Designing tasks for mathematically talented students

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For the Design Research project presented, a learning environment for mathematically talented and interested 7th-grade students was investigated. The results show that the subject matter of graph theory offers both opportunities and means for students to develop their abilities. The data analysis showed likewise how the tasks might be modified in order to impose on their potential and thereby foster students' abilities of a formalized perception and pervasion of mathematical information and of generalization.

Keywords: Talented students, Design Research, RME, task design, graph theory.

INTRODUCTION

Mathematical potential and talent is one of the topics in mathematics education research increasingly attracting attention (Leikin, Karp, Novotna, & Singe, 2013) – with good reason. Not only since PISA and TIMSS – which have impressively revealed the heterogeneity of students within countries and even within single school types – the huge variety of students' abilities is well-known. However, there is only a small number of investigations that show how high-achievers can be supported in order to develop their potential (cf., e.g., Kießwetter & Rehlich, 2005). In the context of mathematics education research, there are certain findings regarding the abilities and characteristics of mathematically talented students (e.g., Krutetskii, 1976). These constitute a stepping stone for the investigation at hand: On the basis of the existing results, a Design Research project (cf. Gravemeijer & Bakker, 2006, p. 1) was conducted in order to develop and refine a learning arrangement for a group of mathematically talented and interested students and to assess, for a concrete subject matter – namely graph coloring –, how mathematically talented students can be challenged and, at the same time, supported in their specific abilities. Therefore it was one main goal to get insights into the abilities and possible difficulties of the students, which are being focused in this paper.

THEORETICAL BACKGROUND

Mathematically talented students and their abilities

One of the most important investigations in the field of abilities of mathematically talented students was conducted by Krutetskii (1976). He found four components being constitutive for mathematical abilities during school age: *Obtaining mathematical information*, i.e. "the ability for formalized perception of mathematical material, for grasping the formal structure of a problem" (Krutetskii, 1976, p. 350); *Processing mathematical information*, which comprises, among others, "the ability for rapid and broad generalization of mathematical objects, relations, and operations" (Krutetskii, 1976, p. 350); *Retaining mathematical information*, meaning memorizing mathematical approaches etc.; and a *General synthetic component*, connecting and interrelating all other components and forming a mathematical cast of mind. Mathematically talented students can be characterized by possessing these abilities to a great extent. Therefore Krutetskii's categories should be considered when learning environments and tasks for these students are being designed.

In the investigation presented, the focus was – on the one hand – on the students' ability to *mathematize* situations and – on the other hand – on their ability to *generalize*, which means being able to (a) recognize similar situations and (b) handle the generalized solution in these situations (Krutetskii, 1976, p. 237). It was investigated in how far the students were able to mathematize and generalize when handling the tasks of the developed learning environment.

Task design and realistic mathematics education

For refining and (re-)designing our learning environment, we focused especially on students' abilities to *mathematize* and *generalize* (see above). In our project, research was not supposed to be separated from a practical perspective, on the contrary: research and design were closely interwoven (Gravemeijer & Bakker, 2006). The Design Research alignment meant a focus on *task design*, which is of significant importance in mathematics education (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013). However, the terms *task* and *task design* vary widely (Watson et al., 2013). The aim of our investigation was to develop and refine *tasks* in their meaning of being "what students are asked to do. Then 'activity' means the subsequent mathematical (and other) motives that emerge from interaction between student, teacher, resources, environment, and so on around the task" (Watson et al., 2013, p. 11). In our case, *task design* means the design of tasks for the above-mentioned purpose as well as possible oral impulses of the teacher and corresponding material.

The design of the learning environment within our investigation is being guided by the domain-specific instruction theory of *Realistic Mathematics Education (RME)*. RME "itself is the result of a long history of design research in the Netherlands" (Gravemeijer & Bakker, 2006, p. 2).

"According to RME, mathematics should be seen as an activity (Freudenthal, 1973), and students, rather than being receivers of ready-made mathematics, should be active participants in the educational process, in which they develop mathematical tools and insights by themselves" (Drijvers et al., 2013, p. 56).

