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We analyse data from two studies in Ireland and Sweden relating to the concept of inverse function. In particular, we consider components of the participants' evoked concept images of this topic when answering open questionnaire questions. The results show that the students' concept images contain algebraic, geometric and more formal components: both Irish and Swedish students describe inverse functions as swapping x and y, as a reflection or a reversal. How various components may or may not enrich students' conceptual understanding of inverse functions is discussed.

Keywords: Inverse function, concept image, university mathematics education.

INTRODUCTION

In this paper, we aim to identify elements of undergraduate students' concept images of inverse functions. The concept of inverse function is usually covered in introductory calculus courses either at school or at university, with a well-developed conception of 'function' necessary for understanding 'inverse function' and deciding when an inverse function exists. However, as Pettersson, Stadler and Tambour (2013) argue, the function concept is a troublesome one for students and Carlson, Oehrtman and Engelke (2010) point out that students who are unable to conceive of a function as a process (rather than taking an 'action' view) have difficulties inverting functions. The concepts of function and inverses are essential for representing and interpreting the changing nature of a wide array of situations (Carlson & Oehrtman, 2005) and to describe the relationships between logarithms and exponentials for example. In this study, to enrich and broaden the data, we considered data from two projects in two countries (Ireland and Sweden). Both projects were focussed on the development of conceptual understanding of the function concept, and both involved collecting data from first year undergraduate students. The two studies included questions on the inverse function concept and we will present an analysis of these data here.

Despite the volume of research into the concept of function (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992), there does not seem to have been much work which concentrates on the topic of inverse function alone. One such paper is that of Even (1992) where prospective secondary mathematics teachers' knowledge and understanding of inverse function were investigated. In an open-ended questionnaire the participants were asked to find. Both the function () and the inverse were given. The answer is straightforward if one uses the idea of an inverse as undoing. However, the results showed that several students did not draw on their conceptual knowledge of the inverse property of undoing and instead used a chain of calculations to get the answer. This tendency to calculate instead of using the conceptual meaning of inverse function may be related to weak conceptual knowledge. However, Even (1992, p. 561) concluded that "a solid understanding of the concept of inverse function cannot be limited to an immature conceptual understanding of 'undoing'", which she claimed may result in incorrect conclusions, e.g. that all functions have inverses.

The conception of undoing is not the only way to look upon inverse functions. Vidakovic (1996) also placed importance both on composing the function and the inverse to get the identity, and on the action of swapping variables. Carlson and Oehrtman (2005) categorise three different conceptions of inverse function: inverse as algebra (swap x and y and solve for y), inverse as geometry (the reflection in the line $y = x$) and inverse as a reversal process (the process of 'undoing'). Carlson and colleagues (2010) showed that students who conceived of inverses as reverse processes were able to answer a wide variety of questions about inverses.

The algebraic and geometric views are considered in a paper from Wilson, Adamson, Cox and O'Bryan (2011). They argue that the common procedure of swapping x and y to find the inverse is confusing for students and can lead to the meaning of the result being obscured, especially for contextual or real-world problems. They contend that both swapping the variables and drawing the graph as a reflection in the line $y = x$ do not take into account the important aspect of the domain of the inverse function being the range of the function and vice versa. In particular this causes problems when the dependent and the independent variable of the function are in different units. Wilson and colleagues (2011) instead proposed the approach of solving for the dependent variable, rather than literally swapping x and y, to reduce confusion and enhance students' conceptual understanding of inverse functions. Attorps, Björk, Radic and Viirman (2013) commented on the geometric view, using GeoGebra to teach inverse functions. The results of their study indicated that several students showed an intuitive conception of inverse functions as some kind of reflection, but lacked the full comprehension of why and where the reflection should be performed.

Bayazit and Gray (2004) reported on teaching inverse functions to Turkish high school students and they observed that the students who showed a conceptual understanding of the inverse function put particular emphasis on the '1–1 and onto' conditions. They suggested that teaching should link the inverse function more explicitly to the concept of '1–1 and onto' as well as to the concept of function itself.

