Coming to see fractions on the number line

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The aim of this paper is to present a didactical sequence that fosters the development of meanings related to fractions, conceived as numbers that can be placed on the number line. The sequence was carried out in various elementary school classes, containing students with certifications of mathematical learning disabilities (MLD). Thus, our didactical aim was to make accessible to all the students of the class, including MLD students, meanings related to fractions using a common didactical sequence for the entire class. Our research is based on a range of different perspectives, from mathematics education to neuroscience and cognitive psychology. We discuss how such perspectives can be combined and provide the theoretical bases to design the didactical sequence, which will be outlined, and which allowed us to implement and strengthen inclusive education.

Keywords: Fraction, number line, artifact, mathematical learning disabilities.

INTRODUCTION AND LITERATURE

The concept of *fraction* is a very difficult one to master: frequently students are unable to reach an appropriate understanding of it, as described for example by Fandiño Pinilla (2007), and they can even come to fear fractions (Pantziara & Philippou, 2011). When children encounter fractions – typically in Italian school this happens in third grade (at 8–9 years of age) – it is the first time they have to treat sets of digits differently than how they treat those given in decimal positional notation representing positive integers. The numerator and denominator of a fraction are two numbers, each of which is bound by the rules that apply to positive integers, but that together represent a new, *single*, number. Learning to see the numerator and the denominator of a fraction together, as a single number is one of the most difficult – if not *the* most difficult – cognitive aspect of fractions (Bobis, Mulligan, & Lowrie, 2013).

Others disciplines, besides mathematics education, such as cognitive psychology and neuroscience, have also been very active in investigating the phenomena of (difficulties in) understanding mathematics (incuded fractions), even if the different interested fields of research have not yet reached sufficiently common grounds for conducting scientific and interdisciplinary studies. In this paper, we consider some results from research in neuroscience and cognitive psychology to ground important design decisions taken during the elaboration of a teaching experiment constructed around the learning of fractions in primary school. In the following paragraphs we will illustrate reasons why it is important to learn (and therefore teach) fractions, both from a didactical (math education) point of view and from the perspective of cognitive science.

IMPORTANCE OF FRACTIONS AND DIFFICULTIES IN LEARNING THEM

From the logical and epistemological points of view, the notion of *fraction* can be seen in different ways: as a linguistic representation of the decimal number obtained from the division indicated (but not calculated) by the number corresponding to the numerator and the one corresponding to the denominator; as an operator where the denominator indicates in how many equal parts a given unit is divided (*each part is called a unit fraction*) and the numerator indicates the number of these to consider.

$$3/4 = 3 \textit{ times } 1/4 \rightarrow 3 \times 1/4$$

Frequently, at least in Italian education, the conception of fraction as an operator is not explicitly identified as a rational number. Only when it is transformed into a decimal number is it placed on the number line.

From the point of view of learning mathematics, fractions constitute an important leap within domain of arithmetic because they represent a first approach to the idea of extension of the set of Natural Numbers. In this sense, fractions need to assume a specific position on the number line (Bobis et al., 2013; Bartolini Bussi, Baccaglini-Frank, & Ramploud, 2013). Teaching the notion of fraction is, therefore, a quite delicate issue and it is ever so important to explore insightful ways of structuring didactical activities around it. In this respect, particularly insightful approaches have been provided, for example, by Bobis, Mulligan and Lowrie (2013). Even if certain basic aspects of the concept of fraction, particularly when seen as the perception of the variation of a ratio, seem to be innate (McCrink & Wynn, 2007), the learning of fractions presents obstacles, not only of a didactical nature. In fact, research in mathematics education (e.g., Bartolini Bussi et al., 2013), has shown how learning about not only the semantic aspects but also the lexical and syntactical ones of fractions involves the overcoming of different epistemological and cognitive obstacles such as:

— Assuming that the properties of ordering natural numbers can be extended to ordering fractions (e.g. assuming that the product/quotient of two fractions makes a greater/smaller fraction).

— Positioning fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011).

From a cognitive point of view, fractions seem to demand more working memory resources than representing whole numbers (Halford, Nelson, & Andrews, 2007). Moreover, fraction knowledge also requires inhibitory control and attention (Siegler et al., 2013), so that the numerator and denominator are not treated as independent whole numbers (Ni & Zhou 2005). With this in mind, it is clear that for a student with MLD (even when "D" stands for "difficulties" instead of "disabilities") the learning of fraction will be a particularly arduous task. In fact, recent studies suggest that dyscalculia, a particular kind of MLD, is rooted specifically in weak visual-spatial working memory and inhibitory control (Szucs et al., 2013).

