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The present study deals with the interaction of mathematical abilities and the role of the mathematical memory in the problem-solving process. To examine those phenomena, the study investigates the behaviour of high-achieving students from secondary school when solving new and challenging problems. Although the mathematical memory accounts for a small part of the problem-solving process, it has a critical role in the choice of problem-solving methods. The study shows that if the initially selected methods do not lead to the desired outcome, the students find it very difficult to modify them. The study also shows that students who use algebraic problem-solving methods perform better than those who use numerical methods.

Keywords: High-achievers, mathematical memory, abilities, problem solving.

INTRODUCTION

Despite increasing emphasis on the education of gifted and high-achieving students, we still have limited empirical data about their mathematical abilities and use of memory functions during mathematical problem-solving. So far, much of the research on mathematical abilities has been conducted on low-achievers (e.g., Swanson & Jerman, 2006). Only a few studies are focusing the mathematical abilities of gifted and high-achieving students (e.g., Brandl, 2011; Vilkomir & O´Donoghue, 2009) or the connection between those students' memory functions and their mathematical performance (Leikin, Paz-Baruch, & Leikin, 2013; Raghubar, Barnes, & Hecht, 2010). Yet no study since Krutetskii (1976) has examined the role of the mathematical memory in the context of able students' problem-solving activities.

BACKGROUND

Mathematical abilities

We are not born with abilities that are explicitly mathematical, but an active contact with the subject may, under favourable circumstances, generate complex mathematical abilities (Krutetskii, 1976). When discussing the subject, we should remind ourselves that mathematics is not a topic defined by sufficient and necessary components and there is no uniform terminology for the abilities that we tend to define as mathematical (Csíkos & Dobi, 2001). Thus, it is not possible to define a structured system of mathematical thinking in which the units are satisfactory to understand the system. A historical review shows that Calkins (1894) concluded – based on replies from Harvard students – that mathematicians have concrete rather than verbal memories, that there are no differences in ease in memorising between mathematicians and other students and that, when doing mathematics, there is no significant difference between men and women. In the early 1900s, mainly because of the dominance of psychometric approaches, the research community's efforts to define mathematical abilities were unsatisfactory. Nevertheless, Binet, Piaget and Vygotsky made relevant contributions to the subject by replacing psychometric approaches with socio-cultural attitudes and thereby showing that abilities are nor static or innate, but qualities that can be assimilated and developed by the individual (Vilkomir & O´Donoghue, 2009).

An essential contribution to the subject was made by Krutetskii (1976) who observed around 200 pupils in a longitudinal study (1955–1966). Krutetskii's analysis of the pupils' problem-solving activities lead to a model of mathematical ability as a dynamic and complex phenomenon, consisting of: a) the ability to *obtain* and *formalize mathematical information* (e.g., formalized perception of mathematic material), b) the ability to *process mathematical information* (e.g., logical thought, flexibility in mental processes, striving for clarity and simplicity of solutions), c) the ability to *retain mathematical information* or *mathematical memory* (i.e., a generalized memory for mathematical relationships) and d) a *general synthetic component*, named a "mathematical cast of mind" (Krutetskii, 1976, pp. 350–351).

Although the above model is often used to identify mathematical giftedness – and studies (e.g., Brandl, 2011; Krutetskii, 1976; Öystein, 2011) show that high-achievers are not necessarily mathematically gifted – Krutetskii indicates that even students performing very well in the learning of the subject, e.g. high-achievers, manifest abilities that can be regarded as proper mathematical abilities (ibid, pp. 67–70).

Mathematical memory

Memory is thought to be critical to both learning and doing mathematics (e.g., Leikin et al., 2013; Raghubar et al., 2010). Research that deals with memory functions was conducted for more than 120 years, but during the first eight decades the topic was almost exclusively examined by quantitative measures (Byers & Erlwanger, 1985). However, in the 1940s, the research shifted focus toward more qualitative terms. Thus, Katona (1940) stated that information related to a method and based on understanding is easier to remember than arbitrary numbers. Later, Bruner (1962) noted that detailed knowledge can be recalled from memory with the use of simple interrelated representations. Although numbers are fundamental tools in mathematics, Krutetskii (1976) underlines that recalling numbers or multiplication tables cannot be equated with mathematical memory; highly able students memorise contextual information of a problem only during the problem-solving process and forget it mostly afterwards. Yet, they can still several months later recall the *general method* which solved the problem. In contrast, low-achievers often remember the context and exact figures related to a problem, but rarely the general problem-solving method. Thus, mathematical memory is a *generalized memory* for mathematical relationships, schemes of arguments and methods of problem-solving (ibid, p. 300).