Mathematical learning should – according to RME – originate from problem situations in realistic contexts, which do not necessarily need to be from the real world (Van den Heuvel-Panhuizen, 2005, p. 2). Based on their activity within these situations, students use their individual concepts in order to handle situations and therefore extend them. In our project, students were supposed to handle the presented tasks and hence the mathematical information (cf. Krutetskii, 1976, see above), and thereby apply their mathematical concepts, develop them and generalize them for being able to apply them in different situations (Krutetskii, 1976).

Graph theory as subject matter for mathematically talented students

The application of graph theory as a subject matter for talented students offers different advantages. Most of these are connected to the large applicability and therefore its huge potential for fostering students' creativity (cf. Leuders, 2007).

In graph theory, the notion of *graph* is fundamental: "A graph G consists of a finite set V, whose members are called vertices, and a set E of 2-subsets of V, whose members are called edges" (Biggs, 1985, p. 158; partially highlighted in the original). As graphs represent networking structures, they can be visualized by points and connecting lines between two points. Especially *graph coloring* is a central scope of application in the focus of our study. "A vertex-colouring of a graph $G = (V, E)$ is a function $c: V \to \mathbb{N}$ with the property that $c(x) \neq c(y)$ whenever $\{x, y\} \in E$. The chromatic number χ (G) is defined to be the least integer k for which there is a vertex-colouring of G using k colours" (Biggs, 1985, p. 172). Due to graph coloring, we can distinguish adjacent vertices and, thereby, create disjoint partitions of vertex-sets.

Research questions

In this Design Research project, the following questions were at the core of the research interests (based on Krutetskii, 1976). These are:

1) To what extent are the talented students able to *mathematize* the realistic problem situations in the given tasks?

2) In how far are they able to recognize the similarity of different situations and *generalize* their solutions?

On the basis of these questions, it is considered how the task design can be refined in order to optimize the learning processes.

DESIGN OF THE INVESTIGATION

Designing the tasks

We developed tasks, according to two kinds of problem situations (cf. Joklitschke, 2014). The first kind of problem situations is rather abstract. Here, a concrete problem situation may be the assignment of persons to different groups. Persons can be represented by vertices – and the do-not-like-relation by edges. If the vertices are being colored, the colors signify the emerging groups. The number of colors then represents the number of groups. A second field of application addresses problem situations with rather geometrical visualizations, for instance the coloring of maps. Here, the dual graph is needed, which includes the information about adjacent areas: every area is represented by a vertex, and adjacent areas are visualized by an edge (cf. Leuders, 2007, p. 141). By finding the chromatic number, we get to know how many colors are needed.

Our tasks comprised certain subtasks and were supposed to serve for a 90 minutes lesson. Two of them are being focused in this paper. As we assumed problem situations with a geometrical representation to be easier to grasp for the students (see Bronner, 2014), the first task was a map task. Students were supposed to find the minimum number of colors for coloring the 16 federal states of Germany (see Leuders, 2007, pp. 132ff, Figure 1). The second task, then, comprised a problem situation without geometrical visualization (cf. Leuders, 2007, pp. 133f). We chose the context of the soccer world championship with national supporters who are – due to rivalries – supposed to be accommodated in different places (Figure 1).

Implementation and data collection at school

The developed tasks were investigated in a group of eleven 7th-graders – aged twelve or 13 – at a German secondary school. This group had already been active for two years and was supposed to give mathematical talented and interested students the opportunity to enhance their abilities. The selection of the students for this course depended on their performance in mathematics (according to the teachers' assessment) and their motivation to participate. Since this course was already well-established, all students had experience in the field of graph theory: They had worked on realistic problem situations (shortest path problems, spanning trees and Euler graphs, developing algorithms) beforehand. The students were used to work in small groups on open problems and to generate the mathematical contents by themselves.

In this investigation, the students worked in groups of four (or three) as they were used to. The group work took 90 minutes. The group work of all groups was videotaped. The analysis focused on one "focus group" because here, the students communicated a lot so that the analysis could be undertaken on a profound empirical base. The videos were transcribed.

