Investigations in magic squares: A case study with two eight-year-old girls

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The paper presents the results of a case study of two eight-year-old girls working together on an activity involving – among others – two magic squares. During the activity we have observed the girls' participation in the task, which led them to the discovery of some properties of operations and, moreover, to mathematical reasoning. Additionally, there were differences in the way the two girls perceived the given activity at particular moments, a fact that may be related to their general views of mathematical activity.

Keywords: Mathematical reasoning, investigations, magic square.

INTRODUCTION

Mathematics teaching and learning is a process that begins from the early years of childhood and takes place in formal and informal settings. Children, even at a small age have access to powerful mathematical ideas, such as mathematization, connections, argumentation, number sense and mental computation, algebraic reasoning, spatial and geometric thinking, data and probability sense (Perry & Dockett, 2002). During most of their time, and especially during play, children are engaged in informal mathematical thinking, which may include reasoning and argumentation (English, 2004). Although there is a consensus on the importance of that informal mathematical knowledge and its contribution to the child's further development, the research on reasoning processes in informal settings is rather limited (e.g., Ginsburg, Inoue, & Seo, 1999). Having in mind these considerations, we designed a case study aiming to study the reasoning processes that will occur, together with the mathematical concepts that may evolve by engaging two girls in a series of mathematical tasks. Particularly, our research questions were the following:

— What aspects of mathematical reasoning can be observed during the particular activity?

— In what ways has the particular activity contributed in the girls' understanding of properties of mathematical operations?

— Which were the characteristics of the girls' participation in the activity?

THEORETICAL FRAMEWORK

The process of learning mathematics can be viewed by many different perspectives. Among them, there are those that focus on the child's activity while doing mathematics and comparing that activity with that of a mathematician. Ponte (2001) talks about "a parallel between the activity of the research mathematician and the activity of the pupil in the classroom" (p. 53).

One of the important activities of the students who are doing mathematics is the mathematical investigations, in which the students rather than solving a problem with clearly-framed questions, are faced with a situation in which the conditions might not be completely clear, thus they might have to search for regularities and relations or even formulate some questions by themselves (Ponte, 2001). During these processes it is highly probable that the students will use some mathematical reasoning in their work. Lannin, Ellis and Elliot (2011) connect mathematical reasoning with nine essential understandings. Among them we find developing conjectures, generalizing to identify commonalities, generalizing by application, investigating why, justifying based on already-understood ideas, and validating justifications. This framework has proved much helpful for the purpose of our research.

Mathematical activities like those described before can be also observed in students at the early stages of their education. In NCTM's (2000) *Principles and Standards of School Mathematics* in the "Reasoning and Proof Standard for Pre-K through Grade 2" we read that the ability for mathematical reasoning "develops when students are encouraged to make conjectures, are given time to search for evidence to prove or disprove them, and are expected to explain and justify their ideas" (p. 122). It is also a known fact that children do use mathematical notions in their informal everyday activities before they enter the formal school system (Ginsburg et al., 1999). English (2004) claims that children during their play are engaged in mathematical reasoning; moreover, from a researcher's perspective, there is an interest towards "the thinking behind children's mathematical responses" (p. 14).

The importance of children's reasoning processes lies in the fact that they are strong facilitators of their learning, even more than specific contents of mathematical knowledge (Perry & Dockett, 2002). But we have to stress here that none of the previous can be achieved without the help of the teacher who – among other actions – has to ask the right questions and choose the proper tasks. Ponte (2001) offers a detailed description of the expectations for the teacher in an investigation class. These vary from the careful selection and design of tasks to decisions concerning time management and class organisation. What is important, however, is that the tasks should be designed in such a way that conjecturing, justifying, generalising, etc. will come up naturally during the students' participation in the activity.

DESIGN OF THE STUDY AND METHODOLOGY

The design of our study was based on our theoretical framework; most of the tasks were taken by a textbook which is aimed to promote interest in mathematics (Lankiewicz, Sawicka, & Swoboda, 2012). Our choices were driven by the following assumptions: the problems should be accessible to a wide range of students on the basis of their prior knowledge; they must be solvable, or at least approachable, in more than one way and without the use of tricks; they should illustrate important mathematical ideas; they should serve as first steps towards mathematical explorations and be extensible and generalisable (Schoenfeld, 1994).

