Reasoning with dynamically linked multiple representations of functions

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In video-taped interviews, teacher-students were asked to describe and explain effects of a parameter a on the standard representations of f(x)=x²+a in a computer based dynamic learning environment. An analysis of one particular interesting yet typical misconception leads to differentiating between surface perception and structural insight. Theoretical considerations based on Duval and Davydov lead to postulating that for a full understanding of the relation between a and f(x)=x²+a, a learner needs to identify to structural analogies between the representations of f. A qualitative analysis of further interviews results in a category model of student responses that can be used for diagnostic purposes.

Keywords: Multiple representations, functions, abstraction, ICT, qualitative analysis.

INTRODUCTION

A typical task connected with the use of mathematical software in classroom is to explore the relations between the standard representations of functions. For example, pupils could be asked to explore the effects of a parameter a in $f(x) = x^2 + a$ on the shape and position of the graph of f by means of a dynamically linked multiple representation learning environment, where the value of a is controlled by a slider (Figure 1).

Usually, it seems sufficient to observe that changing the value of a causes the parabola to move upwards while a gives the distance. But is it really as simple as that? Just describe what seems, literally, obvious?

Duval (2002) argues that for showing a full understanding of the concept of function, a learner needs to be able to change within and between various representations of a function, for example, equation, table and graph. This means that properties of one representation are explained by properties of another.

If a learner is not able to perform such a change then, following Duval's rationale, he does not understand to the full extent, even if his observations within one representation appear to be perfectly valid.

This article begins with a case study that illustrates Duval's concept of understanding functions. A student describes the effects of the parameter a on the graph of $f(x) = x^2 + a$ (Figure 1). She then realises that one of her observations contradicts what she has learned about how graphs and equations connect, but she is not able to resolve this contradiction. She knows a lot, yet she does not quite understand.

So it appears that learning about functions based on visual perception only is not sufficient. For developing a sound understanding a learner needs to turn his attention from the visual properties of the different representations to structural analogies between them. These analogies do not equate to perceivable similarities between the representations. For identifying structural analogies one needs to "see through" the specific appearance of each representation. In this sense, the identification of structural analogies is a form of abstraction. Hence this article then turns to the concept of learning by scientific abstraction by
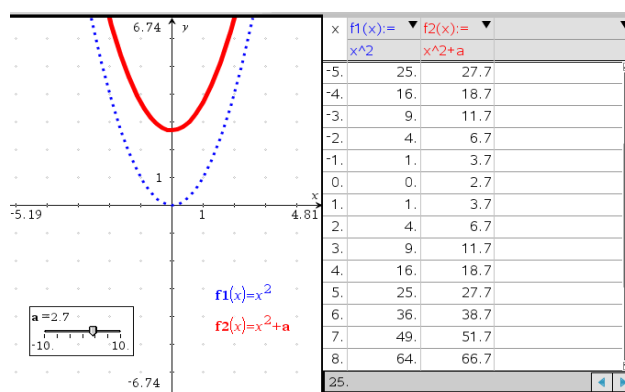

Figure 1: Dynamic multiple representation environment for exploring the effects of *a* on the representations of *f(x) = x² + a*

Davydov (1972). It serves as a suitable theoretical basis for justifying the postulation that learners need to refer to a structural level.

When theoretical considerations lead to expect that learners behave in a specific way then we must be certain that they are able to do so. The last part of this article reports on a qualitative analysis of further interviews with teacher-students where three categories of understanding could be identified, among them references to structural analogies between representations as required by theory.