Studies (e.g., Squire, 2004) show significant distinctions between different types of memory systems. Relating mathematical problem-solving to the cognitive model – by using a simplification – one can say that information is processed (e.g. the problem is solved) in the working memory and is stored (e.g. the problem-solving method) in the long term memory. Long term memory has two subcategories: *explicit* and *implicit* memory, depending on the type of information stored in the respective system. The implicit memory stores information about procedures, algorithms and patterns of movement that can be activated when certain events occur; in mathematical context, the *procedural* memory is a relevant part of this system (Olson et al., 2009; Squire, 2004). The explicit memory stores information about experiences and facts which can be consciously recalled and explained; thus, it is associated with the ability to create mental schemas for problem-solving (Davis, Hill, & Smith, 2000). Thus, we can assume that mathematical memory, as defined by Krutetskii, belongs to the explicit (hence not to the implicit) memory system.

Krutetskii (1976, p. 339) and Davis and colleagues (2000) suggest that proper manifestations of mathematical memory are not observable in the primary grades, because at that age able pupils usually remember relationships and concrete data equally well. Krutetskii indicates that mathematical memory is formed at later stages, most probably on the basis of the initial ability to generalize mathematical material (ibid, p. 341).

Accordingly, the present study (Szabo, 2013) examined the dynamics between mathematical abilities, as defined by Krutetskii, from the following perspectives:

1) The evidence and the interaction of mathematical abilities when high-achieving students are solving new and challenging mathematical problems.

2) The role of the mathematical memory in the process of solving new and challenging mathematical problems.

METHOD

Participants

According to Krutetskii: a) the mathematical memory cannot be observed properly in young pupils or in low-achievers and b) mathematical abilities are manifested by high-achievers. Consequently, the present study focused remarkably high-achieving, 16–17 years old students from Swedish secondary school. The participants attended an advanced mathematics programme and achieved the highest grade in mathematics. The participation was optional; after four months of classroom observations and consultations with their mathematics teacher, three boys and three girls were selected to attend the study.

Tasks

The analysis of a given problem, regardless of the mathematical field it belongs to, indicates the structure of the mathematical thinking needed to solve the problem (Halmos, 1980). Several studies confirm that the most effective way to discern mathematical abilities is to analyse the behaviour of individuals in the context of problem-solving activities (e.g., Gyarmathy, 2002; Krutetskii, 1976). Other results (e.g., Krutetskii, 1976; Öystein, 2011) indicate that individual experience influences students' ways of solving problems. The aim of the present study was to investigate the participants' mathematical abilities, not their knowledge of the subject; thus, to avoid as far as possible the influence of prior experiences, new problems were proposed that were not of a standard nature. After examining the participants' textbooks and consulting their math teacher, the following problems were selected:

Problem 1: In a semicircle we draw two additional semicircles, according to the figure. Is the length of the large semicircle longer, shorter or equal to the sum of the lengths of the two smaller semicircles? Justify your answer.

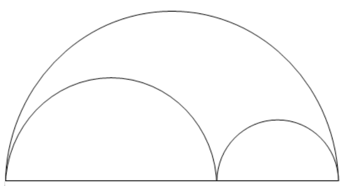

Figure 1

Problem 2: Mary and Peter want to buy a CD. At the store, they realise that Mary has 24 SEK less and Peter has 2 SEK less than the price of the CD. Even when they put their money together, they couldn't afford to buy the CD. What is the price of the CD and how much money has Mary and Peter respectively?

Observations and materials

Classroom interaction affects student's thought process and that interaction is not limited to verbal communication; even gestures or other minor events are affecting the process (Norris, 2002). Krutetskii also underlines that it is difficult to map individual mathematical abilities if pupils are observed in a classroom situation. To avoid these confounding factors, the students were observed individually and, in order to avoid stress, they had unlimited time for completing the tasks. They were asked to write down every step in the process and to "think out loud" whenever it was possible. If a student neither wrote, nor drew or spoke for a while, some of the following questions were posed: What is bothering you? Why do you do that? What do you want to do and why? What are you thinking about? All observations were carried out during a single day and recorded by using a technology which enables to digitalize handwritten notes and related verbal utterances (www.livescribe.com).

Pupils are not used to communicate their thoughts while solving problems (Krutetskii, 1976). To avoid the risk that significant parts of their cognitive actions would not be documented during the process, every problem-solving activity was followed by a contextual interview. The recordings of the interviews and problem-solving activities were transcribed verbatim. Although the unlimited time for solving the problems, no participant needed more than 14 minutes to solve a single problem and the average duration of the succeeding contextual interviews was four minutes.

