Conceptualizing and studying students' processes of solving typical problems in introductory engineering courses requiring mathematical competences

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The project KoM@ING aims to investigate the mathematical skills which are required in technical subjects of engineering bachelor courses. Our subproject is especially interested in the first-year-course "foundations of electrical engineering". In order to do research on this subject we developed the concept of a "student-expert-solution" (SES) which was generated by analysing expert interviews. The SES is supplemented by the required resources and a didactical reconstruction, for example, typical mistakes, alternative solution approaches and learning goals. The SES of a basic engineering exercise will be presented. We find some discrepancies with the standard modelling cycle, as well as some surprising problem solving strategies.

INTRODUCTION

We are interested in the competence expectations that are implicit in these tasks, i.e., expectations that are set up by the instructors of engineering courses, often based on years of experience, but not on any explicit theory of engineering competence. Originally, we intended to focus on mathematical aspects only, but it turned out that a more holistic approach is more appropriate. Based on this, we are analysing how and how well students in first year standard university courses on electrical engineering solve tasks given to them in homework assignments and written examinations Our project is part of the KoM@ING-project, where the modelling and assessment of mathematical and engineering competences is the focus. Our subproject chose a qualitative approach. The tasks we are analysing require knowledge and cognitive resources from mathematics on the one hand and from electrical engineering on the other. The mathematical knowledge is partly based on school knowledge and on the mathematics that students learn in the separate courses on higher math in the first year parallel to the engineering courses. It is well known that there is a mismatch between the mathematics learned in the separate courses, the mathematics at school level and "the contextual mathematics" required in solving engineering tasks (see, e.g., Redish, 2005). The mathematical practices in engineering contexts look far different from those in purely mathematical contexts. The tasks given to the students cannot directly be regarded as mathematical modelling tasks in the sense as this is discussed in mathematics education.

We focus on five tasks from the final exam of the second part of the "foundations of electrical engineering"-course (called the "GET-B" exam), which electrical engineering students are to take after their first year. All of the students' written work was scanned. Moreover the same tasks were given to eighteen pairs of students and their work and communication was video-recorded. With nine of the student pairs we used stimulated recall for extending the base of our analysis. In order to analyse the problem solving processes and the written work of the students we need a didactically oriented task analysis and theoretical frameworks on which we can base this analysis, in other words a "normative solution" of our task.

THEORETICAL BACKGROUND

We consider the following three theoretical frameworks as relevant: The first approach is the modelling cycle by Blum and Leiß (2007), which divides the solving of a mathematical modelling exercise into seven steps: (1) understanding of the task and the underlying situation, construction of the so-called "situation model" (2) simplifying and structuring of the situation: construction of the so-called "real model", (3) translating into a mathematical problem (entering the "world of mathematics"), (4) mathematical work, (5) interpretation of the result in the real world, (6) validating and (7) presenting of the results. The cycle consists of two parts, the "rest of the world" with steps (1), (2), (6), (7) and the mathematics with step (4). The changes between the two worlds happen in step (3) and (5). This modelling cycle description is considered as an idealisation, probably only applicable in school contexts. Nevertheless, this approach is useful for us as a tool to show important features of our "modelling example", which differ even on an idealised level.

The second approach is problem solving by Polya (1949), who divides problem solving processes into four phases: understanding the problem, devising a plan, carrying out the plan and looking back. The third approach is the description of ways of mathematical argumentation and mathematical resources in physics by Bing (2008) and by Redish and Tuminaro (2007). They distinguish between four framings that describe how students justify their results to exercises: calculation (algorithms give exact results), physical mapping (math should represent physics correctly), invoking authority (using of results of the physics-course) and math consistency (similarities to other physics problems solved with math).

We consider the first two approaches as "draft" process models, which will have to be extended and adapted to the specific tasks we are analysing. The third approach discusses the role of mathematical resources and knowledge in solving problems from physics and we use this approach to identify and characterize resources needed by the students. In other words, we consider that the development of theoretical descriptions has to be based on empirical results as well. We ask the task designer and electrical engineering experts to solve the tasks from the perspective of students who well understood the contents of the electrical engineering course. Based on further consultation of subject matter and didactical experts, we (re-)construct what we call the "student-expert-solution" (SES). The SES is used as a basis for sharpening the theoretical description and analysis of the solving processes, resulting in what we call a "theoretically enhanced SES" (TESES). We use this as a tool for understanding first year engineering students' solving problems.