A CASE STUDY

In videotaped interviews, teacher-students of the University of Education Heidelberg were given the task as shown in Figure 1. They first were asked to describe the changes within graph, table and term when the slider is operated. Then they were asked to explain why they thought their descriptions were correct. One student described the effects of the slider as both a translation and a change of shape:

1   Student: [moves slider to the right] well the parabola moves upwards along the y-axis … the um how is it called the width [moves both palms repeatedly towards each other as if clapping hands]
2   Interviewer: yesyes, ok
3   Student: changes … anyway when one moves it to the right … towards the positive … [moves slider to the far right such that the parabola nearly vanishes from the screen] … and when one moves it downwards to the negative [moves slider to the left such that the parabola's vertex nearly touches the lower screen edge] and here the parabola becomes wider but still opens upwards
4   Interviewer: can you explain why the parabola moves up or down when one changes the slider value [points at the slider with a pen]
5   Student: [looks a the slider, murmurs] hm what is a … what … is a [leans back, talks louder] a is … a was something … a is … the y-direction
6   Interviewer: ah
7   Student: when I … upwards … well then it is not a normal parabola any more then it is, like, somehow narrower

When asking the student to explain her observations (line 4) the interviewer, who is the author of this text, ignores that she describes the effects of a as a translation as well as a change of shape of the curve (lines 1 and 3). The student answers by referring to knowledge about the effects of a she probably had acquired at school (line 5). But, again, she immediately points out that the curve looks shrunken (line 7). This seems to be a problem for her as it conflicts with what she has learned about parameters and graphs (meanwhile the value of a has been set to 1):

8   Student: this is simply a normal parabola that has been moved upwards by one, but the width changes too.
9   Interviewer: why do you say 'but'?
10   Student: the parabola does not move upward only but it becomes narrower too, and then there should be something in front of the [x] squared
11   Interviewer: ah, why?
12   Student: because that's how I learned it

That the parabola has been reduced in size is, with respect to the algebraic structure of x^2+a, wrong. The student is aware hereof since she expects a factor in front of x^2 (line 10) which could explain that the curve's shape seems changed. Yet, considering her descriptions of the effects on the graph only, she proves to be a careful observer. The parabola does appear to be shrunken (Figure 2).
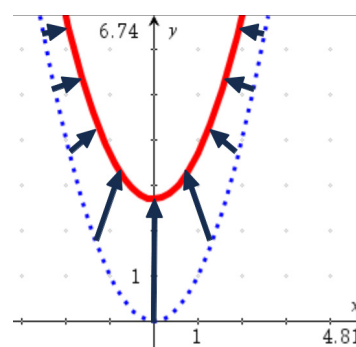
Figure 2: Moved upwards only or shrunken too?

The student realises that her description contradicts what she knows about the effects of a parameter change on the function graph. To resolve this contradiction she needs to explain the effects of a on the graph by means of the specific algebraic properties of the equation, i. e. she needs to perform a coherent representation change. This is achieved by considering the additive structure of the expression x^2+a:

Every single value of $f(x)=x^2$ is increased by a, so each single point $(x|x^2+a)$ of the parabola has been moved upwards from its original position $(x|x^2)$ by a, which affirms that the effect of a can only be interpreted as a vertical translation. So a valid interpretation needs to be based on an analysis of the term structure. Hence algebra plays a decisive role for describing and explaining relations between parameters and functional representations. When a learner describes the effects of a parameter change without giving a sufficient explanation on the base of the algebraic representation it is not clear whether his descriptions are based on his visual impressions only or whether they are based on a structural insight into the situation. This is particularly problematic when these descriptions fulfil the expectations of the teacher: If a learner only mentions a vertical translation by a, does he just reproduce what he sees, or does his description reflect an understanding of the `mechanism´ of how a affects each representation of x^2+a? One risks confusing a correct verbal description with an understanding of the situation, pupils as well as teachers.

Referring to possible restraints of the tool and its use – e.g., lack of visual aids or failing to adjust the window – misses the point this article is trying to make. The fact that misinterpretations can appear indicate a fundamental difference between a concept and its representations. A learning environment with a "perfect" design that avoids misinterpretations does not guarantee that a learner understands how the different representations interact. While giving all the correct answers, he might only reproduce what he sees on the screen, thus confusing the concept with its representations. Each learning environment needs to activate the learner's reflection (Yerushalmy 2005), which means – as it is argued here – that he turns from describing perceptually accessible properties of each representation towards analysing analogies between the three representational forms on a structural level.

