Sharing structures of algebraic expressions through language: A transformation gap

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Collaborating in the transformation of algebraic expressions results in a need to share the structures of algebraic expressions with the help of language. However, little is known about how such structures are conveyed through language – and adopted by the other participants of the discourse. This paper uses a functional-pragmatic framework to reconstruct the patterns behind such discourses in which structures are shared. With this framework, a transformation gap between conveying and adopting structures is identified: Where the first speaker can only refer to the original expression to propose a transformation, the other participants are able to refer to both the original and to a transformed expression for seeing structures.

Keywords: Structure sense; language; discourse; algebra.

INTRODUCTION

Algebraic expressions can be flexibly interpreted. Arcavi (2005) shows the different ways for students to make sense of symbols and symbolic expressions. Drouhard and Teppo (2004) argue that an algebraic expression, especially one with a distributive structure, can be interpreted in many different ways according to a specific context that can be flexibly activated, e.g. as a function. Accordingly, the students' transformation of an algebraic expression is not a mechanical activity that is aimed at a predefined result, but an activity that is guided by the students' individual interpretations of the structures of expressions. Furthermore, these structures in an algebraic expression do not solely exist „on their own", but come *into existence* in the activity of transforming an expression (Rüede, 2012). For example, when transforming the expression $5ab + a - 5b - 1$, there are different ways to apply the distributive law; students may either relate $5ab$ to a or to $-5b$. Each relation gives way to a different notion of the structure of the expression and, through this, a different application of the distributive law.

In a collaborative task of transforming an algebraic expression, students coordinate their joint activity to a large part with the help of language. More specifically, they explicate their interpretations of the structures of the expression in the discourse with the help of language to other students. Being able to use language to describe structures in algebraic expressions helps students to see structures, as language provides the means to share perceived structures with the teacher and with others. However, little is known about how students convey a structure in an algebraic expression to other participants of a discourse through language - and how language affects how the latter ones can adopt these structures.

STRUCTURE SEEING AND ITS CONNECTION TO LANGUAGE

In order to conceptualize the transformation of algebraic expressions not as a mechanical activity, but as activity that is guided by the interpretations of structures in algebraic expressions, the model of structure sense was introduced (Linchevski & Livneh, 1999) and used and refined in various studies (Novotná & Hoch, 2008). As structures of algebraic expressions emerge in the activities *with* the expressions, structure sense is here regarded as structure seeing, that is, as an interpretative activity of relating identified parts of an expression to each other and to the whole expression in order to decide for a transformation of this expression (Meyer, 2014).

When translating a text into an algebraic expression, students link the structure of the text with the algebraic expression they are constructing. There is evidence that the translation of a task into an algebraic expression is linked to the understanding of language (Duru

& Koklu, 2011). A sequential translation of the words of a text from left to right might be a way for students to structure an expression also in a sequential way (e.g. Clement, 1982). However, other evidence suggests that students do not usually translate this way, but rather first look into the meanings of the words before translating (Capraro & Joffrion, 2006).

The link between the structure of the text and the algebraic expression, however, is not direct, but mediated by meta-linguistic awareness, that is, by the ability of a student to "reflect on and analyze spoken or written language" (MacGregor & Price, 1999, p. 451). Based on their empirical results, MacGregor and Price conclude that "this conscious awareness of language structures and the ability to manipulate those structures may be a manifestation of deeper cognitive processes that also underlie the understanding of algebraic notation" (1999, p. 462).

Caspi and Sfard (2012) give insight into the elements of language that can be used for expressing structures in algebraic expressions, like compound noun phrases and objectified processes. While they do not look into students' seeing of structure, they conceptualize different levels of how elements of an algebraic expression can be expressed with language and what level of generality this language indicates. The first level is that of processes, where language is used to express a calculation. The calculation is presented in the order in which it is executed. The second is the granular level around the description of a calculation; it still describes a process but also contains compound clauses that make procedural elements into an object, like "the sum of…is". The third is the objectified level where complex calculations and processes can be substituted with objects or objectified descriptions, e.g. "A product of a sum of two numbers". Caspi and Sfard find out that 7^{th} graders are, under certain circumstances, capable of talking about algebraic expressions on the third level. Their argument suggests that when students, at a higher level, can express complex structures in a more condensed way with language, then more complex structures are available to participants.

