The history of the fourth dimension: A way of engaging pupils in secondary classrooms

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The National Curriculum in England has, over the past decade, been revised a multitude of times. Disengagement of pupils was one of the reasons for revisions. In September of 2014, a new curriculum in mathematics was introduced, aiming to give greater freedom to teachers and schools to construct a curriculum and teaching episodes that are engaging and appropriate for their students. This paper investigates how such episodes can be constructed through the investigations based on historical development of mathematical concepts and how they could easily link to the new curriculum, offering at the same time greater opportunities for pupils' engagement. The history of the fourth dimension is one such possible topic, and the paper suggests a way of using it in a secondary classroom.

Keywords: Fourth dimension, engagement, Schläfli, Stringham, Flatland.

INTRODUCTION

In the decade between 2004 and 2014, mathematics curriculum changed twice, about fifty reports on mathematics education in the country have been published, and seven different Secretaries of State for Education passed through the British Parliament. Each of the changes and reports suggested that the state of mathematics education in Britain is troublesome; the causes were identified, the evidence was given (in either a narrative or analytical format), and of course the suggestions to improve the situation were recommended.[1] The most troublesome of all troubles listed was the perceived irrelevance of mathematics and the lack of desire to engage with it.[2]

When exploring the issue of disengagement, teachers reported that it was the curriculum that narrows down the topics and the lack of choice to engage with different topics from the curriculum that was at the basis of the problem.[3] The organizations such as National Strategies (discontinued 2010) previously tried to help teachers devise teaching episodes and gave suggestions on pedagogy. The new curriculum instead offers an element of autonomy, meaning that schools and teachers are given freedom to choose and design the topics and teaching episodes appropriate to their environments. Likewise, the curriculum itself lists the skills and knowledge to develop in pupils, but gives no (or minimal) guidance as to the choice of topics.

The choice of topic described in this paper – the history of the fourth dimension – arose from two experiences: of working with gifted and talented pupils some years ago on the representations of the fourth dimension in mathematics (Lawrence, 2012) and the engagement reported in alternative curricula, such as, for example, Steiner system, which introduces projective geometry and the study of the fourth dimension in the final years of secondary school (Woods et al., 2013).

The paper thus first gives a historical overview of the topic, and then investigates the possible pedagogical approaches to develop material based on certain principles listed. It concludes by showing how an unorthodox topic such as this, can nevertheless be easily linked to the new curriculum and the skills and knowledge it aims to develop in pupils. The engagement is expected but not yet empirically proven; this is proposed as a possible future study, and some initial experiments from schools in which the teaching pedagogy is trialed are described.

HISTORY OF THE FOURTH DIMENSION

Whilst the concept of the fourth dimension was developed in the nineteenth century, the origins of it could be traced as far back as the antiquity in the most possibly wide sense of conceptual development. Aristotle for example, discussed it in *De Caelo*,[4] and Ptolemy denied and disproved it but nevertheless mentioned and contemplated upon it (Cajori, 1926, p. 397; Heiberg, 1893, p. 7a, 33). John Wallis, although writing this whilst considering geometric interpretations of quantities he was developing in the context of algebra, wrote (Wallis 1685, p. 126):

> A Line drawn into a Line shall make a Plane or Surface; this drawn into a Line, shall make a Solid: But if this Solid be drawn into a Line, or this Plane into a Plane, what shall it make? A Plano-Plane? That is a Monster in Nature, and less possible than a Chimaera or Centaure. For Length, Breadth and Thickness, take up the whole of Space. Nor can our Fansie imagine how there should be a Fourth Local Dimension beyond these Three.

With the French Revolution, some revolutionary mathematical thinking happened, and Lagrange in particular, spoke of three coordinates to describe the space of three dimensions, introducing time as the fourth, and denoting it *t* (Lagrange, 1797, p. 223).

The reader is reminded that this can by no means be an exhaustive study, but is a sketch of the history of the fourth dimension and the narrative given is but a thread that will later be examined in possible educational setting and application to teaching.

Let us then trace further historical development. The first such opportunity was the example of Möbius and Zöllner. Möbius (1827) first spoke about an object getting out of a dimension it belonged to in order to perform a spatial operation. If one had a crystal, structured like a left-handed staircase, how would one get its three-dimensional reflection? Zöllner (Johann Friedrich, 1834–1882) further simplified this. If one has a circle and a point outside of it, how can one get the point into the circle without cutting or crossing over the circumference?