Data analysis

The *general synthetic component* in Krutetskii's model, i.e., the "mathematical cast of mind", is difficult to observe during occasional problem-solving and is typical for mathematically gifted students (ibid, pp. 350–351). The participants were certainly high-achievers, but not tested for mathematical giftedness and besides that, only two tasks were proposed to them during the present study. Thus, the general synthetic component was not focused in the study. Conversely, the *ability to generalize* is frequently used when pupils establish mathematical memories (ibid, p. 341); thus, the study examined the presence of the ability to generalize mathematical information during the students' activities. A rigorous a-priori analysis of the proposed tasks led to an identification-model for the present study, which focused the following abilities from Krutetskii's framework: *obtaining* and *formalizing* mathematical information (O), *processing* mathematical information (P), *generalizing* mathematical relations and operations (G) and *mathematical memory* (M).

The digital recording of the problem-solving activities resulted in an exact linear reproduction of the pupils' written solutions, drawings and verbal utterances. This was very useful when performing *qualitative content analysis* of the empirical material, inspired by Graneheim and Lundman (2004) and van Leeuwen (2005). The students' solutions were analysed by identifying, coding and categorising the basic patterns in the empirical content. At first, the method highlighted those abilities that were directly expressed in the empirical material, i.e. the *manifest content*. After that, the *latent content* was analysed, by combining data from observations and contextual interviews. I exemplify this with data from Linda, who – when solving Problem 1 – looked at the task, drew some semicircles and whispered for herself:

> Linda: Thus, eh... Oh, and here we are after all just using what radius they have and such. One would...

After this device, which occurred after 30 seconds from start, she solved the problem by not saying that much and it was not possible to decide if mathematical memory was present at the time or if she only used her ability to obtain and formalize mathematical information. Later, when analysing the latent content, the following sequence from the contextual interview referred to the above mentioned episode:

> Linda: And then you express it simply as that, well, expressing their different diameters as something of each other.
> Interviewer: Yes.
> Linda: It is similar to another task that I like very much...
> ...
> Linda: Like there, when solving that, the first thing to do... it is making formulas... How different triangles and squares... how the inside of it looks.
> Interviewer: Yes... hum.
> Linda: Just like there, if you express different sides through... and take one side minus the other, just like in that problem...

The statement "It is similar to another task that I like very much" and the following explanation, relating the actual problem to an apparently different task – with a context of triangles and squares – show the evidence of an (explicit) memory for a generalization, i.e. mathematical memory in the actual device. The combined analysis resulted in a matrix there every device which lasted at least one second during the observed activities, and the time period for its occurrence, was related to the mathematical abilities focused in the present study. The matrix displayed both the interaction between the focused abilities and the occurrence of the abilities, measured in seconds, during every particular problem-solving activity. The matrix also indicated that some devices were actually related to two interconnected abilities.

RESULTS

The interaction of the mathematical abilities

Every student confirmed that the problems were new and challenging, which was a key issue in the design of the study. The analysis displays that the students' problem-solving activities contain three main phases. All activities start with an *initial phase* which encloses both the ability to *obtain and formalize the mathematical information* and *mathematical memory*; these abilities are intimately connected and it is difficult to differentiate them. Directly after the initial phase, follows a phase where the ability to *process mathematical information* is prominent. Nevertheless, every activity ends with a different phase of *processing mathematical information*, where the students are checking their results. Beyond these three main phases, the observed mathematical abilities interact in irregular and unstructured configurations.

The analysis also shows that if the chosen method does not lead to a direct solution of the problem, students become stressful and discontinue processing the mathematical information; they return to the initial phase, which is once again followed by a phase of information processing. Some participants went through this shifting of phases three times. The stress was most evident at Problem 2, where three of those four students who used similar methods made the same error when solving the inequality $2x - 26 < x$. All three activities include the incorrect sequence "$2x - 26 < x$ gives $x - 13 < x$", before returning to the initial phase. All participants were familiar with inequalities; thus, one may naturally wonder why high-achievers make seemingly simple errors. The interviews reveal that the stress occurred when the formalization led to inequalities instead of the expected equations:

Earl: That´s I was a little surprised when it was... on the inequality you solved it.
Erin: It is always difficult to start thinking outside the box... It feels like your mind goes blank.
Linda: Because I get so... When I start with equations ... then I really want to solve it with equations.
Sebastian: This kind of tasks usually requires an equation.

Thus, it seems that the stress was due to the selected method, i.e. equation-solving, lacked those procedures that are necessary when solving inequalities.

The problem-solving methods used by the students

The problem-solving methods could be divided into two categories, as identified during the a-priori analysis of the proposed tasks: *algebraic* respectively *numerical* methods. Consequently, it was possible to distinguish 7 algebraic and 5 numerical methods among the 12 processes. All of the algebraic methods – despite different approaches – led to correct solutions. In contrast, when applying numerical methods, the problems were not solved in a proper way.