Our present work on fractions is part of a broader body of research (Robotti, 2013; Baccaglini-Frank & Robotti, 2013; Baccaglini-Frank, Antonini, Robotti, & Santi, 2014) that has the objective of building inclusive curricular material, grounded theoretically in research in mathematics education and in cognitive psychology, appropriate for *all* students, including those with MLD.

CONCEPTUAL FRAMEWORK

A large number of studies associated short-term memory (STM) and working memory (WM) with mathematical achievement for students and expert (see reviews in Raghubar, Barnes, & Hecht, 2010). Moreover, non-verbal intelligence, addressed to general cognition without reference to the language ability (DeThorne & Schaefer, 2004) also seems to be strongly related to mathematical achievement (Szűcs et al., 2013). These (and similar) findings suggest that non-verbal intelligence may partially depends on spatial skills (Rourke & Conway, 1997). Thus, spatial processes, performed on the base of spatial skills, can be potentially important in mathematical performances, where explicit or implicit visualization is required. Moreover, research in cognitive science (Stella & Grandi, 2011) has identified specific and preferential channels of access and elaboration of information. For students with MLD these are the visual non-verbal, the kinesthetic-tactile and/or the auditory channels.

Studies in mathematics education as well, although with different conceptual frameworks, have highlighted how sensory-motor, perceptive, and kinaesthetic-tactile experiences are fundamental for the formation of mathematical concepts – even highly abstract ones (Arzarello, 2006; Gallese & Lakoff, 2005; Nemirovky, 2003; Radford, 2003). In this regard, within a semiotic perspective, Bartolini Bussi and Mariotti (2008) state that the student's use of specific artifacts in solving mathematical problems contributes to his/her development of mathematical meanings, in a potentially "coherent" way with respect to the mathematical meanings aimed at in the teaching activity.

Thus, in this paper we aim to describe examples (activities) of inclusive math education (Ianes & Demo, 2013), constructed referring to the math education domain as well cognitive psychology and neuroscience domains.

The goal of the activities we will describe, was to realize a sequence that would favor, for *all* children (including those with MLD) the development of mathematical meanings of fractions as numbers that can be placed on the number line. The sequence of activities was designed, realized and analyzed taking into account the following principles:

- the importance of an epistemological analysis of the mathematical content
- the role of perceptive and kinesthetic-tactile experience in mathematical concept formation as well the visual non-verbal, and auditory channels of access and elaboration of information, in particular in children with MLD
- the role of social interaction, verbalization, mathematical discussion;
- the teacher as a cultural mediator.

Following these principles, as we will describe later, particular artifacts (like paper strips, rulers and scissors) were identified with the intention of using them to help mediate the meanings at stake in the activities.

METHODOLOGY AND SEQUENCE OF ACTIVITIES

The sequence of activities was designed by 22 primary school teachers and 1 supervisor (the first author) composing a study group. The activities were carried out during a pilot experimentation, which involved 22 classes (nine 5th grade classes, six 4th grade classes, and seven 3rd grade classes), before being revised for an upcoming full-blown study. In this paper, we will report on the pilot experimentation carried out in the 3rd grade classes. Students worked in small groups.

The sequence of activities asked to work with differents artifacts such as A4 sheets of paper, squared-paper strips or represented squared-paper strips in notebooks. At first, students were asked to represent fractions on squared-paper strips, then to represent squared-paper strips in their notebooks and to represent, upon these strips, fractions. At last, students were asked to represent fractions on the number line (see below). As described below, the teachers also included moments of institutionalization and discussion (Bartolini Busssi & Mariotti, 2008) based on some critical episodes.

The activities concerning the sequence are:

1) Partitioning of the A4 sheet of paper: this activity involves dividing the A4 sheet of paper, chosen as a unit of measure, in equal parts, by folding and using the ruler; the procedure allows for the introduction of "equivalent fractions" as equivalent surfaces, and of "sum of fractions" for obtaining the whole (the chosen unit, that is, the A4 sheet).

2) Partitioning of a strip of squared paper. This activity involves three sessions:

   A) Given a certain unit of measure, position it on the strip; then, position some fractions on the strip (1/2, 1/3, 1/4, ...) according to the given unit of measure (see Figure 1). The objective is to represent, on the same strip, different fractions, introducing reciprocal comparison.

   B) Given different units of measure on different strips, on each strip a same fraction is represented (1/2). The objective is to make explicit the dependence of the unit fraction upon the chosen unit of measure (1/2u).

   C) Given a squared strip, choose appropriate units of measure to represent different fractions on that strip (e.g., 1/3 and 1/5). The objective of this activity is to find the *lcm* (least common multiple) between denominators as the appropriate unit of measure.