ANALYTICAL FRAMEWORK AND RESEARCH QUESTION

One way of studying students' conceptions is to use the theory of *concept image* (Tall & Vinner, 1981). This theory has, for a number of decades, proved to be a useful tool in analysing undergraduate students' conceptions of mathematical concepts (e.g., Bingolbali & Monaghan, 2008; Wawro, Sweeney, & Rabin, 2011). A concept image is defined to be the cognitive structure associated to a concept and includes the individual's interpretations of characteristics and processes that the individual connects to the concept. It also includes examples, intuitive ideas, mental images and, if known, formal definitions and theorems. The cognitive structure is built up successively through the individual's meetings with the concept. When meeting tasks involving the concept different parts of the concept image can be activated; the part activated at a particular time is called the *evoked concept image*.

In this study we are looking for components in the students' evoked concept images. Our research question is: What characteristic elements can be found in the evoked concept image of inverse functions of first-year university students?

METHODOLOGY

The Irish study

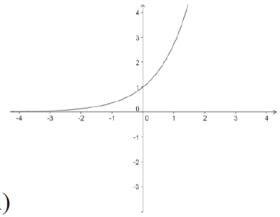
The data from Ireland involved students' responses to one of twelve questions on a concept inventory instrument designed to investigate undergraduate students' understanding of the concept of function (the full instrument can be found at http://staff.spd.dcu.ie/breens/documents/ConceptInventoryforFunction.pdf). First year Humanities, Education, and Finance students taking calculus modules (taught by the first and third authors) in two Irish universities were asked to voluntarily complete the inventory at the end of their module. 100 students took the test, 65 of whom answered at least part of Question I (see Figure 1).

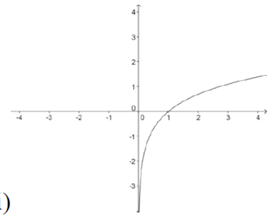
The second level syllabus followed by these students mentioned inverse functions solely in the context of inverse trigonometric functions and the textbooks did not contain formal definitions or geometric representations of inverses. Inverse functions were initially discussed in both university modules (recall these were taught by the first and third authors) as reverse processes, and the role of bijectivity in determining whether an inverse exists was identified. A formal definition of inverse (for all x in the domain of f and all y in the range of f) was presented, and means of adjusting the domain or codomain of a function to make it bijective and thus invertible was discussed. The graphs of a function and its inverse as mirror images of each other in the line $y = x$ were explored, while the algebraic method of finding an inverse was acknowledged when presented by students.

The Swedish study

The Swedish data were collected as part of a project aiming to explore students' development of their un-

I. (a) Look at the graphs (i) and (ii) below. The claim is that graph (ii) represents a function which is the inverse of the function represented by graph (i). Explain what this means and state whether you think the claim is true.


(i)


(ii)

(b) Look at the graph (iii) given below and draw, on the same axes, a graph to represent its inverse if you think there is an inverse. Explain clearly why you think there is an inverse or not an inverse.

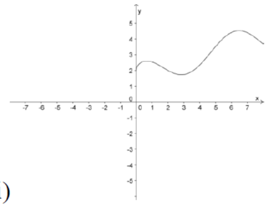

(iii)

Figure 1: The questions in the Irish study (I)

derstanding of a threshold concept (Pettersson et al., 2013). A study group, in total 18 prospective secondary teachers who were enrolled in courses in mathematics, was observed during their second semester of teacher education. In the second level syllabus in Sweden inverses are not explicitly included although they might be mentioned. At university these students were introduced to inverse functions in the course 'Vectors and Functions'. Because of lecture observations we know that the inverse function was defined by . Both algebraic and geometric aspects were mentioned and also that the function needs to be 1–1 for an inverse to exist. In a subsequent calculus course inverse trigonometric functions were included and the need to restrict the domain of the function to ensure it is 1–1 was discussed.