METHODOLOGY

In order to get a detailed solution for the exercises we conducted interviews with the task designer and electrical engineering experts from the institute at the University of Paderborn, which is responsible for the GET-B, using the Precursor-Action-Result-Interpretation (PARI) method, a task-based interview technique conducted with experts of the task (Means et al., 1995). The aim of these interviews is to identify the explicit and implicit expectations of competences. The half-structured PARI-interview consists of three phases: In the first phase, experts have to do the exercise without any interruptions, but they are told to think aloud while writing down solutions. In the second phase, the interviewer goes through the written solutions with the expert in order to reconstruct the reasons for the way the exercise was solved and identify the used resources. In this phase experts need to justify each step of their solutions and make explicit the knowledge they used. The last phase is a didactic reconstruction of the exercise, which consists of two parts. In the first part the experts' view on students' expected solutions is solicited. This part contains questions on alternative solutions to the exercise, typical mistakes of students after their first year and possibilities to validate the results. In the second part the interviewer asks for the reasons for assigning the exercise and possible variations for exercises on the topic, aimed at making explicit the implicit competence expectations.

The interview is the foundation of the student-expert-solution (SES), which is the best solution an electrical engineering student could achieve with the knowledge presented in electrical engineering and mathematics lectures prior to the exam (i.e., the knowledge after their first year of studies). In the next step the student-expert-solution is subdivided into categories, i.e., phases and cognitive resources in a deductive approach based on the three mentioned theoretic approaches. This document consists of a two-column table: the SES in first column and related theory-based comments in the second column; it is called the "theoretically enhanced" student expert solution (TESES). The TESES is used as theoretical instrument to analyse the transcribed solution processes of our pairs of students. The participating students attend degree-relevant courses in electrical engineering or industrial engineering, and they were at the end of their first year when the study was conducted. Nine pairs were filmed while they were working on the exercises and were talking about the way they solved them.

EXEMPLARY RESULT: A SES FOR ONE TYPICAL TASK

A sample exercise

For illustration, we present one of the exercises of the mentioned exam, which deals with magnetic circuits, and its theoretically enhanced student-expert-solution. This section gives the problem setting and a short overview of the solution. The exercise consists of six subtasks and starts with the following sketch of a magnetic circuit:

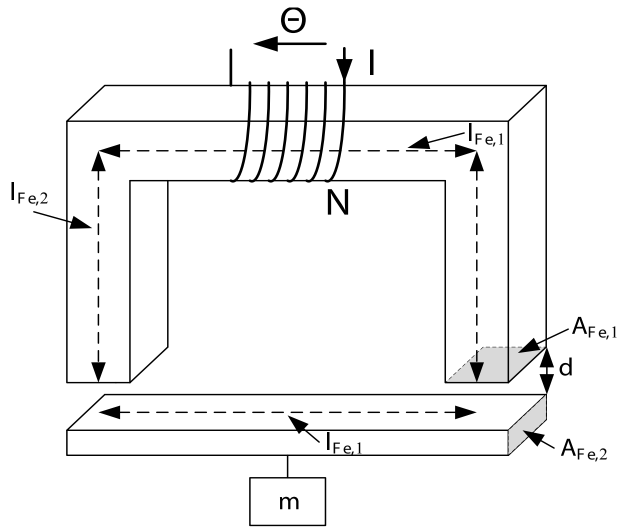

Figure 1: Sketch of a magnetic circuit consisting of two iron cores

The magnetic circuit consists of two iron cores with different cross section areas. The winding on the U-core has N=100 windings und is flown through by an electric current of I=10A. At the places of minimal distance between the two cores there should be a joint that behaves like an air gap. The exercise gives the following data for the iron cores: $l_{Fe,1}$=50 cm, $l_{Fe,2}$=30 cm, $A_{Fe,1}$=150 cm², $A_{Fe,2}$=60 cm², μ_r=1000.

Subtask 1: Sketch the equivalent electric circuit diagram of the magnetic circuit und simplify it as much as possible. Solution: The three parts of the U-core (the upper part and the left and right parts) and the lower iron core each give constant reluctances and can thus be summed up to one reluctance R_{Fe}. R_L, the reluctance in the air gap, is dependant on the width of the air gap d and has to be doubled, because there are two joints between the two parts.

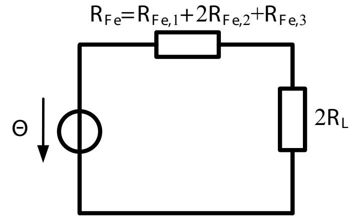

Figure 2: Equivalent circuit diagram

Subtask 2: Describe the total reluctance $R_M(d)$ as a function of the variable d. Solution: Because of subtask 1 the total reluctance can be written as the linear function of d: $R_M(d)$=R_{Fe}+$2R_L \cdot d$ using calculated values for R_{Fe} and R_L.