My study is focused on the language that students use to express the structures of algebraic expressions in order to share them with others, and how this affects which structures are adopted by others or put to the fore in an algebraic expression - and which are disregarded. In order to look into this question, I use a functional-pragmatic framework which is based upon an activity-theoretical notion of linguistics. Within a functional-pragmatic analysis, the genesis and transformation of the propositional content in the discourse can be traced back to the linguistic actions and 'linguistic reality' of participants.

METHODOLOGY: A FUNCTIONAL-PRAGMATIC ANALYSIS

The aim of a functional-pragmatics analysis has been formulated by one of its main representatives as follows:

> In short, the fundamental aim of Functional Pragmatics is to analyze language as a sociohistorically developed action form that mediates between a speaker (S) and a hearer (H), and achieves – with respect to constellations in the actants' action space […] - a transformation of deficiency into sufficiency with respect to a system of societally elaborated needs. (Redder, 2008, p. 136)

Functional Pragmatics regards a communicative act as driven by a purpose. Speaker and hearer equally participate through speaker-actions and hearer-actions in this 'purpose-guided' activity. The inner structure of the speech acts of both speaker and hearer are synchronized with respect to „topics, focus of attention, previous (speech-)actions, etc (p. 138)" between an extralinguistic reality (depicted by P), the mental reality of speaker ($\prod_s$) and hearer ($\prod_h$) and the linguistic reality (p) (Redder, 2008).

In this study, the extralinguistic reality (P) is the structures of algebraic expressions, while the aim of the functional-pragmatic analysis is to reconstruct the linguistic reality (p), in this case, the structures that are actually shared in the discourse. The students' joint action of transforming an algebraic expression is a „transformation of deficiency into sufficiency" in relation to the „extralinguistic reality" of algebraic expressions - it is about the speaker transmitting identified structures of algebraic expressions ($\prod_h$) to the hearer ($\prod_h$), so that speaker and hearer arrive at a mathematically acceptable transformation („sufficiency").

Functional pragmatics provides a tool to analyze how structures of algebraic expressions are made available and are picked up in a communicative act for the purpose of finding a mathematically adequate transformation. This reconstruction involves two steps. The first step is the reconstruction of the *surface progression* of the discourse. The second step is the reconstruction of the *pattern* behind this surface progression. In Functional Pragmatics, the pattern is what guides the participants' actions without them being explicitly aware of it.

Reconstruction of the surface progression of the discourse

The linguistic actions of the speaker and hearer are continually influencing each other; at a given time, both participants' actions have an equal potential to influence the other's actions. There are three qualitatively different categories to distinguish these structures of linguistic actions:

— the *organization of the discourse* (Ehlich, 2007, p. 71), that is, how linguistic actions coordinate each others actions;

— the *action potential*, that is, the potential of the speakers action to bring something about in the hearer or vice versa, e.g. bringing about a deeper understanding (Ehlich, 2007, p. 71), this is connected with the purpose behind a speaker's actions;

— the *propositional content* of the linguistic actions.

A functional-pragmatic analysis starts with separating the parts of the transcript in line with these three categories. First, the organization of the discourse is in focus. It is reconstructed by looking into those parts of the transcripts where speaker and hearer coordinate each other's actions (e.g. by expressing interpersonal relations like „I'm writing, you dictate". This is driven mainly by linguistic categories.). Second, action potentials that are realized in a discourse are carved out. In the activity of transforming an algebraic expression, one can think of a situation where only procedural action potentials are realized. Accordingly, the activity would be about a calculation and about arriving at a result - this would relate to Caspi and Sfard's (2012) processual level. In the analysis of action potentials, both linguistic and didactic categories can be put to use. Third, building upon the previous steps, the propositional structure of the discourse is reconstructed. This includes the reconstruction of the linguistic realities of speaker and hearer ($\prod_s$ and $\prod_h$). For this, didactic categories have to be used, as this is on the plane of the mathematical knowledge that is in focus in the discourse. In this step of the analysis, relations, dependencies, references to previous linguistic actions etc. are in focus.