UNDERSTANDING MULTIPLE REPRESENTATIONS BY ABSTRACTION

Does the student from the case study understand the effects of a on the representations of x^2+a? She gives a careful description of what she sees on the screen. And she knows, too, how the effects of a parameter on a function graph can be read from a given equation. But she cannot resolve the conflict between her obser-

vations and her knowledge. In this sense, she shows a lack of understanding.

So what does she – or a learner in general – need to achieve so that he or she shows a full understanding of the relation between parameters and the representations of function? Giving a description of his observations only is not sufficient as we have argued, even if the description is correct. He needs to give reasons for why his observations are valid, i.e., why they are consistent with the whole of the multiple representation environment. Reasoning by referring to well-known rules might be acceptable, but in the case of the student more was needed to clarify the conflict between what she saw and what she knew. It would be helpful to explain the mechanism of how a and representation of x^2+a connect. This means to refrain from reproducing visual information but to analyse analogies between the representations on a structural level. Focussing on structures instead of surface leads to a cognitive activity that is central for the learning of mathematical concepts: abstracting.

In the context of learning with a dynamic multiple representation environment abstracting means extracting the essential information from representations by conceiving structural analogies between representational forms while eliminating irrelevant surface properties. By referring to the concept of scientific abstraction by Davydov (1972) and others we will show that conceiving structural analogies is achieved by identifying invariants in the dynamic multiple representation environment while the structure of the algebraic representation is decisive.

Mathematical concept formation as the result of abstraction

Mitchelmore and White (2007) identify two different approaches to abstraction among theories of mathematical learning, which they call empirical and theoretical.

Empirical abstraction refers to a cognitive representation of knowledge that results from identifying common properties in a set of examples. "Abstracting is an activity by which we become aware of similarities […] among our experiences. […] An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class" (Skemp 1986). However, empirical abstraction that is limited

to a perceptual analysis of real or cognitive objects can hardly explain the formation of such concepts that meet the scientific requirements of generality and precision. Hence, for the formation of a scientific concept, a theoretical basis is needed which supports argumentation that is independent from perceptual evidence.

Theoretical abstraction: To be valid beyond experience, knowledge needs to be developed within a theoretical system of its own, which comes with specific symbolic representations and rules of argumentation. Following Vygotsky (1934/1986), this symbolic representation form does not need to resemble any physical features of the knowledge that it represents. In fact, perceptually or otherwise empirically accessible properties are unsuitable for forming an abstract concept. "A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole [...] Such a concept, in contrast to an empirical one, does not find something identical in every particular object in a class, but traces the interconnection of particular objects within the whole, within the system in its formation" (Davydov 1972/1990, 255).

For continuing our case analysis the concept of theoretical abstraction – or "scientific abstraction" as Davydov puts it (1972/1990) – appears to be suitable: The effects of a on x^2+a can be described as a function $a \rightarrow x^2+a$, which is a function different from f. Its properties can only be derived from changes within the representations of f. The change within the graphical representation of f appears to be a translation and a dilation, here the student is perfectly right. To decide whether this interpretation conforms with the rest of the multiple representation environment, the algebraic expression of $a \rightarrow x^2+a$ needs to be taken into consideration. Its additive structure decides which of the two interpretations of the effect of a on x^2+a is valid. So it is knowledge about symbolic algebra that forms the necessary theoretical basis for understanding relations between the multiple representations of a dynamic learning environment. However, the student is not able to apply her knowledge about algebra to explaining how, or whether at all, her descriptions are valid for the whole of changes within the multiple representations of $f(x) = x^2+a$.

Abstracting as conceiving structural analogies

In a teaching concept of "ascending from the abstract to the concrete" based on Davydov, Giest (2011) states that, with each new learning process, "initial abstractions" are gained from examining a learning material that allows change and variation. From varying the material, invariants become apparent that initiate the necessary reasoning for identifying a constant structure within change. With a dynamic multiple representation environment (e.g., Figure 1) the necessary variation here is twofold. First by operating the slider, thus changing the visible appearance of each representation and second, by switching between the three representation forms. These changes correspond to Duval's (2002) forms of representational changes that characterise a full understanding of the concept of function.