The students were given seven tasks in total and the session, which took place in one of the students' house, lasted for two hours. Both the authors of the paper were present in the session and both were known to the two girls. The first author, who will be referred to as Researcher in the transcripts, was the one who provided help and guidance to the students. Her roles were consistent with those described by Ponte (2001) and NCTM (2000) and can be summarised in the following:

a) propose challenging questions for the students;

b) support and evaluate students' progress by promoting a balanced participation in the activity;

c) think mathematically by asking new questions and by becoming involved in mathematical reasoning;

d) supply and recall information;

e) promote students' reflection.

The two students, Ania and Magda, were both eight years old at the time of the study, and they were fellow-students in the second grade of a public primary school in Rzeszow, Poland. They both had good marks

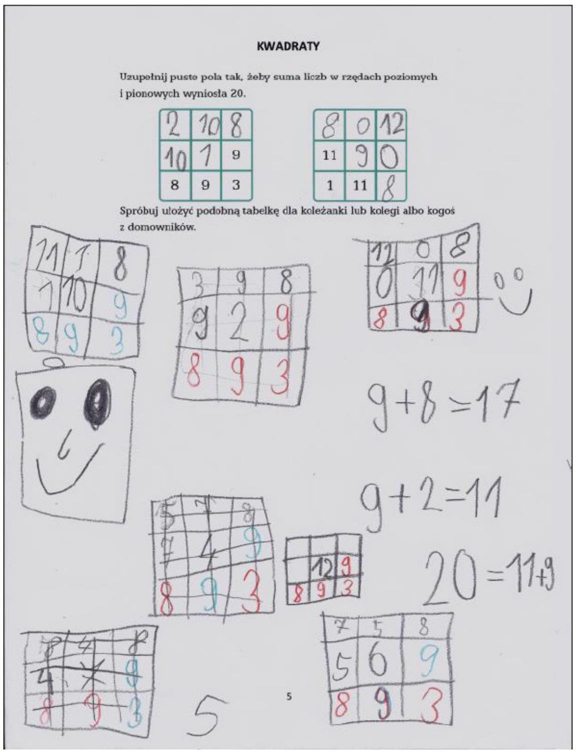

Figure 1: The worksheet containing the magic squares and part of solutions

in mathematics and they volunteered to participate in the research.

The analysis in the present paper focuses on a task related to a particular type of magic squares, which are partially filled, contain equal sums horizontally and vertically (but not diagonically) and the same number can appear more than once. This task was chosen because it fulfils the assumptions mentioned before and, particularly it is aimed to promote conjecturing and justification. Moreover, it has led to a rich discussion and engagement of our students. This was the fifth task in the row and Figure 1 shows the worksheet that was given to the students, together with some of the girls' solutions.

In the top of Figure 1 we read: "Complete the empty fields so that the sum of the numbers in rows and columns is 20". And then: "Try to create a similar table for your friend or somebody from your family".

The analysis of the girls' activity was done according to our research questions and was based on our theoretical underpinnings. Particularly, we firstly tried to locate any manifestations of Lannin and colleagues' (2011) essential understandings that are related to mathematical reasoning:

- developing conjectures,
- generalizing to identify commonalities,
- generalizing by application,
- conjecturing and generalizing using terms, symbols, and representations,
- investigating why,
- justifying based on already-understood ideas,
- refuting a statement as false,
- justifying and refuting the validity of arguments,
- validating justifications.

The analysis of the episode has shown that not all of the above were manifested in our students' interactions, which was somehow expected, since some of these understandings (e.g. generalising by the use of representations) were rather advanced for our eight-year-old participants.

Another useful analytic framework was Brandl's (2011) mathematical giftedness model, which consists of abilities specific to mathematics and general personality traits. The former include mathematical sensibility, memory, structuring, generalising and the reversion of mathematical processes. The latter include intellectual curiosity, willingness of exertion, joy in problem solving, perseverance and frustration tolerance. Although the model refers to mathematical giftedness, we have found it useful for characterising the participation of our students. Finally, throughout the discussion we have located the conjectures that are related to properties of the specific magic squares, as well as numbers and addition.