Reconstruction of the pattern that guides the students' linguistic actions

The reconstruction of the surface progression of the discourse is the starting point for analyzing the elementary propositional basis. The elementary propositional basis is the reconstruction of the knowledge that is independent of the speakers and hearers actions. The reconstruction of this knowledge leads to a notion of the linguistic reality of p. In this case, the linguistic reality p encompasses the structures of an algebraic expression that are shared. The relations between p, P, $\prod_s$ and $\prod_h$ constitute a pattern in the discourse, in this case, the pattern of sharing structures of algebraic expressions in a discourse.

In generalizing such a pattern, an apparatus can be reconstructed. An apparatus describes general patterns in discourses; it is assumed that patterns are generalizable to other discourses of the same kind. In this study, the analysis aims at reconstructing discursive patterns of structure seen in regard to the question how structures are conveyed and adopted in a discourse. The pattern presented here, however, can only be regarded as a first approximation of a general pattern. The pattern has to be tested for its generality in other qualitative and quantitative studies.

Background for the study

The case study of Max and Tim presented here is part of a larger design research study. The larger study aims at promoting the students' structure seeing by providing scaffolds and requiring the students to negotiate structural elements of algebraic expressions (Meyer, 2014). Tim and Max are 8th graders from a German middle school; they were chosen for the teaching intervention based on a previous assessment that showed that they possessed a basic understanding of the transformation of algebraic expressions, but a lack of understanding of the underlying structures of algebraic expressions. The teaching intervention consisted of three tasks with subtasks. It took 1.5h and was supervised by trained interviewers.

In the episode presented here, Tim and Max worked on the second task that required them to transform a sequence of algebraic expressions by applying the distributive law. The episode represents the third expression in this sequence; the students thus already applied the distributive law to the two previous – structurally simpler – algebraic expressions. With each expression, the students are given the original formulation of the distributive law $ab + ac = a(b + c)$ as a structural reference. The students already acquired marking strategies and were able to use them to mark structures and to better communicate about structural elements. In the here presented episode, the students start to work on the expression $ab + ac + bd$.

EMPIRICAL RESULTS: SHARING STRUCTURES THROUGH LANGUAGE

As a first step of a functional-pragmatic analysis, a transcript is divided into segments (letters) and sections (numbers); these segments and sections are heuristically refined during the analysis. The already refined segmented transcript is given here; the later analyses refer back to this transcript.

1	Tim:	[1a][*writes down expression ab+ac+bd that is given in the task*] [1b]Hm, [1c]do you want to do this?
2	Max:	[1d]Mhm [*confirming*]
3	Tim:	[1e]Ah, wait, |[2a][*starts to write down the transformed expression a(b+c)+bd*], [2b]I would do it this way, [2c]simply for the reason, [2d]because of course *a* is there two times [*points at the a's in the original expression*], [2e]thus *ab* and *c* [*points at a, b and c in the transformed expression*]
4	Interviewer:	[2f]Mhm [*confirming*]
5	Tim:	[2g]Just taken times [*german for multiplicating*]. [2h]And there [*points at bd*] just only the *b* times *d* is taken. |
6	Interviewer:	[3]Ok
7	Max:	|[4a]Yes, [4b]I believe that too, [5a]because only there is *a* [*points at ab and ac*], [5b]and there is no *a* in front [*points at bd*]
8	Interviewer:	[5c]Mhm
9	Max:	[5d]This is why one just has to put this in brackets [*points at b+d*] [5e]and this other one [*points at bd*] comes behind the brackets. |

The action potential and the organization of the discourse

The *organization of the discourse* that is established by Tim is oriented at Max and at the Interviewer. At first, Tim organizes his actions in relation to Max. In 1a, he writes down the task while at the same time addressing Max: "Do you want to do this?" (1c). On the one hand, Tim wants Max to participate in the solution of the task. On the other hand, he is delaying the discourse, so that he 'squeezes out' some time to think about the algebraic expression at hand. The invitation to participate is held up in 2b, where Tim explains: "I would do it this way" - this discursive action implies that there might be another way and that Max is expected to propose one. Later on, Max directly relates to Tim's transformation of the expression in 2a, confirming it (4a). With this he indicates, that the propositional core of his speaker actions are related to Tim's reasoning. In addition Max organizes his speaker actions in relation to Tim's actions (2b), by suggesting that *he* "believes that too" (4b).