The four students of this focus group were additionally interviewed in semi-structured partner post interviews. Here, every group got the same tasks and questions. The students were interviewed in pairs of two as this was expected to foster their communication and give them safety. The interviews took place two weeks after the group work. The interview guide comprised questions on the approaches that the students had worked on before as well as two new, but analogous problem situations (one problem regarding map coloring and one problem regarding partitioning). The

Task 1: Color selection

GeoPaint Inc. wants to include a colored map of Germany with its federal states in their assortment.

a) What should the company consider? Note the necessary criteria and try to color the map.

b) As the company has to order each color separately, the map gets cheaper when they use as few colors as possible. How many colors are needed?

c) In order to develop a computer program that helps designing the colored map, GeoPaint has to create an illustration that represents the neighborhood of the federal states. How could such an illustration look like?

Task 2: A Peaceful World Championship

At the soccer world championship 2014, many European teams compete. But the national supporters partly do not stand well with each other. Therefore, fans with a special rivalry shall be accommodated in different locations. Now, the organizing committee has to find a suitable allocation of the national supporters. These are the conditions:

- German supporters do not stand well with fans from the Netherlands and England.
- ...

How many locations do the organizers need to ensure a peaceful meeting? How can you represent the problem adequately?

Figure 1: "Map coloring task" and "Soccer world championship task"

semi-structured interviews – which allowed not only to inquire their approaches to the same tasks, but also to go into detail to see certain differences between the pairs, especially in their inferential reasoning – took 30 minutes each. The students were supposed to comment on their approaches by thinking aloud. The interviews were videotaped and transcribed afterwards. For the data analysis, both, the transcripts of the group work and of the interviews, were taken into account.

ANALYTICAL FRAMEWORK

For investigating students' individual approaches, we used a framework that arises from a philosophical perspective, founding on ideas of Kant, Frege, Wittgenstein, Heidegger, and Brandom's (1994) theory of *semantic Inferentialism*. Based on these philosophical notions, a theoretical and analytical framework was developed for mathematics education (e.g., Schindler, 2014), whose applicability has been shown for different subject matters (e.g., Schindler & Hußmann, 2013). The philosophical background signifies that individual approaches and students' concepts can only be understood in their use, i.e. in reasoning processes. Therefore, the data analysis is being conducted on the basis of three crucial analytical elements, which are: individual commitments, inferences and focuses (see below).

The concept of *language game* plays an important part in the theoretical background. For our analysis it is important to analyze the use of concepts in these language games, since concepts and their meaning can – in this holistic perspective – only be understood by means of the role that they play here: "grasping a concept involves mastering the properties of inferential moves that connect it to many other concepts" (Brandom, 1994, p. 89). The basic elements in this framework are propositions that individuals hold to be true and explicate, like for example "Four colors are enough to color the map". These individual *commitments* constitute the building blocks in our data analysis, which is carried out turn-by-turn on the basis of the transcripts at hand.

Furthermore, the reasoning process itself is being analyzed, as it is crucial for reconstructing students' understanding. *Inferences* embody the reasoning process, as they constitute the relation between commitments if a student entitles a commitment with another commitment, like e.g. "Five colors are one too many, thus four colors are enough to color a planar graph". For our data analysis, it is important that these inferences, as well as the commitments, do not have to be formally correct but to be held as true by the individual student. Besides commitments and inferences, *focuses* – as individual categories that are used to pick up, select and handle the information at hand – are being reconstructed in our data analysis. Students can – consciously or not – utilize e.g. concepts, properties, or other entities as categories, such as the number of colors and partitions. Via the analysis of these three elements (students' focuses, the commitments and inferences), we analyzed how students mathematize the realistic situations and in how far they are able to generalize (see results).

RESULTS

An insight into students' mathematization process

In general, the analysis revealed that students showed an enormous performance handling the maps task and mathematize the information at hand. First, they focused on painting the map with colored pencils. While doing so, they did not yet gain access to the formal structure of the problem and some stuck to the real-world conditions of the problem situation. For example, they committed to "The company can blend colors, then they do not have to buy so many of them". During their work, their focus became more and more structure-oriented. It was only when they started with the task to support a computer program (see task 1c, Figure 1), that they focused on graphs for the first time.

Sasha:	Well, we have to do this with a graph, right?
Tim:	Yes. I already worked on two or three graphs, but they do not yet work out.
Klara:	(reading the task out loud) The neighbors of federal states…
Sasha:	Yes, let's see.
Klara:	So, simply a graph?
Sasha:	Yes.