3) Placing fractions on the number line. The fractions, considered to be lengths of segments with origin in 0, are placed on the (positive) number line using the idea at the basis of the operator conception of fractions (developed in point 2). Since the right endpoint of the segment on the number line is labeled with a fraction, it will also assume the meaning of "number", as do all the other whole numbers on the line. Different fractions will be associated to a same point on the line, and will be used to revisit the meaning of "equivalent fractions".

ANALYSIS AND DISCUSSION

In this section we present an analysis of points 2 and 3 of the sequence, and in particular the transition from point 2 to point 3, which we consider the most significant in order to place fractions on the number line. Our objective is to highlight how the meaning of fraction evolved, thanks to the use of the tools (squared strip of paper, and number line) and to the designed tasks.

ACTIVITY 2, SESSION A. A certain unit of measure is given (for instance, a unit measure corresponding to 15 squares). The students are asked to position it on the *strip of squared paper* and to place and color on the strip unit fractions like 1/5,1/3, etc.

Figure 1: Four strips of squared paper where students had postioned a certain unit of measure and had defined unit fractions and colored fractions (4/5, 2/3, 5/3, 7/5)

With respect to the kinesthetic-tactile aspects that characterize activity 1 (partitioning of the A4 sheet of paper in equal parts), the manipulation of the artifact "squared strip" becomes a prevalently perceptive experience, in which the main channel for accessing (and possibly producing) information is the visual non-verbal one. Therefore the task (implicitly) requires the use of a procedure in which the fraction is conceived as an operator: the students partition the strip and produce linguistic signs associated to the name of the fraction expressed in verbal language ("Un mezzo" – tr. "One half"), in verbal visual language (the writing "un mezzo" – tr. "One half") and arithmetical language ("1/2"). The teacher institutionalizes the relationship between the different signs (partitions of the strips, visual verbal, visual non verbal, and arithmetical signs) in terms of rational numbers. Thus, the construction of meaning related to the notion of rational number, is based on the interplay between different types of semiotic sets (Arzarello, 2006). Note that the task was completed by all groups of students.

ACTIVITY 2, SESSION B. Each group of students is asked to choose a unit of measure, reproducing it on a strip and placing the fraction 1/2 on the strip. Then, their strips are compared. The dependence of the fraction on the unit of measure, observed comparing the results of the different groups of students, becomes explicit during a classroom discussion, from which we include an interesting excerpt:

Student 1: Maybe we made a mistake.
Student 2: No, we did not make a mistake, I am sure I folded the unit in half, so it's ½.
Student 3: We shouldn't look at the length, because each group chose a different unit of measure. [...]
Students 4: Because doing ½ is cutting in half, so if the units are different the halves are different [...] we have to be careful because to understand which counts more we can't put them one on top of the other like we did for the placemats.

Here a shared meaning is being developed for the fraction as an operator on a chosen unit of measure (1/2u). Note that the kinaesthetic-tactile approach in which the strips were put one beside the other is no longer effective for comparing fractions.

ACTIVITY 2, SESSION C. The task asks to choose a unit of measure to represent on the same strip different unit fractions like 1/3, 1/6, 1/8, 1/2, 1/4.

The parameters defining the situations are such that the situation makes it necessary to choose 24 squares (corresponding to the *lcm* of 2, 3, 6, 8 and 4) as the unit of measure. In fact the children do not simpy looked for the unit of measure spontaneously, generally using trial and error methods, but they also check the efficiency of their choice. Moreover, positioning on a single strip different fractions, makes the ordering of fractions quite similar to that of the other numbers that are perceptively evident (Figure 2).

We note here that for the different fractions on the strips (Figure 2), the teacher asks to also associate color to the verbal, figural and arithmetical representations. The reason is that, as suggested by Stella and Grandi (2011), the verbal channel is not the preferred one for most students with MLD. Color becomes a tool supporting working memory and possibly also long term memory, through which the meanings developed