The Swedish students participated voluntarily in three questionnaires relating to their understanding of the concept of function (Pettersson et al., 2013). For the present paper, focussing on the concept of inverse function, the answers on three of the questionnaire tasks were analysed (see Figure 2). In a questionnaire at the end of the course on vectors and functions, the students were given the question S1. In the questionnaire given at the end of the semester, not far from the end of the calculus course, the students were asked two questions on inverse functions, S2 and S3.

S1: Make a mind-map or a concept map with the word inverse function as starting point. Bring in other words/concepts that you can associate with inverse function. Please also indicate how you think that the words/concepts are interrelated.

S2: What is required for a function to have an inverse?

S3: In the picture (see below) you see two graphs of two different functions. What can you, based on the picture, tell about these functions?

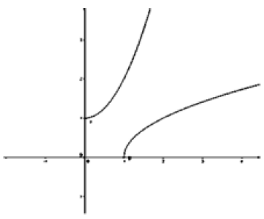


Figure 2: The questions in the Swedish study (S1, S2 and S3)

Coding

In each country the students' responses were coded using a grounded theory approach: that is the students' responses were read multiple times, codes were assigned, and these codes were then grouped into categories. At the coding stage, the responses to the tasks above were initially coded by one of the researchers, and then checked by another before the agreed codes were grouped into categories. The two sets of categories which emerged were then compared to check for consistency.

RESULTS

Results from the Irish study

We first considered the students' answers to I(a) above, i.e. the students' explanations as to what it means to say one function is the inverse of another. Only 4 students explained the concept correctly, 50 students gave an explanation which contained errors or was incomplete, while 11 answered true or false with no explanation. An example of a correct answer was:

> Let be the function in (i). Let be the function in (ii). The claim states that . It claims that is a reflection of through the line $y = x$. I agree with the claim. (Cathy)

The categories of components that arise in the evoked concept images of inverse function are shown in Table 1. Note that some students are counted more than once here if their answer referred to two or more of the concept image components identified. For example one student's answer was coded using 'opposite' and 'reflection' and so now appears under both 'Opposite' and 'Reflection' in Table 1.

We can see that the most frequent conception is 'reflection'. This category also includes responses from students who used the term 'mirror'. Its close associates of 'opposite' and 'symmetry' are also frequent. Seeing the inverse as a reverse process is common, while only 5 students gave anything resembling a concept definition of an inverse in answer to this question. There were 13 responses categorised in the 'other' category, these include: 6 responses which refer to a feature of the given graphs, 2 responses which mention 'folding', 1 response for 'domain and range interchanged', and 1 'example'. The remaining responses in the 'other' category are not mathematically relevant. We have included both correct and incorrect notions within each component of the concept image in Table 1; for instance, although 33 students used 'reflection' or 'symmetry' in their explanations, only 9 students correctly described the line of reflection or symmetry as the line $y = x$. Two referred specifically to reflection in the origin and 3 to reflection in the x-axis, while a further 6 spoke of a reflection without being specific. Students used 'opposite' when describing their concept image of an inverse in different ways: 6 of them used the word in a way that suggested reflection, and 3 mentioned graphs; 7 talked about functions being the opposite of each other; 2 spoke about opposite values. Some examples of responses in different categories are given in Table 2 below.

When asked if the claim given in I(a) was true, 41 of the 58 students who answered this question were able

Conception	Reflection	Opposite	Reverse	Symmetry	Definition	$1/f$	Swap x and y	Other
Total	27	17	10	6	5	3	1	13

Table 1: Conceptions emerging in response to I(a)

Conception	**Sample explanations given by students**
Reflection	Inverse is a reflection of the function. The inverse is that function mirrored through the line $y = x$.
Opposite	Inverse means they are exact opposites of each other, they cancel each other out.
Reverse	It is the function in reverse.
Symmetry	The inverse of a function is the function given through symmetry in the line $y = x$.
Swap x and y	When the x and y coordinates swap, e.g. here the point (1, 2) becomes (2, 1).

Table 2: Sample responses from Irish students

to correctly identify that the graphs shown did indeed represent inverse functions.