RESULTS

The task which is the focus of the study was the fifth in the row. For the purpose of the present paper we focus only on the first part of the task, which was to fill in the missing fields of the two magic squares. Mathematical activities of particular interest for our research are written in italics. Our notes are written in brackets. The discussion that follows took place few minutes after the worksheet was given to the students, since they needed some time to comprehend the task:

Ania: It will be here 12, for example 12 [the sum of 9 and 3 which are in the last column], 2 plus.... in order to be equal to 10, then 2+8. We have to put 8.

Magda: And here we can [put] 1 and add 10 [in the middle column].

Ania: [checking] Uhm. Yes. And here 8 [writing what she calculated before]. And here it would have to be 2 [in the top left].

Magda: and 10 [writing 10 in the last field, i.e. the middle in the first column].

Researcher: Is it ok?

Ania: Yes. 8+9+3 equals 20; 10+1+9 also equals 20; 2+10+8 also equals 20.

Researcher: Magda, how did you know that there [in the middle field] has to be 1?

Magda: Because I knew here and here I knew that it will be [showing the middle row and middle column]

Ania started with the last column in which only one number was missing; thus, she chose to begin with the easiest part of the task. Magda focused on the middle field; in this case the situation was open because both middle row and column had two empty spaces. Magda probably chose 1 because that was the number that added up to 10 (because of 9 in the bottom field). She *made a conjecture*, while Ania *validated* it. Both girls were engaged in solving the task and were *monitoring each other*. The researcher asked the question "why" in order to make Magda justify her choice. But she had difficulties in expressing her way of thinking. Up to that moment the girls were not aware that 1 in the middle field is not the only solution.

Thus later on, the question of the researcher "Can another number be in the middle?" surprised the students, since they thought that they had completed the first square. It created a cognitive conflict, since they were probably never faced a problem with more than one correct solution. It made them thinking for a while and the first answer of both of them was "no". After that, Ania had a second thought. She *made the conjecture* that in the middle you can put the number 2. But she quickly *refuted the hypothesis*: "If here would be 2, then not. Here has to be like that". She showed the other numbers in the middle column: 9 and 10 and she *concluded* that 2 does not fit to them. Because the completed square was misleading the students, the researcher asked them to draw another one – the same with the one given in the task. Then she repeated the question: "Can something else than 1 be in the middle?"

Ania: No, because for example if here was 2...
Researcher: yes...? [showing interest]
Ania: then here it would equal 11 [with 9]... [thinking for a while] ... and here (the last column) you have to add 8 for sure. Here you have to add 8 for sure! [repeating and writing 8]
Researcher: Yes..?
Ania: And if here was 11, then 11, then you would have to add nine...? [unsure]

Ania was engaged in solving the problem. It was a real *challenge* for her and she demonstrated *intellectual curiosity*. She was not convinced that number 2 is not adequate although she had *rejected* it before. Therefore, she wanted once again to check if number 2 can be put in the middle of the square. This resulted in *discovering* that in the last column has to be 8 "for sure", which is later expressed in the sentence: "Here always has to be 8, because it can't be a different number". This is an expression of *generalization* accompanied by an *explanation* which is not justified. While Ania was trying to *investigate* the situation with number 2, Magda seemed to not be interested in the problem anymore. She proposed to move to the second magic square given in the task (see Figure 1). Ania firmly answered: "No, Magda. Wait, now we do that" which showed her *perseverance*. This happened few times during that task, which demonstrates Ania's *willingness of exertion*.

After filling the square the researcher wanted to engage Magda, so she asked "When are we sure that the square is correctly filled?". This provoked a *justification* by Magda: she drew lines on all columns and rows and *explained*: "When all squares [she means sums] will be correct. You have to make operations". After that the girls were convinced that number 2 can be put in the middle of the square. Ania also added: "But 1 also can be" by which she wanted to stress that there are two correct solutions. After that Magda proposed to check number 4. The solution made them enthusiastically state: "Here can be 4 as well!" Another discovery encouraged their further investigations: Magda noticed that she can use the previously filled squares to fill the next ones:

Magda: [filling the square with 7 in the middle filed] It's so easy. Look [writing 4 and 8 at the top and showing the previous square with 4 in the middle; laughing]: From that. Because here everything is opposite!