Subtask 3: Describe the inductance L of the setting as a function of d, the width of the air gap; assess the width of the air gap for which the inductance is maximal, and calculate the value of this maximum. Solution: The formula for L is $L(d)$=$N^2/R_M(d)$, i.e., the width of the air gap d is part of the denominator and the fraction becomes maximal, if d=0. We get the maximal value by insertion of d=0.

Subtask 4: Calculate the magnetic flux density b_L in the air gap.

Solution: We get the formula for the magnetic flux density by combining two formulas for magnetic fluxes.

Construction of the associated theoretically enhanced student-expert-solution

The first tool to analyse the exercise is the modelling cycle by Blum & Leiß (2007). We have developed several modifications of this cycle in order to better describe students' processes including identifying the resources required during the solving process. We summarize first and go into details later:

— Instead of constructing a real model as suggested in the modelling cycle students rather need strategies to understand conventionalized sketches and use them to mathematize the electrical engineering problem.

— Instead of entering the "world of mathematics," they enter into a "mathematics of physical quantities" with special resources: those resources are not solely based on pure mathematics learned in school or university mathematics courses.

— The authenticity of the problems offers strategies to validate the results.

We subdivided the solution process of subtask 1 and 2 into four phases. The two subtasks can be preliminary assigned to what is called (1) understanding of the task and the underlying situation, construction of the so-called "situation model" and (2) simplifying and structuring of the situation: construction of the so-called "real model". However, the competence requirements are quite different. The students do not do idealisations and simplification themselves, but they have to understand the given sketches as a "real model" whose idealisations will remain largely unconscious to them. The following idealisations, which were expressed in the interviews with the experts, are implicit: We only look at the magnetic behaviour of the test arrangement in an idealized static situation. The inductance of the windings, leakage fields and non-linear magnetic behaviour of the test arrangement are disregarded. Dynamic effects caused by the motions in the system are disregarded: Because of the changing of the distance between the two iron cores the energy changes and hence a non-linear ratio between power and force arises. The students are socialised into a world of certain real models. Idealisations often stay implicit and the students are often not aware, that there are idealisations. That is similar to physics students who use mass points without being aware of the idealisation, or geometry students that use dimensionless points. Students are to learn to "read" the sketch of Figure 1 and later find the fitting equations for this figure.

In contrast to saying that students should draw a picture or diagram of their own choice for understanding a situation (Polya, 1949; Blum & Leiß, 2007), the diagrams of Figure 1 and Figure 2 are very conventionalized in electrical engineering and constitute a specific "notational system", which is part of the tools of the discipline. Students who have not understood this might have difficulties if they approach the tasks and try to develop their own idealizations based on general physical knowledge – they could try to understand all the physical mechanisms and then be overwhelmed by the real situation. A further requirement is that students are familiar with the technical terminology (concepts) of electrical engineering in order to understand terms like the "magnetic flux" or "reluctance". Subtask 1 of the exercise requires using the "method of the equivalent electric circuit diagram", which helps to eventually mathematize the situation. The sketch of the test arrangement (Figure 1) has to be translated into an equivalent circuit diagram (e.g., Figure 2) using special rules for translation, which were expressed in the previous section. Equivalent electric circuit diagrams form a second notational system in electrical engineering. They are part of the acquisition of a domain-specific "graphical language" (similar to free-body force diagrams in mechanics or Feynman diagrams in quantum mechanics).

The third phase consists of setting up the equation for calculating the total reluctance with the help of the equivalent circuit diagram, which was generated in the first subtask. Once again, this is a translation task into which students have been socialized – idealized electrical or magnetic circuit diagrams are translated into sets of equations using the so-called Kirchhoff rules, just like free-body force diagrams are translated into vector equations using Newton's Laws, and Feynman diagrams are translated into path integrals using Feynman rules. The modelling of physical situations as idealized graphical diagrams and subsequent "mathematization" using sets of algorithmic translation rules is a common theme among the most powerful theories in physics and engineering. As mentioned above the reluctances in the iron and the air gap have to be added to execute this step, which requires forming of a set of equations between known and unknown physical quantities. There are also some differences to the modelling cycle in this step: A set of equations between physical quantities (numbers with units instead of just numbers) has to be set up. The student does not enter the "world of mathematics", but instead the "mathematics of physical quantities" with electrical engineering meaning. We are also convinced that many modelling problems at school level equally do not enter the "world of pure mathematics," but remain contextual mathematics with quantities.