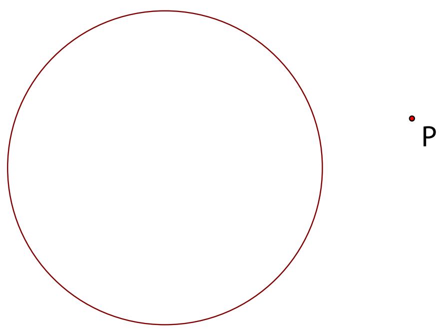
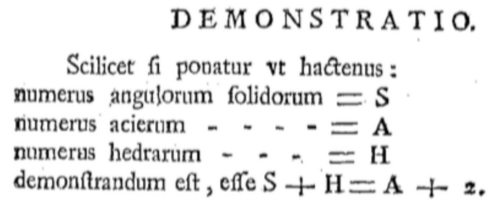


Figure 1: Zöllner illustrates that in order to perform certain operations in space, objects must exit their current dimension (Zöllner, 1878)

In 1852, Ludwig Schläfli (1814–1895), a Swiss mathematician published a book *Theorie der vielfachen Kontinuität, (Theory of Continuous Manifolds)*, in which he wrote about the four dimensions. Schläfli looked at *Elementa doctrinae solidorum* published in 1758, in which Euler described for the first time what was to become known as *Euler's characteristic*, the expression which conveys the information that in all convex solid bodies the sum of the solid angles and the number of faces is equal to the number of edges add 2.


DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum = S
numerus acierum - - - - = A
numerus hedrarum - - - = H
demonſtrandum eſt, eſſe S + H = A + 2.

Figure 2: Euler's characteristic first described in *Elementa doctrinae solidorum*, 119

We now usually denote Euler's characteristic by Greek letter chi and describe it for convex polyhedra $\chi = V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces in a polyhedron. If we further analyze the formula we notice that we begin from the first variable which counts points (point we earlier took to represent 0^{th} dimension); the second variable which numbers the edges in a solid, (representing line, 1^{st} dimension) and the third variable, numbering the faces of a solid, (polygon is bound part of a plane, representing the 2^{nd} dimension).

Schläfli (1852) showed that this formula is also valid in four dimensions or indeed any higher dimension. We will get there – but let us first look at how he first defined a system which would describe any regular polytope in any dimension.

There is only one polytope in 1^{st} dimension, a line segment, and the Schläfli symbol denoting this is {}. Regular polygons in two dimensions are, for example, triangle {3}, square {4}, pentagon {5}, etc. Remembering that he only used these symbols to denote regular polytopes, we continue. In three dimensions the five regular polyhedra, Platonic solids, can be described as {3, 3} – tetrahedron has three-sided polygons that meet three at each vertex; {4, 3} – cube has four-sided polygons three of which meet at each vertex; {3, 4} – octahedron has three-sided polygons four of which meet at each vertex; {3, 5} – icosahedron has three-sided polygons five of which meet at each vertex; {5, 3} – dodecahedron has five-sided polygons, three of which meet at each vertex (Schläfli, 1852).

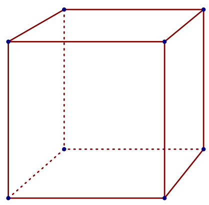


Figure 3

Schläfli realized and showed that Euler's characteristic can be represented in a slightly different form: he stated that $V - E + F - C = 1$, in effect stating that the number of vertices, minus the number of edges, plus the number of faces, minus the number of cells, equals 1. Cell in three dimensions is a solid (convex); so for a cube this formula would be:

8 (vertices) – 12 (edges) + 6 (faces) – 1 (cell) = 1

He then showed that the minus plus pattern continues with even dimensions (0th, 2nd, etc.) having positive value and odd (1st, 3rd, etc.) negative. This was a big breakthrough: by extending the validity of Euler's characteristic to the fourth and any other higher dimension, Schläfli showed that it was possible to calculate various characteristics of four-dimensional polytopes if we had certain other information. This also meant that the four-dimensional solids could be now identified, classified, and studied.

The mathematical description of generating the fourth (and higher dimensions) was first given in an elegant way by William Stringham (Stringham, 1880, p. 1):

> A pencil of lines, diverging from a common vertex in n-dimensional space, forms the edges of an n-fold (short for n-dimensional) angle. There must be at least n of them, for otherwise they would lie in a space of less than n dimensions. If there be just n of them, combined two and two they form 2-fold face boundaries; three and three, they form 3-fold trihedral boundaries, and so on. So that the simplest n-fold angle is bounded by n edges, $\frac{n(n-1)}{2}$ faces, $\frac{n(n-1)(n-2)}{2}$ 3-folds, in fact, by $\frac{n!}{k(n-k)!}$ k-folds. Let such an angle be called elementary.