Obviously, the student from the interview meets the first of Duval's requirement at least partially. Within the graphical representation, she gives two pertinent interpretations of the effects of the parameter change on the actual function graph, and from the algebraic representation she can read correct information about the effects of parameters on function graphs in gene. But she does not fulfil the second criterion of Duval. She is not able to make a coherent change between the algebraic and the geometric form of representation here. Or to say it with Giest: She is not able to identify the necessary invariants within the multiple representation environment.

Conceiving structural analogies as identifying invariants

In Figure 1, the invariant in question is not visible, it becomes apparent as changes between and within representational forms. Considering the additive structure of the term of x^2+a, the invariant is characterised by the common operator $+a$. In the equation, the invariant is the summand $+a$ that redefines $f1(x)=x^2$ to $f2(x)=x^2+a$ (Figure 3). In the table, the invariant is the constant difference between the values of $f1$ and $f2$ in all table lines. In the graphical representation, the invariant is the constant vertical distance between the two graphs of $f1$ and $f2$, which is the same at all points. Thus, the invariant has a specific meaning in each representation form, yet, in each form, it can be visualised by an arrow with constant direction and length. Especially the arrows from the geometric representation form show that the effect of a on the graph of x^2+a must be interpreted as a translation only.
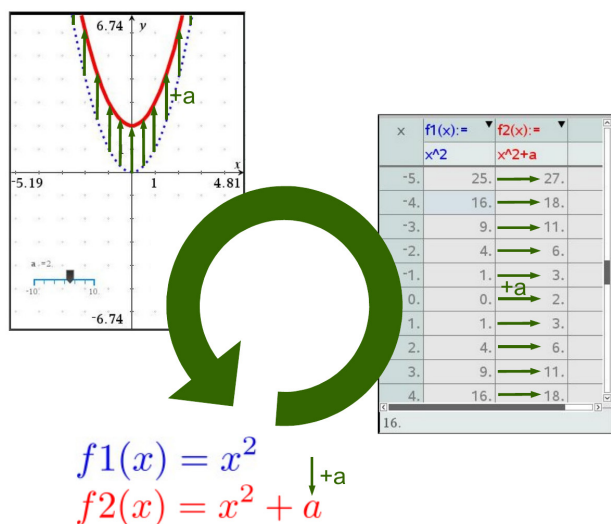
$$f1(x) = x^2$$
$$f2(x) = x^2 + a$$

Figure 3: The invariant + *a*, identified in all three representation forms, shows that the effect of *a* on *x*² + *a* is indeed a translation

We can sum up now: To show understanding of the relation between a parameter *a* and the representations of *f(x)+a* a learner needs to identify the operator +*a* as an invariant within each representation and between all representation forms of *f*.

This seems expecting much from pupils. However, next we report on a study where references to structural analogies in further interviews could in fact be identified.

CATEGORIES OF REASONING: AN INTERVIEW STUDY

Aim and methods

Together with the interview from our case study further interviews were analysed with the aim of categorising students' answers regarding to what extent they showed structural insight into the relations between graph, table and term of *f*. The interviewees were teacher-students of the University of Education Heidelberg from their first to their third year of study, all having selected mathematics as one of their compulsory subjects. The interviews contained questions about various tasks about for exploring the relation between parameters and quadratic functions. The task from Figure 1 was the first. All were accompanied by dynamic multiple representation learning environments, prepared in advance by means of the TI-Inspire CAS software on a laptop. Apart from two questions – the initial one that asked for a description of how the given parameter affected the appearance of graph, table and equation and one that asked the students to explain why they thought their observa-

tions were correct – no other questions were fixed in advance.

From the case study and subsequent theoretical considerations above, two a-priori categories were formulated, "reason with reference to a rule" and "reason with structural reference." Roughly, the first comprises all statements where students refer to what they have learned and believe to be generally true, while the second covers all statements that refer to invariants as described above. A refined definition will be presented in the results section. The research questions were as follows:

(A) To what extent can the descriptions and explanations from the interviews be assigned to the two a-priori categories?

(B) How did interviewees exemplify reasons with structural reference?