In the next phase the total reluctance is calculated using the previous derived formula. As an example, in Figure 3 we reproduce the calculation of the constant part of the total reluctance (i.e., the reluctance of the iron core). The shorthand $\mu=\mu_r\mu_0=1000\mu_0=1000\cdot 4\pi\cdot 10^{-7}$ Vs/(Am) is used for the

$$\begin{aligned}
R_{Fe} &= R_{top} + 2 \cdot R_{left} + R_{bottom} \\
&= \frac{1}{\mu}\frac{l_{Fe,1}}{A_{Fe,1}} + \frac{2}{\mu}\frac{l_{Fe,2}}{A_{Fe,1}} + \frac{1}{\mu}\frac{l_{Fe,1}}{A_{Fe,2}} \\
&= \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}}\frac{0.5\mathrm{m}}{0.015\mathrm{m}^2} + \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}}\frac{0.3\mathrm{m}}{0.015\mathrm{m}^2} + \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}}\frac{0.5\mathrm{m}}{0.006\mathrm{m}^2} \\
&= 26.526\frac{10^3\,\mathrm{A}}{\mathrm{Vs}} + 2 \cdot 15.916\frac{10^3\,\mathrm{A}}{\mathrm{Vs}} + 66.315\frac{10^3\,\mathrm{A}}{\mathrm{Vs}} \\
&= 124.673\frac{\mathrm{kA}}{\mathrm{Vs}}
\end{aligned}$$

Figure 3: Calculation of the total reluctance

permeability of the metal, where μ_0 is the vacuum permeability and μ_r is the given relative permeability.

In the fourth phase, the resources of "mathematics of physical quantities" are needed, i.e., the management of units as well as techniques and strategies for the transformation of fractions containing physical quantities. Often these have neither been part of school nor university mathematics, and students tend to make many mistakes when attempting to apply these resources. One central resource is the "management of units", which includes the "handling of powers of ten". Those resources are typically required in physics and physics-related subjects if tasks require to calculate numerical values of physical quantities.

We define the "management of units" as the manipulation of base and derived physical units. All physical units in this exercise can be expressed in the base-units meter (m, for lengths), kilograms (kg, for masses), seconds (s, for time), and Ampere (A, for electrical current). E.g., the unit Volt (for electric potential) can be expressed as $(m^2kg)/(s^2A)$ using base units. The handling of powers of ten, which are expressed by letters in front of units, is also part of the management of units. The students have to translate the letters like k or m (for milli) to powers of ten, in this case 10^3 and 10^{-3}, respectively. The powers of ten then have to be multiplied, divided and potentiated using the rules for potentiation. For example, they need to realize that $6cm^2$ is equal to $6 \cdot (10^{-2}m)^2 = 6 \cdot 10^{-4}m^2$, instead ignoring that the power of ten is squared alongside the unit and arriving at $6 \cdot 10^{-2}\,m^2$. In the last step the base units have to be translated in a summarising unit and the powers of ten into the right letter in front of the units.

The techniques and strategies for the transformation of fractions containing physical quantities include handling of algebraic and arithmetic terms with fractions. As seen in Figure 3, there are also compound fractions (especially due to units), and fractions have to be transformed in order to be able to add them.

Also resources of electrical engineering in a narrower sense are needed to solve the second subtask: students must know the formula for the reluctance, which is $R=1/\mu \cdot l/A$, where µ is the permeability, l is the length, and A is the cross-sectional area of the conductor. They have to insert the right values for each part of the iron core as well as for the two air gaps in order to apply the formula to the situation.

For the third subtask, students need to recall the formula for the inductance. Initially, students have to find a formula, which only contains known parameters from the problem setting, in this case the formula $L(d)=N^2/R_M(d)$. The maximum value can be determined using knowledge presented in the GET-B-lecture, namely, that the value for the inductance decreases the farther the two iron cores are moved away from each other. So the inductance is maximized if there is no air gap, i.e., the width of the air gap is zero. Alternatively $L(d)$ can be interpreted as a function of d, where the students can use techniques from mathematics to find the maximum (minimising the denominator). These two different types of reasoning for finding a solution are often applicable, i.e., reasoning by either calculation or physical mapping, see Bing (2008).