The general structure of the students' mathematical abilities

The analysis emerged in a matrix there every device in the problem-solving activities was related to at least one mathematical ability. On the other hand, the analysis revealed that at some devices there were two interrelated abilities present at the time – thus the methods of observation and analysis used in this study were not sufficient to differentiate those interrelated abilities. Accordingly, the ability to process mathematical information (P) is present at 52 % of the total time of the students' activities (see also Table 1). Obtaining and formalizing mathematical information (O) – solitary or in combination with other abilities – is present at 45 % and mathematical memory (M) at 17 % of the total time (Table 1).

According to the a-priori analysis of the tasks, the ability to generalize mathematical information (G) could be detected when numerical solutions were developed into general solutions. Consequently, when numerical solutions were presented by the students, they were asked if they were able to generalize the obtained results. The analysis shows that none of the participants has been able to generalize the obtained numerical solutions; thus, the ability to generalize mathematical information could not be observed in this study.

The role of the mathematical memory in the problem-solving process

The analysis demonstrates that the mathematical memory is present predominantly at the *initial phase* of the process, at a relatively small proportion. In isolated form – at 5 % of the process – the ability is present in the manifest content and it is mainly used for recalling mathematical relationships and problem-solving methods. In the latent content, the ability is present during 12 % of the process, mainly in combination with the ability to obtain and formalize mathematical information (O with M) (Table 1).

Despite of its minor proportion, the mathematical memory is essential to students' achievement in the problem-solving process, because: a) the students selected their methods in the initial phase of the process and b) the students found it very difficult to modify the selected methods. Although they started over the process by returning to the initial phase, none of them abandoned the initially selected method.

DISCUSSION

One of the study's main objectives was to map the interaction of high-achieving students' mathematical abilities during problem-solving. Three main phases of the problem-solving activities were identified: the *initial phase*, the *subsequent phase* of processing the

O	O with P	O with M	P	P with M	G	M
33 %	2 %	10 %	48 %	2 %	0 %	5 %

Note: O = the ability to obtain and formalize mathematical information; P = the ability to process information; G = the ability to generalize mathematical information; M = mathematical memory

Table 1: Average time for mathematical abilities, according to the total time of the problem-solving process

information and the *ending phase*, where results are checked by once again processing the information. Despite the limitations of the study, the chronological order of the mentioned phases emphasize to some extent Polya's (1957) model for problem-solving, which consist of four phases: a) *understanding* the problem, b) *devising a plan* in order to solve the problem, c) *carrying out the plan* and d) *looking back*. Thus, the study indicates that high-achievers solve new and challenging mathematical problems according to the ground stones in Polya's model.

According to the results, the role of the mathematical memory – despite its relatively small presence in the process – is critical, since the participants selected their methods at the start of the process and did not change them later, e.g. when the formalization led to inequalities instead of the expected equations, the participants returned to the initial phase but did not abandon the selected method. A selection of an improper method caused stress, time delay and errors during problem-solving. Thus, it seems that the participants experience a close and rigid interrelation between problem-solving methods and included procedures, i.e. they are not acting flexibly when solving new and challenging problems. By confirming the findings of other studies (e.g., Brandl, 2011) – where typical high-achievers are characterised by being dutiful, nonflexible and conformist – the results indicate that these participants were high-achievers but probably not mathematically gifted. In contrast, mathematically gifted students are described as flexible, high-level problem-solvers and out-of-the-box-thinkers (e.g., Brandl, 2011; Krutetskii, 1976; Leikin, 2014). Hence, the study confirms some qualitative differences in problem-solving between high-achievers who are not essentially mathematically gifted and mathematically gifted students.

However, the inflexibility of the participants can also be explained by two main functions of the cerebral cortex, where working memory operates. One function is to assemble all new information in relation to previous experiences (Olson et al., 2009). Thus, we can assume that at the initial phase, when obtaining and formalizing the information, the students are influenced by previous experiences (e.g. mathematical memory) and act as they are used to, e.g. by starting problem-solving with equations. Another main function of the cerebral cortex is to automate all knowledge (Olson et al., 2009). Yet, automated processes are rigid and extremely hard to modify during an on-going activity. Therefore, it seems that equation-solving is an automated process for typical high-achievers and that the interpretation of new information in the light of past experiences affects their possibility to think flexibly in unusual situations.

Finally, it has to be mentioned that the present study confirms Krutetskii's (1976) observation that during the initial phase it is extremely difficult to distinguish the ability to obtain mathematical information from the mathematical memory. Since information-units stored in the long term memory systems are retrieved at extremely high speed to the working memory (Olson et al., 2009), the methods used in this study were not sufficient to differentiate the information-units related to respective abilities. For a better understanding of the interaction of the mathematical abilities there is a need of further studies. One possible access is to design studies where the structure of the mathematical ability is examined with approaches from several research fields, e.g. by combining qualitative research methods with practices from cognitive neuroscience. In that way, we would possibly be able to answer the questions that were not possible to be answered in the present study.

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