In response to I(b), 47 students attempted to draw an inverse function. 45 of these students sketched a reflection of some sort; Table 3 shows the distribution of these attempts.

Task I(b) was answered correctly by 11 students, that is, they were able to say that the function did not have an inverse and were able to give a reason for their answer. These reasons were: fails the horizontal line test (6), not 1–1 (6), inverse would not be a function (1) – illustrating a further component of the concept image held by these students. Note that two students said the function was not 1–1 and also illustrated using the horizontal line test which accounts for the numbers adding to 13. A further 4 students correctly stated that the function had no inverse but did not give a reason. Two students said that the function did not have an inverse but did not give a complete explanation, for example one of them said that "it is not just a line/angled line". The students who said that the function did have an inverse gave a variety of explanations for their answer: for instance, one said that every function has an inverse, while another said that the function was "defined for its whole domain". We saw that the conception of inverse function as a reflection in the line $y = x$ could be misleading with four students making remarks such as "There is an inverse as possible to draw line $y = x$ and reflex (sic!) images".

Students who gave correct and complete answers on I(b) offered a variety of answers to I(a) illustrating a variety of elements of concept images associated to inverse functions. These were: definition (2), reflection (3), reverse (1), opposite (2), no answer (3).

Results from the Swedish study

In the analysis of the survey, we first categorised the components of the evoked concept images that were exposed in the answers to task S1 (11 answers) and S3 (12 answers), see Table 4. The categories included also incorrect answers. Answers that contained more than one component were counted into more than one category.

To illustrate the components that arise in the students' concept images of inverse function we have picked answers from three frequent categories. In the category 'reflection' a correct answer to S3 was given by Anna: "They [and] are each other's reflections in the line ." Another student, Peter, gave through the mind map in S1 an explanation which is partly correct: "Reflection of a function in a line gives us an inverse function." Like several students Peter omitted the line in which the reflection takes place. Helena wrote correctly: "A kind of reflection of the function. Sometimes the reflection is not a function and then there is no inverse. [...] The reflection is performed in " But Helena also wrote: "A function can also be reflected in other ways, e.g. in or in the x-axis." That is of course true if you just talk about the graph, but it is irrelevant for inverse functions. Helena was the only student mentioning reflections in lines other than

The following excerpt from Anna, exposes a component of concept image categorised as 'example': " and is an example of inverse functions." Anna also drew a graph of the two functions and the line . Furthermore, Anna suggested that has no inverse. Other students also mentioned these functions and some students commented that if we restrict to , then the inverse exists. Answers which can be seen as exposing a component of concept image categorised as 'swap x and y' were given by e.g. Anna and Bob, who wrote more or less the same phrase "change places for x and y".

Type of reflection	In (0, 0)	In x-axis	In y-axis	In $y = x$	In $y = -x$	In $y = 2$
Total	22	13	3	3	1	3

Table 3: Answers to I(b)

Conception	Reflection	Example	Reverse	Graphical features	Swap x and y	Opposite	Definition	Other
Total	16	10	6	5	4	2	1	3

Table 4: Conceptions emerging in response to S1 and S3

Helena gave a more detailed version: "To get the inverse to , we switch x and y ."

The students often gave answers with several parts and each part was connected to one of the categories. To give an idea of a complete answer we present the answer from Frida, who gave a comprehensive explanation:

> The property for can be transformed into , e.g. ; . To 'raise' has the inverse process 'cube root'. is a reflection of in the line . when (inverse processes). There is a mapping of x such that the obtained value gives the original value, ; .

The analysis of task S2 (10 answers) was done with the same procedure as the other tasks. The categories of components of concept image evoked were: 1–1 (6 answers), continuous (3), reverse (2) and graphical features (1). For example, Dora stated "the function needs to be able to run backwards" (categorised as 'reverse') and continued with the following description, which was categorised as '1–1':

> A 'regular' function needs only to satisfy the requirement that each x is mapped to one y. But several x can have images at the same y. To have an inverse function it must additionally meet the requirement that each y can be traced back to only one initial x.