Stringham tried to illustrate this in his paper in the following manner (Figure 4).

The study of the dimensions became something of a vogue in the 19th century and many a famous mathematician, from Graßman (1844), Riemann (1854), Clifford (1873) and Cayley (1885) to name but a few, wrote on it. But how to translate this into a classroom experience for teenagers? A novel from 1884 may give us some guidance on that.

NARRATIVE ABOUT LIFE IN DIFFERENT DIMENSIONS

The introduction to 'dimensionality' could be given and illustrated finely through the metaphor about *Flatland*, the 19th century English mathematical novel written by Edwin Abbott Abbott (1838–1926), a London schoolmaster and Shakespearean scholar. Flatland is a land that is flat. It is (Abbott, 1884, p. 2):

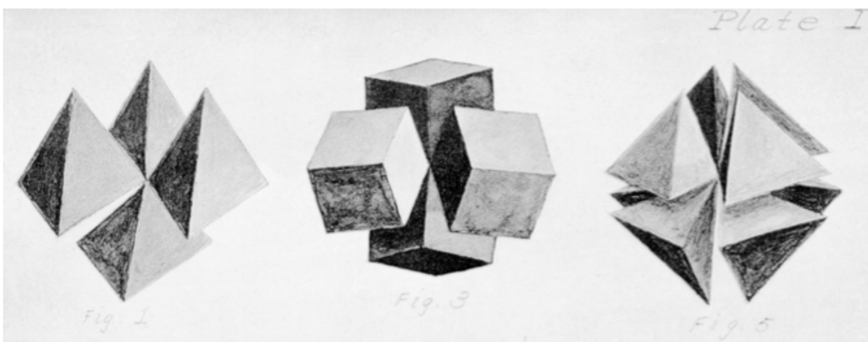


Figure 4: These represent 'respectively summits, one in each figures of the 4-fold pentahedroid, oktahedroid, and hexadekahedroid, with the 3-fold boundaries of the summit spread out symmetrically in three dimensional space' (Stringham, 1880, p. 6)

> like a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it, very much like shadows – only hard and with luminous edges – and you will then have a pretty correct notion of my country and countrymen...

Flatland, whilst it gives opportunities for many discussions to be brought into the mathematics classroom, also offer a good introduction to the bigger questions that are not easily dealt with in mathematics education. Abbott for example raises the question of ethics and the place of women in the world as he knew it. The two dimensional beings who are stuck in the two dimensional reality are also stuck in the belief that women should be treated in a different way to men. One of illustrations from this strange world shows that very clearly.

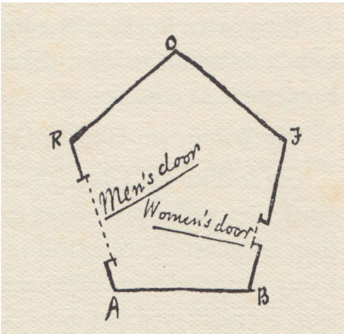


Figure 5: The most common construction for a house in Flatland, with separate doors for men and women

Apart from the social dimension, Flatland further offers ample opportunities to introduce some higher order thinking about 'big' questions in mathematics – what is the nature of space for example and how many (real) dimensions does it consist of? The question of dimensionality is in the book introduced with an almost mystical experience that the main protagonist, Square, has when he meets Sphere. First the Square saw Sphere through Sphere's intersection with the Flatland, but eventually Sphere spoke the Square. And the 'mystical' wasn't that after all as Sphere explains (Abbott, 1884, p. 77):

> Surely you must now see that my explanation, and no other, suits the phenomena. What you call Solid things are really superficial; what you call Space is really nothing but a great Plane. I am in Space, and look down upon the insides of the things of which you only see the outsides. You could leave this Plane yourself, if you could but summon up the necessary volition. A slight upward or downward motion would enable you to see all that I can see.

Of course Sphere is, similarly to Square, stuck in his own world of limited dimensions and when, towards the end of the novel, Square regurgitates the analogy between dimensions and speaks of projections of fourth dimensional bodies in three dimensions, the Sphere explains a simple "Nonsense!" – of course there are no higher dimension than that which he could experience.

What Flatland offers is an introduction to the discussion about dimensions in mathematics that can lead to asking pupils to imagine a life in dimensions different to the ones they are used to. What if they lived in two dimensions? How would they see the friend sitting next to them? Equally what if they lived in four dimensions? What would they and their friends look like?

SOME SERIOUS MATHEMATICS, BUT HOW TO DO IT IN THE CLASSROOM?