For the analysis, six interviews were selected which, on first view, promised a sufficiently large range of students' observations and explanations. The categories were developed by means of a qualitative content analysis (QIA, Mayring, 2010). The QIA is a systematic method for text analysis guided by pre-set coding rules. In the variant of the deductive category application these coding rules are derived from the relevant theoretical framework, which then are applied repeatedly by trained coders with the aim of refining the rules by enhancing inter-subjective comprehension. The coders were three teacher-students who did not take part in the interviews selected for coding. As mathematics students they were able to understand the specific terminology of coding rules while, as students, they still relied on a very precise formulation of the rules to agree on common coding.

Results and discussion

Apart from the two a-priori categories two more emerged from the coding process. First a category "proposition" that separated observations prompted by the interviewer's request to describe from explanations prompted by the request to explain, which often were very similar. For this, the well-known Toulmin model of argumentation was introduced to the coders; the model itself was not object of analysis here. Second a category "reason with reference to an example" which became necessary to allow for reasons that explained rules by referring to single values of

x or *a*, mostly *x=0* or the actual value of *a*. Eventually, the third and most successful coding round in terms of intercoder reliability was based on the following rules:

— Category A "Proposition": All statements or gestures that can be seen as answers to the interviewer's initial prompt "Describe what you see when you operate the slider" and that show characteristics of a general rule. They refer to the effects of changes of the slider on the representations of the given function. If the interviewee modifies his propositions later during interview, these modifications are coded within this category too.

— Category B.1 "Reason with reference to a rule": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer to a rule or sound like one. Such rules often refer to connections between the parameter and the curve's shape or position. The statement does not refer to single parameter values but shows that the interviewee implies a global validity. Such rules do not need to be stated explicitly but the interviewee can also indicate that he knows of such a rule (e.g. "that's how I have learned it"). Even when the interviewee shows understanding at a structural level in some other part of the interview, reasons

with reference to rules or knowledge are to be assigned to this category.

— Category B.2 "Reasons with reference to an example": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer a single value of x or to the actual value of the parameter or the actual state of the visible graphical configuration. These statements often refer to connections between the actual value of the parameter or a single value of x and the curve's present shape or position. Even when the interviewee refers to rules or shows understanding at a structural level elsewhere, reasons with reference to rules or knowledge are to be assigned to this category.

— Category B.3 "Reasons with reference to structure": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that apparently refer to an invariant between different representation forms or within one representation. The invariant here is the value of *a* which takes on a specific appearance in each representation: In the algebraic representation it is the summand +*a*, in the numerical representation it is the difference between the old and new
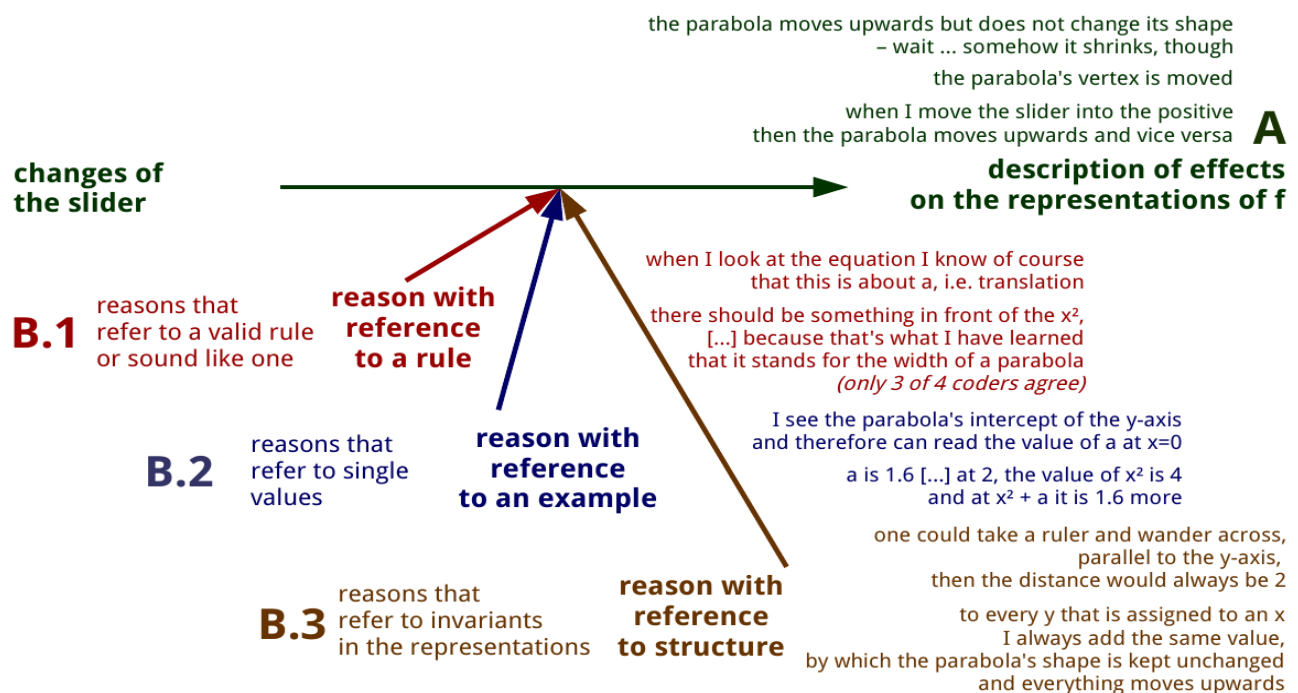