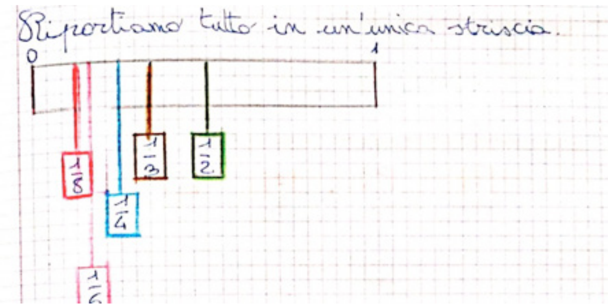

Figure 2: different fractions on the same strip

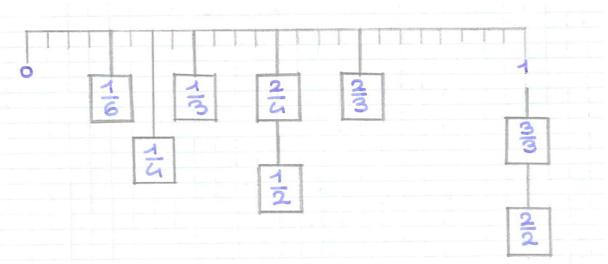

Figure 3: Fractions on the number line

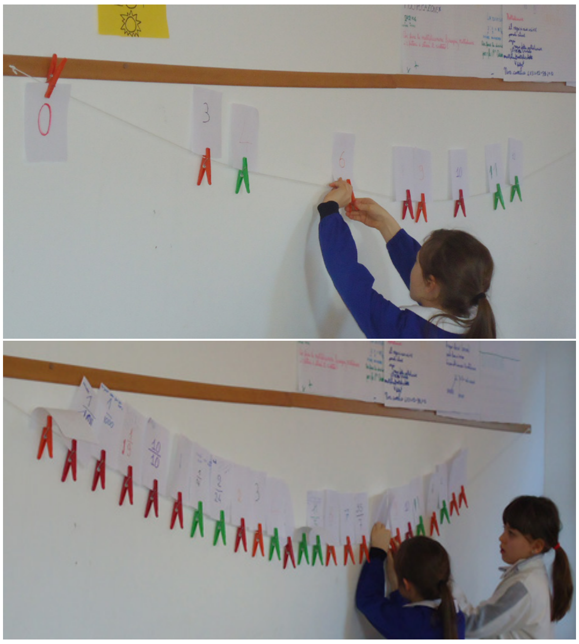

Figure 4: String on the wall where the position of the unit is made to vary dynamically sliding the corresponding label attached with a clothes' peg

can be recalled and used (Baccaglini-Frank & Robotti, 2013).

ACTIVITY 3. The objective of this activity is to place and order fractions on the number line (Figure 3) that has been partially constructed in Activity 2 and that is now used by the teacher as a tool of semiotic mediation (according to Bartolini Bussi & Mariotti, 2008). This transition is fundamental: the representation of the artifact "strip of paper" becomes a mathematical sign that represents the mathematical object "the number line".

Actually, from now on color is no longer used and the labels are referred to points on the number line. We can therefore claim that fractions here have assumed the role of rational numbers. The teacher could take advantage of this transition to construct a new tool of semiotic mediation, developed from the preceding artifacts. The "narrow strip" now becomes a concrete artifact (Figure 4): it turns into a piece of string on the wall, upon which 0 is placed at the left end and the position of the unit is made to vary dynamically sliding the corresponding label attached with a clothes' peg. The dynamic component of this artifact recalls certain software (such as AlNuSet, GeoGebra, Cabri2...) of course with evident differences, including the fact that as the unit (the position of the paper card with written "1") is made to vary, the positions of the other whole numbers and fractions do not vary dynamically at the same time or automatically, as a consequence of the new placement of the unit: their motion requires a specific action in order for the numbers on the line to maintain the desired mathematical relationships. This feature can actually be exploited to foster the students' appropriation and *active control* of important mathematical meanings at stake, such as the density of rational numbers or the ordering of fractions on the number line during activities like this one.

CONCLUSIONS

We have outlined particularly significant (and delicate) passages of the sequence of activities, showing how the transition was guided. Initially the students were exposed to a somewhat traditional conception of fractions as operators in the context of partitioned areas (the "part-whole" meaning described in Bobis et al., 2013). This idea was soon re-invested in a slightly different context: the areas became strips that gradually lost their "fatness" and were narrowed down until they become (oriented) segments indicating distances from the origin of the number line. The power of an approach like the one described resides in how such a transition can be gradual and continuous, if the teacher manages to keep alive the situated meanings that emerge throughout its unraveling. This is in fact what happened, and the children (including those certified with MLD) came to deal with fractions as numbers on the number line, without hesitating to compare them, place equivalent fractions on the same point, and add them, according to meanings they had developed using the strips of paper that still had an area.

In summary, the analysis of the teaching intervention has shown that students have elaborated personal meanings consistent with the mathematical meanings related to fractions. In particular, the strip was used as instrument of semiotic mediation to develop the meanings related to fractions as operators and, then, to the ordering of fractions, to equivalent fractions and finally to equivalence classes. The use of the strip, the string and color (for a certain period of time), has had a key role in favoring the construction of the number line as a mathematical object. On the number line fractions, associated with points, could assume the role of rational numbers being representatives of equivalence classes. Finally, it is possible that this kind of construction of meanings related to fractions might also support the management of procedural aspects involved in operations with fractions, as various researches both in mathematical education and in cognitive science have already suggested (Siegler, 2013; Robotti, 2013; Robotti & Ferrando, 2013). Further studies are needed to explore and to confirm this hypothesis that we consider significant both for research and for teaching.

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