The *action potential* revolves around transforming the expression in a way that is acceptable to the interviewer/teacher, while making themselves understandable to each other. As Tim's actions are directed at the interviewer, who is regarded as a knowledgeable teacher, he likely wants to give correct reasons and to use "mathematical" language (the literate register). Furthermore, Tim needs Max to understand and approve of his transformation, so that they together can arrive at a common solution. Thus, he has to make himself understandable to Max. On the other hand, the purpose of Max' actions is to confirm Tim's transformation of the expression. As shown above, the discourse is organized in a way that requires Max to give an individual perspective. Accordingly, Max has to relate to Tim's actions but also has to give his own perspective on the transformation.

The propositional content of Tim's linguistic actions

The basis of the *propositional content* is the written expression $a(b + c) + bd$ that Tim writes down in 2a. The following speaker actions all relate to this expression (depicted in Figure 1; relation diagrams are an element of a functional-pragmatic analysis). In his first actions, Tim explains the first part of the expression, namely $a(b + c)$, by saying "because of course *a* is there two times" (2d). The following speaker actions depend on this latter speaker action with "thus" and "just".

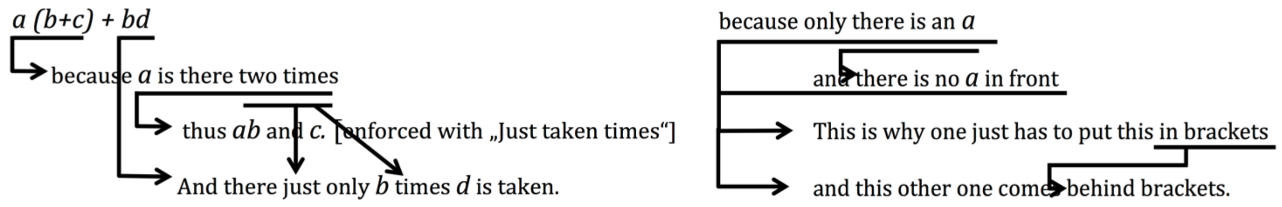

Figure 1: Reconstructed surface progressions of sections 2 and 5

"Thus" refers to the propositional content "*a* is there two times", while "just" is connected to the propositional content "*ab* and *c*" (2g). Tim gives two reasons for the expression $a(b+c)$ that build upon each other. In his later speaker actions, Tim also explains "*bd*". He is connecting this action with "and" to his previous actions, while at the same time he uses deictic gestures/words ("there") to connect his action to the algebraic expression (2h). The expression "just only" stands in contrast to "two times". In this way, Tim expresses that in case of *bd*, there are no two variables. This is further indicated by using the phrase "taken times" (multiplication) for explaining both $a(b+c)$ and *bd*.

The algebraic expression in Tim's *linguistic reality* has two cornerstones. The first cornerstone is how often a certain variable is there. This is expressed by "*a* is there two times" and "only the *b* times *d*". The second cornerstone is the multiplication; on the one hand, "*ab* and *c* [...][are] just taken times", on the other, "*b* times *d* is taken". Both the first and the second part of the expression rest on this cornerstone. This way it connects the first and second part of the expression.

The relations of the cornerstones show how Tim structures the algebraic expression. "*ab* and *c*" is dependent on "two times". In other words, leaving out one *a* is a result of *a* being there two times, where leaving out *a* is explained by the multiplication. These relations are expressed by Tim as causal relations through "because", "thus" and "just". The second part of the expression *bd* is structured in the same way as the first part. This time, there is only a single variable *b*. The multiplication is not based on leaving out a variable. In summary, Tim structures the expression according to how many times a variable occurs, and how this results in a certain form of multiplication - one time by leaving out a variable (transformation of the first part) and one time by not changing the variables (transformation of the second part).

The propositional content of Max's linguistic actions

The basis of the *propositional content* of Max' linguistic actions is both the original expression and its transformation. In 5a, Max relates deictically to the original expression, pointing at the expressions *ab* and *ac* and referring to them with "there" ([5a]"because only there is a [*points at ab and ac*]"); in 5b, he refers the same way to *bd*. In both 5a and 5b, he argues for the existence/absence of *a* in the subexpressions of the original expression. In 5d and 5e, Max deictically relates to the transformed expression, but at the same time builds an argument upon the perceived properties of the original expression (indicated by the arrow that encompasses both 5a and 5b in Figure 1), saying "this is why one just has to put this [*points at* $(b+d)$] in brackets" (5d). Max uses the same relations in structuring the second part of the expression (5e). Both parts of the expression, $(b+c)$ and *bd* are connected with "and" and by stating the position of *bd* in relation to $(b+d)$ ("behind the brackets").