Figure 4: Categories of reasoning while exploring the effects of a parameter *a* on the epresentations of *x²* + *a* in a computer based learning environment

function value in each table line. In the geometric representation it is the vertical distance between the graphs of both functions at each point. It differs from category B.2 insofar as statements here do not refer to single values of a but to any value of a in general. It differs from category B.1 insofar as a has been globally (i.e., for all values of x) identified within a representation.

The layout of Figure 4 places the four categories at appropriate places within the Toulmin model of argumentation. The proposition (category A) corresponds to Toulmin's "claim" which are descriptions of the effects of slider changes. The three categories of reasoning (B.1, B.2 and B.3) are placed as warrants into the diagram. All categories are illustrated by statements taken from interviews after little linguistic polishing. Generally a statement was considered exemplary when all four coders including the author agreed. One statement for category B.1 is an exception which, to the author, still appears to be a significant example for this category.

For answering research question (A), both a-priori categories were suitable for categorizing students' answers when accompanied by the two more categories as defined by the coding rules above. As for question (B), the two statements cited here indicate a range of possible structural references from evoking dynamic images – here a ruler that moves vertically across the coordinate plane while measuring a constant distance between the two graphs – and a more static view on how the different representational forms connect – here pointing out that, for all x, a is added to the corresponding $f(x)$, which explains the congruence preserving effect of a. The fact that structural references were in fact observable shows that such references can be expected from students. However, these statements also show that references to structural analogies between representations do not need to be as formal as indicated above. The "moving ruler" argument is a convincing example.

The category system from Figure 4 covers responses from the case $a \rightarrow x^2 + a$ only, where the invariant is easily identified as the constant vertical distance between the two curves or the constant difference between the two function values, each illustrated by an arrow with constant length (Figure 3). With parameters in other places of the algebraic expression this is different: For example, with $b \rightarrow b \cdot x^2$ or $c \rightarrow (x + c)^2$ other (mis)

interpretations of the effects of the parameter can be expected. However, while coding rules need to be adjusted to these cases, the need for a structural reference still holds on theoretical grounds.

Prospects and consequences

Presently, standardised interviews are being developed for diagnostic purposes based on these results. Apart from diagnostic use, these results may be significant for classroom teaching too. Nearly all students and, in recent interviews, pupils reported that they had little, if any, experience with computers in school. Many were not able to explain the movements on screen which, to them, were totally new. These results plead for an extended use of dynamic software in mathematics teaching. In a dynamic environment, insufficient conceptions about function representations become apparent and can be dealt with openly. Last, these results show that, for an exploratory learning with computer based dynamic multiple representations too, a sound basic knowledge in algebra is necessary. Knowledge about term structure turns out to be essential as it plays a decisive role when validating the explorations.

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