By *identifying commonalities* she made a *conjecture* which later resulted in *generalizing by applying* it into another pair of squares (2 and 9 in the middle). The outcome inspired her to continue. Ania *developed* Magda's *observation* and "Everything is opposite" was elaborated to:

Ania: ... every number is changing with something. (...) for example 7 with 4, 4 with 7 [showing squares with 4 and 7]; 9 with 2, 2 with 9 [showing the squares with 2 and 9] 1 with 10 [the first square with 1 in the middle], 6 with 5 [the square with 6 in the middle]".

The researcher moved their focus to the sum of these pairs of numbers and this led them to a common *discovery*:

Magda: 7+4 is 11 [square with 7 in the middle]. Here is also 11 [square with 4 – showing 7 and 4]
Ania: Every square 11, 11 here also 11. All it has 11.

The second cognitive conflict appeared when the students were discussing what numbers could be put in the middle of the square. The first ideas that "all" numbers can be put (Ania) and "even 100" (Magda) as a synonymous of a big number were quickly *rejected* and *reformulated* into "all up to 10!", which was clarified by Ania: "it means all one's [she means one-digit] numbers together with 10". To the researcher's question about number 11, both girls answered "no". The justification of Magda was that "If in the middle would be 11 and here 9, then it would already be equal to 20 [in the middle column] ... and here [empty field] we have one more. It seems that zero is not treated by the students as a number: you have to add something in the field, but there is nothing to add. But as soon as they realized that zero can be put in the empty field, they *applied* that understanding: "And 11 will be changing with 0!":

Researcher: So, can 12 be in the middle?
Ania: yes
Magda: yes
Ania: And 11 will be changing with 0!
Researcher: Uhm. And what will 12 be changing with?
Magda and Ania: hmm
Ania: 12? So I will do it a small one [drawing a new square]
Researcher: Ok, so quickly and then we will move to the next task
Ania: it's a pity. It's so nice that task... 12. But 12 cannot be because...
Researcher: Why 12 can't be?
Ania: Because ... because 12+9 is already 21!
Researcher: ok...
Ania: So 12 can't be
Researcher: Magda, can't it be?
Magda: No
Researcher: Hm. So what numbers can we put there in the middle?
Ania: up to 11.
Magda: from 1 to 11 [simultaneously]

Number 11 in the middle was immediately rejected, while with 12 they were more cautious. Only the thought about exchanging with another number made them remember the initial condition about sum of 20 which was used by Ania in her *argumentation*. The work at the first square was completed by the range of numbers that can be put in the middle in order to fill the whole square. The range was partially complete, since it did not include zero, although it was mentioned and used.

In the second square the students mainly *applied* their own *discoveries* (with some modifications according to the new conditions) and used the *argumentation* developed in the first one. It is interesting that Magda started filling the square by number 9 (which means that you have to add 0) which was the extreme case in the first square and it took them some time to accept it. Generally, they were working with a big enthusiasm and *complemented each other*. After filling the given square they did not draw any other squares because they expressed everything verbally:

Magda: Here for example 8 [in the middle] and add here 1 [above 8]
Researcher: Aha (...)
Researcher: And if here we put 7 [in the middle], then what will be here? [above]
Magda: 2
Researcher: And if 6?
Ania and Magda: [loudly] 3!
Researcher: How do you know that 3?
Ania: Because when it was 7 then it was 2. And if we decrease it more then it will be 3
Researcher: And if we put 5?
Magda: Then 4 (...)
Researcher: What's the biggest number we can put in the middle?
Ania: 20?
Researcher: Can we put 20 in the middle?
Magda: For me 9, because 10+11 it would be already 21.

We can notice that both students made a significant progress; they were more confident in *making conjectures* and *giving justifications*. Moreover, their discoveries gave them satisfaction which can be described as *joy in problem solving*.