The structure of the algebraic expression in Max' *linguistic reality* has only one cornerstone. This cornerstone is the (non-)existence of ɑɑ in the subexpressions of the original expression. However, Max' conclusions that are based on the original expression are, additionally, supported by features of the transformed expression. Thus, his conclusions are circular: Max' justification why one has to transform the original expression into $(b+d)$ requires that $(b+d)$ is already given as a transformation. In this way, Max' justification dissolves the logic *of the process* of transformation. In summary, in Max' notion of the distributive law $ab+ac=a(b+c)$ the two *a*'s on the left side directly result in the bracketed expression $(b+d)$.

Pattern of sharing algebraic structures in a discourse

At first sight, it seems that the linguistic realities of Tim and Max have one element in common, namely the existence of the variable *a*. However, on closer inspection, Max' reasoning is based upon the existence and non-existence of *a* in the subexpressions, while Tim's reasoning is based on how often *a* (or *b*) exists in the original expression. Thus, in the course of this short episode, Max has 'only' picked up one aspect of Tim's structuring of the expression.

At second sight, the linguistic realities of Max and Tim are both based on the original and the transformed expression. However, when comparing the propositional content of Tim's and Max' discursive actions, it becomes apparent that Tim's speaker actions depict a process, in the sense that they relate to his steps of the transformation: Each step relates the transformations to features of the original expression. In other words, the mathematical objects, on which he builds, are located in the original expression, while his deductions are about properties of the transformed expression.

In contrast, Max' speaker actions are abstracted from the process of transformation. While his arguments are based upon objects in the original expression, they require at the same time the existence of the transformed expression. Thus, Max' reasoning is based upon the equation as a whole, that was established by Tim and that links the original expression and the transformed expression. Max' structuring may be a result of the organization of the discourse. As shown above, the discourse requires Max to relate his structuring of the expression to Tim's transformation and reasoning behind this transformation. Perhaps, in the logic of the discourse, Max has no other option as to include the transformed expression into his reasoning.

In a broader perspective, the pattern behind Tim's and Max' negotiation of the structure of the expression can be described as a transformation gap. This transformation gap describes that the one discursive participant who comes up with a first transformation has different means to justify his structuring than the hearer, who can work with both the original expression and the proposed transformed expression. Accordingly, while the speaker needs to connect the original expression with the transformed expression in order to give reasons for his transformation, the hearer can base his reasoning on the original expression *together* with the transformed expression. As a result, at a given point in a discourse about algebraic structures there may exist two different structures of an algebraic expression: One structure that follows the logic of the transformation, and one that disconnects from this logic and focuses on the transformed expression together with the original expression. To the participants, these structures may seem compatible or even equal, while in reality these structures are very different.

DISCUSSION

Although just one case has been discussed in this paper, the reconstructed pattern of conveying and adopting the structures of algebraic expressions in a discourse may yield significant consequences. For example, teaching interventions that focus on promoting students' structure seen by implementing activities of negotiating different structures of algebraic expressions have to account for the transformation gap. One way to do this could be to connect algebraic expressions more strongly to other mathematical objects like functions, so that the different ways of structuring an expression come into light in reference to this object. The possibility of translating the algebraic expression into another representation may further act as a scaffold.

The functional-pragmatic analysis has proven its potential to look into the foundations of the students' ways of conveying and adopting structures in algebraic expressions. The reconstruction of the surface process, resulting in the reconstructed linguistic realities of speaker and hearer, as well as the reconstruction of the pattern behind this, may also be suitable for addressing other research questions, where "deficiency [is transformed] into sufficiency" in students' discourses.

In further studies, the generality of the here presented pattern has to be addressed. It has to be addressed, if it is part of a larger structure-sharing-pattern that is typical for discourses about algebraic expressions. A more systematic view on different discourses of sharing structures may show, if such a pattern is common to discourses about algebraic structures. This might also lead to a deeper understanding of students' resources in regard to structure seeing and of coordinating transformations.

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