In this excerpt, Sasha at once commits that they can focus on a graph. Tim agrees and therefore acknowledges Sasha's commitment. Klara seems to be in doubt if this focus is adequate, as she asks "Simply a graph?".

But after Sasha's conclusive confirmation, everyone of them starts drawing a graph (focus: graph).

In this dialogue, it is interesting that all students immediately focused on an *edge* representing a neighborhood, which is adequate from a mathematical perspective, as it implies constructing the dual graph (Figure 2). As Leuders (2007, p. 140) shows, it is also possible to choose the borders to represent the edges and the vertices to represent the "corners" of three countries bordering. But the students seemed to immediately realize that "it is not of interest to know the exact shape of country's frontiers [...], but only to know who is adjacent to whom" (Leuders, 2007, p. 138, translated by M.S./J.J.). During their prolonged activity, in which they constructed the dual graph, they focused on the numbering of the countries, i.e. vertices, in order to keep track of the neighborhoods and to communicate about them more easily. The focuses on (dual) graphs, on numbering, on edges representing the neighborhood reveals that the mathematization process was successful in the first task.

Similarity recognition and generalization

The second task on the soccer world championship and national supporters represented more of a challenge for the students. Here, they had much more trouble to capture the formal structure of the problem situation and to recognize the similarity to the first task. When dealing with the task, students proceeded intuitively and tried splitting the supporters into distinct groups.

Alex: Well, we can definitely put Russia, Greece and Switzerland together.
Tim: I suggested the same.
Alex: Because they do not oppose each other.
Tim: With whom does Germany *not* get on well? Germany does not get on well with the Netherlands and England, but they get on well with Spain and France, don't they? Yes, Germany does not get along with the Netherlands. Therefore, I would put Germany together with Spain and France.

In this heuristic approach, the students' common focus is on dividing the sets of supporters into groups. In mathematical terms, it is the concept of partition, and the idea is to find an optimal (i.e. smallest possible) partition. The students justify their partitioning via harmonies or compatibilities.

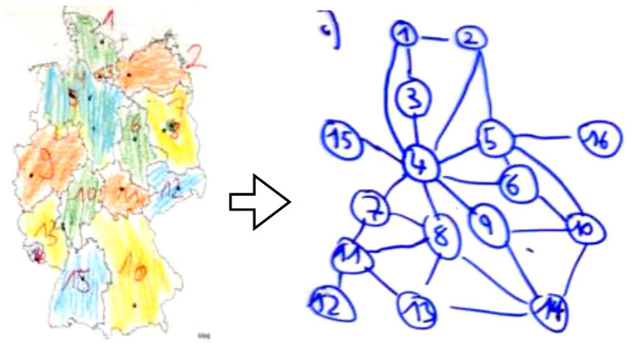

Figure 2: Students' drawings

In this excerpt of the group work, the students focused on splitting the sets of supporters (i.e. building partitions) and on the number of locations needed. They did not mention that the formal structure of this problem is similar to the first one; that the sets of supporters can be depicted – just like the countries – as vertices and the rivalries as edges – just like the borders. They did not seem to realize that both problem situations are about objects and their (binary) relations to each other and that the mathematization is obvious (cf. Leuders, 2007, p. 138). On the contrary, the focuses of the students in our investigation remained in the real-world situation in the second task. When the teacher tried to help students focusing on graphs, they disliked this focus and committed for instance to "But that would only be the same that we did with the other approach" or "That only brings the same result that we can find otherwise". They did not see a reason for focusing on graphs.

Then, in an additional, further task, students were confronted with a problem situation that dealt with four groups of national supporters, which each have a rivalry against each other. Here, students were again *explicitly* asked if it is possible to depict this with a graph and if there are edges, which necessarily have to cross over each other.

Tim: Well, I have drawn one without crossings. Where one can see that everyone hates everyone. (...)
Sasha: Huh? How do you want to draw that with a *graph*?
Tim: Four points and connect them one to each other. (...)

Sasha: But how do you want to draw *rivalry* with a graph?
Tim: Well, because if they are connected then they do not like each other. And here, everyone is connected to everyone.
Sasha: But this is actually the other way round!
Alex: Yes, I would also say that...
Tim: Yes, okay. So, simply four points. That also works. (...)
Sasha: (while writing, talking to himself) Mhm, but that's right.