> Like most students Dora did not use any mathematical word for injectivity but our interpretation is that she understood that the function must be 1–1 and had grasped the difference between the definition of a function and of an injective function.

DISCUSSION

The data presented here come from studies in two different educational systems and as such are quite rich. The tasks given in the two studies touched on the same content but differed in several ways. In spite of that, the answers revealed similar components of the concept images evoked; in particular, the notions of reflection, reversal and injectivity were found to be important. We saw that students' concept images contain algebraic, geometric, as well as more formal components. However, very few students in either study gave a comprehensive explanation (similar to Frida's above) or attempted a formal definition of an inverse function (such as Cathy as seen previously) in response to the tasks assigned.

The intuitive conception about inverse functions as reflections noted by Attorps and colleagues (2013) emerged also in this study: 33 of 58 Irish students who stated if the claim in I(a) is true or not mentioned reflection or symmetry, and 16 instances of reflection were observed in the Swedish students' answers for task S1 and S3. In keeping with Attorps and colleagues' findings, many of the Irish participants also failed to correctly describe the reflection with only 9 students identifying it as being in the line $y = x$. It could be argued that the nature of the Irish question and, in particular, the graphs shown, prompted a reflection conception of inverses. However, it is surprising then that many students inaccurately commented on reflection in the x- or y-axis. The Swedish students showed a clearer understanding on this point and the students who mentioned a line of reflection correctly gave the line $y = x$.

None of the Irish students mentioned the necessity of 1–1 or onto properties of a function when explaining what an inverse is. But in answering task I(b), 10 of the 11 students who answered correctly were able to identify the lack of injectivity of the function (articulated as 'not 1–1' or 'fails the horizontal line test') as a reason for an inverse not to exist. However, only 2 of these 11 students had given correct and complete answers to I(a), while 6 of the students gave answers to I(a) that contained errors or were incomplete and 3 failed to give any explanation. Thus, it does not seem reasonable to argue that the students who emphasised the 1–1 property in relation to inverses had a more robust concept image or a showed a greater conceptual understanding of inverses as Bayazit and Gray (2004) may have assumed. For the Swedish data, when asked specifically what is required for a function to have an inverse, 6 students explained the necessity of a function being 1–1. We found evidence that at least Dora, despite not using any mathematical words for injectivity, understood that the function must be 1–1 and had grasped the difference between the definition of a function and of an injective function. From our experience it seems that many students find this distinction difficult; it may be that study of the inverse function concept could be used to reinforce students' understanding of function itself.

We did not find evidence in the Irish data to support Even's (1992) claim that a naïve conception of inverse functions as 'undoing' may result in incorrect conclusions, such as that all functions have inverses. Only one student revealed this particular misconception and that student did not mention reverse processes at all. On the contrary, four of the students, whose conception of an inverse was as a reflection, believed that the ability to reflect the graph of a function in a certain way confirmed it was invertible. However, Dora who used the words "be able to run backwards" gave additional requirements for a function to have an inverse as mentioned above.

The studies have, despite different tasks given to the students, evoked similar components of students' concept images. The students in both studies were in the beginning of studying mathematics at university level. Breidenbach and colleagues (1992, p. 251) remark that progress in cognitive transitions "is rarely in a single direction", thus it is not surprising that the concept images that emerged are complicated, with many overlaps between categories and variations within categories. Indeed, Bingolbali and Monaghan (2008) found that students' concept images of derivatives evolved over the course of a semester and were influenced by both the lecturer and the students' area of primary study. Learning more about students' progress in developing their concept images could inform our teaching preparation and help us to provide greater opportunities for students to gain valuable insights into the concepts of function and inverse function. At the very least, being aware of the concept images that students may hold and their likely consequences could inform teaching; for example lecturers could refer to the definition at different stages during the course as a means to develop intuition (Wawro et al., 2011), and engineer cognitive conflicts in order to give students opportunities to refine their concept images and deepen their understanding.

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