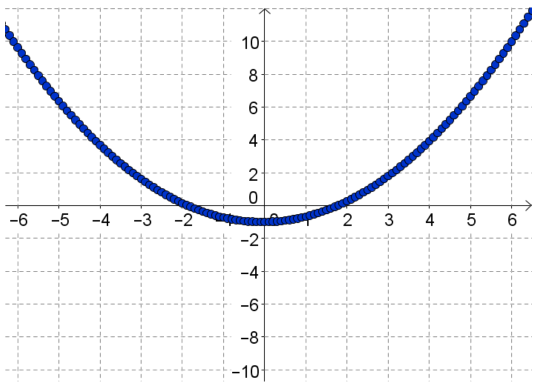

Figure 9: The difference-quotient-Z_{10}-function of f

It emphasizes the *global view of a function* and its (discrete) *difference-quotient-function*, and from the beginning—while working with Z-functions—it draws the attention to the relation between *function* and *difference-Z-function*. Thus, it prepares the understanding of the relation between a function and its derivative function.

Level 5: The local rate of change

The preceding steps to the access to the derivative emphasized the *global view* of the function and the difference-quotient-Z_n-function. The next step will be the concentration on the local view of a function while seeing the relation to the local rate of change of a function.

We continue with any real function f, choose a fixed value $z_0 \in \mathbb{Z}_n$, or even a generalized value $x_0 \in D \subseteq \mathbb{R}$, and consider the sequence of the difference quotient for a real-valued function f with respect to the value of x_0 for $n = \{1, 2, 3, \ldots\}$:

$$n \to D_n(x_0) = \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}}.$$

Thereby, we obtain a—special—sequence of difference quotients for the function f at the point $(x_0, f(x_0))$. For $f(x) = ax^2 + bx + c$ we get:

$$D_n(x_0) = 2ax_0 + b + \frac{a}{n}.$$

Now, the sequence $D_n(x_0)$ can also be interpreted—considering the graph of f—as the sequence of the slope of the secants through the point $(x_0, f(x_0))$.

Seeing the construction of the derivative of a function f in a special point of the graph of f as a sequence of slopes of secants, the discrete-Z_n-functions with growing n provide a basis for the calculation of the *local rate of change*.

CONCLUSION

The way presented here can be seen as a preliminary stage of the—nowadays already traditional—intuitive limit concept approach with continuous functions. The advantage of the discrete way described here is that working with continuous functions–subsequently after the discrete approach and the working with sequences or Z-functions–and the development of the concept of derivative can be based on a content-oriented level of understanding of the limit concept. To develop this level–beyond an intuitive level of understanding–is the main reason while explicitly working with sequences or discrete functions. Applying this idea of discrete actions or calculations, the concept of derivative of continuous functions is developed on mathematical aspects and properties of the limit or the approximation process and not only on intuitive perspectives.

It is expected that the proposed strategy prepares the concept of *local derivative* of a function at one point and gives a chance of a better understanding of this approximation process because of the possibility of the stepwise construction of this process. But it is also expected that the parallel presentation of sequences (or discrete functions) and their difference sequences (or functions) allow also a well-founded understanding of the concept of a *(global) derivative* function. The aim of the proposed concept is the better understanding of the concept of derivative. It is part of the project *"ABC – A discrete Approach to the Basics of Calculus" (see Weigand 2014)*. It is partially—concerning the first levels—empirically evaluated, theoretically extended to a global concept regarding the access to the derivative, and now needs to be evaluated in an authentic classroom setting. It does not—and cannot—avoid the "cognitive conflict" (Tall, 1992) or the necessity of a conceptual change from the discrete thinking used in working with sequences, difference sequences, and rates of change or difference quotient sequences to working with limits and derivatives of real functions. But it successively develops and explains the approximation processes for an understanding of the derivative concept between the intuitive level, nowadays widely used in classrooms, and the formal mathematical level at university. It emphasizes the processes of understanding the concepts; it is a "procept" (Gray & Tall, 1994; Tall, 2013) for an access to the derivative.

The next step in the frame of this project is the construction and development of classroom, learning or teaching units and the empirical evaluation of the results. This will be done in the near future.

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