Here, we see that the students acknowledge the focus on graphs. Tim directly seems to generalize – probably unconsciously – the idea of *dissociation represented as edges* from the first task (boundaries as edges) to the second task (rivalry as edges). Sasha and Alexander, however, are obviously not convinced: Sasha asks for reasons and explanations, but Tim's reasoning is not convincing for them. Instead, the group focuses on *compatibility as edges* – and not on the similarities between the two kinds of dissociation and of the two problem situations. When writing his findings down, Sasha then acknowledges Tim's focus: Afterwards, he continues drawing dissociation as edges – but this does not become subject of discussion anymore. Sasha's final commitment as well as his drawing might indicate a generalization process.

In the post interview, however, neither Tim nor Sasha saw the similarities between the geometrical and the abstract problem situation. The frequent changes of focuses as well as the low level of inferential reasoning during the lecture series go hand in hand with the lacking generalization. However, an impulse of the interviewer encouraged the similarity recognition in one case (Sasha). This was initiated by the interviewer's explicit focus on similarities between the maps problem and the problems with the football supporters.

Interviewer: Are there any similarities between the two problem situations?
Sasha: (Looking straight ahead, then looking out of the window, beginning to smile and looking back to the interviewer) Yes. Because here (pointing onto the map), adjacent countries were supposed to have different colors. And here, it is effectively the same. Because, then the colors are the locations, and the locations just differ from each other.

The interviewer's question seems to foster the student's similarity recognition: After thinking about the question, he affirms the similarity. The inferences that he makes underpin that he understands the reason: He is able to see the functional similarity of countries and locations and commits to "The colors are then the locations". This indicates that students *are* able to focus on graphs and on coloring in problem situations without geometrical visualization, but this needs specific support.

Consequences for the task design

Task 1: One of the prominent results revealing from the above-mentioned data analysis is that the mathematization process was fostered by the sub-task to think of an illustration for a computer program. Here, students were able to focus on graphs easily. This was very useful for systematic approaches in which the students focused more structurally and systematically and thus fostered the mathematization process. This indicated that this task does not need modifications in this regard.

Task 2: It was not easy for the talented students in our investigation to grasp the formal structure of a graph in problem situations, which do not have a geometrical visualization. On the one hand, this indicates that these tasks have a huge potential for talented students to exert themselves, because it was shown that these students can gather the mathematical structure when having the right focuses. On the other hand, the analysis reveals that the tasks and even the impulses of the teacher did not prompt the students to perceive the structure and to generalize their focuses and commitments from the geometrical problem situations. In order to foster these abilities in this context even more, the task has to be modified: There have to be more reflective aspects that make the students think of similarities. Possible impulses would be e.g. "Look for similarities of both problem situations. Is it possible to solve the second task with the same strategy that you used in task 1?". For generalization purposes, it seems to be important for the students to recognize the *focus on dissociation* in both kinds of problem situations. But without a profound inferential reasoning, even the right focus is not sufficient for students' similarity recognition and especially their generalization.

CONCLUSION

The overall aim of the empirical investigation at hand was to explore the students' approaches and abilities (esp. mathematization and generalization) for refining the design of a learning environment.

The results indicate that the students were able to perceive mathematical information and the structure of problem situations formally (cf. Krutetskii, 1976). But this ability strongly depended on the tasks presented. It was manifested as highly-developed in problem situations which had a geometrical visualization, whereas in problem situations without a direct geometrical visualization, students were less able to see the mathematical structure. In the consequence, it was not easy for the students to generalize their focuses from a problem situation with a geometrical visualization to one without the latter. Our findings indicate that Krutetskii's framework constitutes a profound basis for developing tasks for talented students: On the one hand, the students showed the abilities to a certain extent, on the other hand, it was shown how these abilities can be fostered even more.

The results of the investigation revealed many aspects which can be used for optimizing the tasks and support students in their specific abilities: for instance, giving adequate impulses that lead students to focus on graphs; and fostering a purposeful and conscious reflection about similarities of problem situations much more explicitly.

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