We will now turn to examining points we have so far mentioned on our way from Aristotle to Flatland and suggest a way of constructing a narrative and teach-

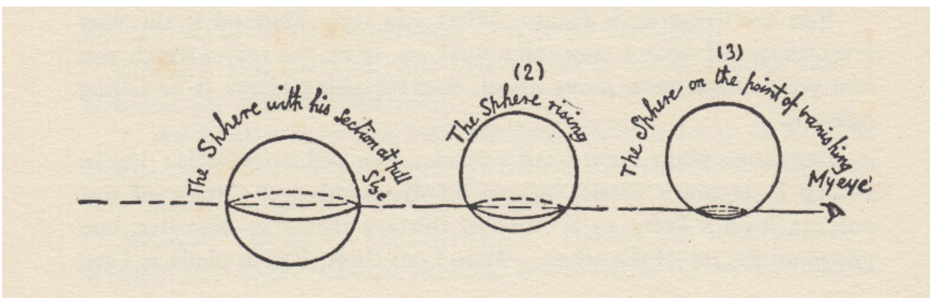


Figure 6: The diminishing sphere leaving projections in Flatland, its cross sections being circles, p. 72

ing episodes that could be used to engage teenagers in a classroom setting.

A Firstly, the trace we plotted of discussions about dimensionality stretch different times, cultures, and geographies. By analogy there could be some new mathematics born just right here and now, from their own thoughts and ideas. We mention philosophers and mathematicians that span twenty three centuries. The development of mathematical ideas can be employed in the classroom in various ways: from construction of timelines to open-ended discussions about the nature of mathematical inventions, and the contributions that are possible in this field.

B We begin with defining of the fourth dimension by musing about our ability to describe it mathematically. Both Wallis' and Lagrange's descriptions could be used in the classroom to give possible interpretations from different context and offer an insight into how new mathematics begins. In this way the way to model behaviors and attitudes may begin to emerge, and pupils may feel emboldened to pose new questions, mathematical exploration is thus brought closer to the classroom practice. A teacher can illustrate this by examples and geometric diagrams from original works some of which are mentioned above and listed in bibliography, and some interpretations illustrate also this paper.

C We meet with the study of Platonic solids. This is a rich field in the history of development of mathematical concepts that also spans centuries, and can be investigated in the classroom in a number of ways. For example, construction of Platonic solids, their representations in art (Emmer, 1982), and the study of Platonic solids by Euler are some instances that could be used as starting points for activities. Students can derive formula for Euler's characteristic and further investigate it in the light of Schläfli's extension of it.

D We come across various mathematical descriptions and symbols – the development of notation and formulae gives pupils opportunities to engage with the process and attempt to do the same/similar themselves in their own contexts. Teachers can work with pupils on Euler's, Schläfli's, and Stringham's algebraic formulations and discuss their different approaches: Schläfli symbols are also an interesting way of presenting geometrical entities.

E Finally the depiction of how life would exist in different dimensions and the consequences for our world are well narrated with the help of Flatland. We come to the practical construction of that narrative – suggestion is to mimic a dialogue by examining existing dialogues in the book and one such example may be that between Square and Sphere, appearing throughout the novel.

CURRICULUM LINKS

At this point it would be good to see whether we can, after all, establish some connections between the history of the fourth dimension and the new National Curriculum (DoE, 2014). We propose to state the NC

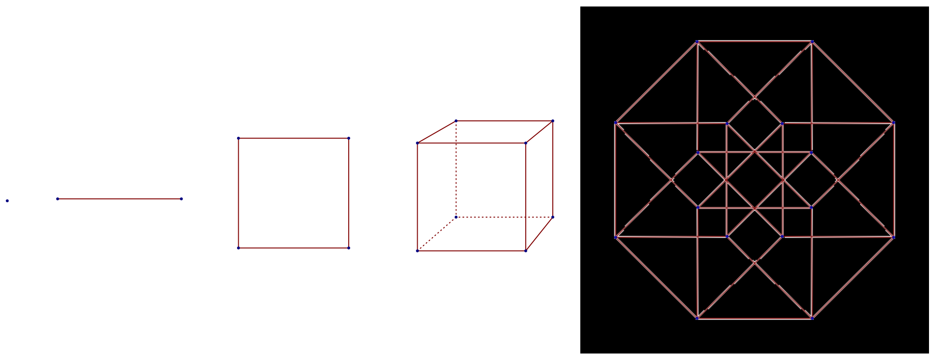


Figure 7: Here is an opportunity to use dynamic geometry software in illustrating different dimensions: starting from the 0th dimension, represented by point, we generate one-dimensional object – the line segment. Further by moving the line segment perpendicularly to itself we generate a square, a two-dimensional object. By moving the square perpendicularly to itself, we generate four-dimensional object, a tesseract (image by the author).

description and 'answers' from the 4^{th} dimension as follows.