Summing up, the particular activity had invoked many aspects of mathematical reasoning, which may be categorised according to our methodological framework into understandings related to mathematical reasoning (Lannin et al., 2011) and general personality traits which support mathematical activity (Brandl, 2011). The former may be further categorised into *interactive understandings*, which are directly oriented to the partner and *personal understandings* which have a more personal character (although they might be also directed to the partner or the researcher). Characteristic cases of interactive understandings in our study were the monitoring of each other and the development/elaboration of the other's observation; personal understandings included the development of conjectures and the generalization by application. During the episode that we analysed a number of conjectures were articulated, validated and eventually elaborated:

1) In the middle field you can put 1.

2) In the top right field always has to be 8.

3) In the middle field you can put 2 (in the sense of another solution for the square)

4) "Everything is opposite" – the addition of two numbers is commutative.

5) The sum of the two numbers in the upper fields of the middle column is constant and equal to 11.

6) 11+0 equals 11. Any number plus zero equals that number.

7) In the middle field of the first square you can put all numbers from 0 to 11.

8) The sum of the two numbers in the upper fields of the middle column is constant and equal to 9. Decreasing one number makes the second increase.

9) The biggest number we can put in the middle of the second square is 9.

It is obvious that some of them are related to properties of the specific character of our magic squares and some properties of addition: identity property, commutative property and associative property.

The personality traits that we located in our mathematical activity were: perseverance, willingness of exertion, intellectual curiosity and joy in problem solving. Moreover, we have noticed significant differences in our students' participation regarding not only their understandings of mathematical reasoning, but also the personality traits. Particularly, Ania was more willing to investigate the situation, developing new conjectures and perseverant. Magda, on the other hand, usually wanted to move on when she was faced with a novel situation; however, whenever she developed her own conjectures she experienced satisfaction. Summing up, we may say that Ania was better in proposing new ideas, while Magda was better in justifying their conjectures and monitoring Ania.

CONCLUSIONS

Our case study has provided us with interesting and valuable data that go in line with the relevant literature. We have observed that our students have been able to articulate sound mathematical conjectures, which supported their mathematical reasoning. Moreover, both students were engaged in the task and have demonstrated signs of mathematical sensibility and intellectual curiosity, but in a different degree. As we mentioned, Ania was better in proposing conjectures, while Magda was better in monitoring; thus, it seems that the students' roles although not identical, were somehow complementary to each other. This fact was very helpful for the "flow" of the activity and the outcome of their investigations. The researcher's interventions, mostly in the form of questions, were also vital for the girls' investigations, promoting their mathematical reasoning and fostering their reflection.

The magic squares have thus proved a useful tool for promoting our students' investigations and we believe that it can also be used for the discovery of properties of numbers and addition.

REFERENCES

Brandl, M. (2011). High attaining versus (highly) gifted pupils in mathematics: a theoretical concept and an empirical survey. In M. Pytlak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 1044–1055). Rzeszów, Poland: University of Rzeszów.

English, L. D. (2004). Mathematical and analogical reasoning in early childhood. In L. D. English (Ed.), *Mathematical and* *analogical reasoning of young learners* (pp. 1–22). Mahwah, NJ: Lawrence Erlbaum Associates.

Ginsburg, H. P., Inoue, N., & Seo, K. H. (1999). Young children doing mathematics: Observations of everyday activities. In J. Copley (Ed.), *Mathematics in the early years* (pp. 88–99). Reston, VA: National Council of Teachers of Mathematics.

Lankiewicz, B., Sawicka, K., & Swoboda, E. (2012). *Matematyka [Plus] 1, ćwiczenia rozwijające zainteresowania* [Mathematics [Plus] 1, Exercises developing interest]. Warsaw, Poland: Nowa Era.

Lannin, J., Ellis, A. B., & Elliot, R. (2011). *Developing essential understanding of mathematical reasoning: Pre-K-Grade 8.* Reston, VA: NCTM.

National Council of Teachers of Mathematics (2000). *Principles and Standards of School Mathematics.* Reston, VA: Author.

Perry, B., & Dockett, S. (2002). Young children's access to powerful mathematical ideas. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 81–112). Mahwah, NJ: Lawrence Erlbaum Associates.

Ponte, J. P. (2001). Investigating in mathematics and in learning to teach mathematics. In F. L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 53–72). Dordrecht, The Netherlands: Kluwer.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53–70). Hillsdale, NJ: Lawrence Erlbaum Associates.