In the fourth part, students need to recall the formula for the magnetic flux density b_L, which is $\Phi=b_L \cdot A_{Fe,1}$, i.e., the product of the magnetic flux density and the relevant area. In this formula only $A_{Fe,1}$ is known, but there is another formula to calculate Φ, namely $\Phi=(N \cdot I)/R_M$, i.e., the product of the number of windings and the electric current divided by the total reluctance. In the second formula all physical quantities are known, and by combining the two formulas students can calculate the value for b_L. They get $b_L \cdot A_{Fe,I} = (N \cdot I)/R_M$ and with the help of this, $b_L = (N \cdot I)/(R_M \cdot A_{Fe,I})$.

This example shows the typical characteristics of "equation management". Initially students recall the relevant formulas containing known and unknown physical quantities. Then they start to transform these equations in order to get unknown quantities with the help of known quantities. This can either be done systematically by writing down all possibly relevant formulas at the beginning of the solution process, or step-by-step by starting with one formula and in each step trying to replace unknowns with the help of known physical quantities. This task is not rule based like solving systems of linear equations. It is not necessary to derive all the equations from general formulas in electric field theory presented in the lecture. This is a didactic reduction compared to the lecture, which was communicated in the exercise classes accompanying the lectures.

After finishing the calculations experts and students used various metacognitive strategies to validate their results:

— Validating of the result with the help of its unit in dimensional analysis: As the units of all physical quantities are known, one can check, if the units on the one side of each equation are the same as the units on the other side.

— Validating of the result with the help of its magnitude: In many cases the lectures, the problems in the exercise classes or previously done experiments show the order of magnitude of the resulting value.

— Check whether all information was used: it is a tacit agreement in the GET-B-course that all information that is mentioned in a problem is needed to solve it.

In contrast to the modelling cycle the implicit assumptions of the model are not questioned after finishing the exercise. On a positive note, in contrast to many school students' behaviour, validation takes place in a limited efficient manner, because the students expect tasks and results to be realistic, whereas in school mathematics often unrealistic, unauthentic assumptions and questions are prevalent.

PRELIMINARY RESULTS OF ANALYSING STUDENTS' SOLUTIONS

As we have not yet completed the analysis of students' work, we like just to point out two surprising results, where students employed special strategies that we did not anticipate. In the third subtask of the problem, the maximization of the inductance, all student pairs at first took the detour that when they read the word "maximum," they thought they needed to differentiate the term and find the roots of the first derivative. This does not lead to a solution, because the maximum is at the boundary of the interval; and since the differentiation itself is not easy, this approach could lead to further mistakes. This may be considered as a didactic obstacle in the sense of Brousseau having origin in school mathematics. In the fourth subtask, some pairs described the physical processes with the help of differentials like dA or dV, i. e., by application of university mathematics. Such argumentations are often used in exercises from field theory, which contain applications of Stokes' and Gauss' theorems for integral vector calculus. The magnetic flux through an infinitesimal area was expressed as $d\Phi = b_L \cdot dA$, where b_L is a function of the position. Students then used $b_L \cdot dA$ to mathematize the problem by seeing that the total flux can be computed using a surface integral. We observe a typical use of "differentials" in modelling physical problems that is not legitimated by mathematical theory. Since b_L is constant here, the integral method leads to the same simple formula as above, but in more general situations, the integral is mandatory. The phase of equation management was successfully preceded by a phase of deriving equations from more general principles. While unnecessary in this case, the approach was welcome as it shows further competencies. It depends on the course whether these are part of the expected range of competencies or whether students are just trained in "equation management."

FIRST CONCLUSIONS AND OUTLOOK

In summary we can state that our analysis shows that it is helpful to modify the modelling cycle as a theoretical tool for describing solution processes of students in engineering contexts:

— The component "cognitive resources" has to be added to the modelling cycle.

- One cannot distinguish between mathematics and the "rest of the world".
- Electrical engineering "does not exist" without mathematics.
- The setting up of equations is inseparably interwoven with the process of getting mathematical results. A division into two separate phases (setting up the model, mathematical solution) as in the modelling cycle is not adequate.

Furthermore, we have just started to use the TESES we described for analysing students' work. We will validate and extend our research results by developing additional student-expert-solutions for the remaining exercises of the GET-B exam, which require the higher mathematics taught at university level to a much greater extent than our above example. The content of these exercises includes for example ordinary differential equations of first and second order (in a task on oscillating circuits) or complex numbers (in a task on alternating current), which are the result of applying Kirchhoff rules to electrical circuits with time-varying currents through resistors, capacitors and inductances.

We will moreover analyse task solutions from written examinations to the GET-B-course in order to confirm, refine and enhance the results from the analysis using the SES and the video studies with the students. We also plan to interview teaching assistants from other universities. Although the content of the lectures is nearly identical between different German universities, there seem to be many differences in the expectations of competences, which can be made explicit by these interviews.

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