***NC** (National Curriculum description): Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems.*

***4dh** (4th dimension history lesson for the classroom):* Whilst mathematics is often described as a method and a way of exploring and solving the big questions of life and existence, the pupils in the classroom report a different experience (Smith, 2004). The history of the fourth dimension can give a concrete and a very tangible view of mathematical development that occurred in the pursuit of the question of existence of higher dimensions as we have shown on above examples.

***NC**: Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas.*

4dh: An example of dimensionality which can be described both visually and algebraically is the way Stringham described the n^{th} dimension. Stringham says that, the simplest n-fold angle is bounded by n number of edges, and so on. Stringham's formulations are related to number theory and figurate numbers. An investigation can be conducted on the similarities and certainly representations of the two corresponding strings of formulae. We can here return to Wallis' studies in context of algebra.

***NC**: Move freely between different numerical, algebraic, graphical and diagrammatic representations, including of linear, quadratic, reciprocal functions.*

4dh: The descriptions of the fourth dimension include all of the above as we have seen in this short paper.

CONCLUSION AND FURTHER DEVELOPMENTS

From the listed history of the fourth dimension, the materials are being developed for the use in the classroom by the author and current cohort of trainee teachers studying with me, in the context of three very different educational settings.

Institution A is the first school where we have started working with the pupils. It is a mainstream boys' school, has a high achievement record in mathematics, and is comparable to specialized and grammar schools (i.e. the study of history, classics, and high achievement in mathematics are the norm). The school A poses the challenges in terms of producing the material that would engage and stretch pupils' abilities so the material developed aims to link geometry and algebra as seen from our mention of Wallis, Euler, and Stringham. It appears that pupils in this setting are very keen to engage with mathematics, and the study of the 4^{th} dimension is seen as a way of stretching the pupils to study beyond the curriculum.

Institution B is a national institution promoting the study of mathematics for the gifted and talented pupils. Whilst many pupils in this setting have some idea about the dimensionality, exercises offering them possibility to represent them in different ways as described above given pupils not only the sense of achievement, engagement, and enjoyment, but their further interest and individual research is also noted.

Institution C is a special school for disabled pupils whose abilities in functional mathematics are low and hence the mathematical curriculum is narrowed down to teaching basic skills such as financial functioning. The challenge in this school was to provide a curriculum that engages and instigates an aesthetic appreciation and enjoyment of mathematical ideas lest mathematics is perceived by both pupils and teachers as a discipline which is only functional or arithmetic bound.

By working within these three very different institutions, we trialed the three aspects of mathematics that we have identified as possible principles of developing teaching programmes for the new curriculum. These are:

a) extending the most able pupils by modeling mathematical practices from the past

b) engaging pupils by learning mathematical skills and understanding via 'big' questions of mathematics

c) teaching mathematical appreciation via aesthetic experience rooted in mathematical concepts.

The preliminary conclusions are that researching a historical development of mathematical concept can give opportunities for multiple settings, differentiation in terms of possible levels of achievement in mathematics, and cover different aspects of modeling mathematical practice with different types of pupils.

Finally, after identifying the links with the curriculum so easily, an idea is forming that the new programme for secondary school mathematics can indeed be entirely based on historical development of mathematics, offering for the first time an educational programme that would be truly meaningful and engaging and give pupils a glimpse of the big questions of mathematics about the nature of space and time that many past mathematicians had been enthused by.

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ENDNOTES

1. See in particular http://mathsreports.wordpress.com/2013/01/05/homehome/ for summary and text of these. Accessed 1st October 2014.

2. For example Adrian's Smith's report on the state of mathematics education in 2004 was followed by the establishment of National Centre for Excellence in the Teaching of Mathematics in 2006. See Lawrence (2009).

3. See for example a project report on student disengagement in English secondary education by Sonia Sodha and Silvia Guglielmi (2009).

4. Aristotle says (2012, 268a, pp. 10–15): "A magnitude if divisible one way is a line, if two ways a surface, and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are, and that which is divisible in three directions is divisible in all." He however rejected the possibility of an extension of this thinking: "All magnitudes, then, which are divisible are also continuous. Whether we can also say that whatever is continuous is divisible does not yet, on our present grounds, appear. One thing, however, is clear. We cannot pass beyond body to a further kind, as we passed from length to surface, and from surface to body." (Aristotle, 2012, 268a, pp. 25–30)