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Heuristics and mental flexibility in the problem solving processes of regular and gifted fifth and sixth graders

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This study reports on a mixed-methods design analyzing the problem solving processes of gifted and regular fifth and sixth graders (age 10–12). The analysis focuses on heuristic techniques, indicating that the regular pupils need more heuristics to be equally successful as the gifted ones. This finding is explained by the theory that heuristics can be used by the regular students to compensate a lack of mental flexibility.

Keywords: Mathematical problem solving, heuristics, intellectual flexibility.

INTRODUCTION

Within problem solving, which is a very important part of mathematics, heuristics play an integral role. It is generally assumed that the use of heuristics is related to problem solving with successful problem solvers using a greater number and variety of heuristics than less successful ones. But there are still some issues unresolved as these results have mostly been obtained with regular students. The number and appearance of heuristics in the processes of gifted or creative problem solvers might differ from those of regular problem solvers. In this article, the problem solving processes of two groups of fifth and sixth graders are analyzed and compared: one group consists of pupils from regular lower secondary schools while the other group consists of pupils that have been very successful at mathematical competitions.

THEORETICAL BACKGROUND

The theoretical framework of this article focuses on an integral aspect of mathematics that is important for gifted as well as for regular students: on mathematical problem solving and particularly on the use of heuristics in problem solving. After justifying the classification of gifted students by their participation in competitions, different aspects of problem solving are discussed. This implies the use of heuristics, their function of compensating a lack of intellectual flexibility, as well as reports on their trainability.

Mathematical competitions provide the opportunity for students to actively engage in mathematical problem solving and to compete with other problem solvers. Successful participants of such competitions are mostly mathematically gifted and talented students (Bicknell & Riley, 2012); a fact which should be true especially for younger students as they have less opportunity (only regarding their age) to compensate missing talent by training. Classifying the successful participants as gifted also fits Renzulli’s (1978) classical definition of giftedness as these students show above-average abilities as well as task commitment and creativity by solving the problems posed at mathematical competitions.

The term “problem solving” has different meanings ranging from solving routine tasks to working in perplexing or difficult situations (e.g., Schoenfeld, 1992). This article refers to the latter interpretation, problem solving as working on non-routine tasks. This implies that the attribute “problem” depends on the solver, not on the task. A task is a problem for a person that does not know any procedures or algorithms to solve it. Instead of algorithms that just have to be followed step by step, heuristics can help solving problems by ordering or reducing the search space and by helping to generate new ideas (Rott, 2014). In this article, heuristics are considered as methods, mental tools, or mental operations such as “working backwards” or “drawing a figure”. The use of heuristics is thoroughly

Bruder (2003) and Bruder and Collet (2011) – drawing on the work of the psychologists Lompscher and Hasdorf – describe the qualities of creative and intuitive problem solvers: One main characteristic of these problem solvers is their intellectual flexibility that allows them (amongst others) to easily consider different aspects and to focus on important parts of problems. Bruder then divides intellectual flexibility into five groups of actions, namely Reduction, Reversibility, Consideration of Aspects, Change of Aspects, and Transferring. For the actions of each group of intellectual flexibility, she presents heuristic actions that are able to help less flexible problem solvers overcome their lack of intuitive problem solving skills. For example, the intuitive skill of structuring facts can be compensated by creating a table; the intuitive skill of reversing relationships can be substituted by working backwards.

Research results regarding the trainability and use of heuristics can be summarized as follows: In the 1960ies and 70ies there have been several studies in America showing (often weak) positive correlations between the use of heuristic strategies and performance on ability tests as well as on specially constructed problem solving tests (Schoenfeld, 1992). Newer studies support this supposed relationship between the use of heuristics and success in problem solving (e.g., Komorek et al., 2007; Rott, 2012). Most of these studies did not only measure the number of heuristics used and the participants’ success in problem solving, but also conducted some sort of training. The results show consensually that usage of heuristics can be accomplished by training. However, these trainings have often been limited to small groups of problems with unknown transferability of strategies to other problems (cf. Schoenfeld, 1992); additionally, these trainings have been limited to regular (school and university) students without specifically addressing gifted students. The findings of these studies seem to support the claim of Bruder that learning heuristics can (at least partly) compensate the abilities of intuitive problem solvers in regular students. But we do not know enough about the abilities of creative or gifted problem solvers to really draw such conclusions.

The research intention of this article is to further explore the problem solving abilities of gifted students and to compare them to those of regular students. Can these two groups be distinguished by the number and appearance of the heuristics in their processes? In the short run, such a comparison can help us to better understand the way in which heuristics work. In the long run, this research can help us to better teach problem solving in schools (for gifted and regular students).

DESIGN OF THE STUDY

The aim of the research presented in this paper is to explore the problem solving behavior of fifth and sixth graders (aged 10 to 12) by analyzing and comparing the processes of two groups of pupils: novices and experts.

“Novices”: The so-called novices were regular pupils from secondary schools in Hanover, Germany that took part in the first four terms of the support and research program MALU for fifth graders, which lasted from November 2008 to June 2010 (with 10 – 15 pupils each term). These pupils came to the University of Hanover once a week for 1.5 hours and worked on problems for about half of this time. Ability tests and a consideration of school grades as selection criteria ensured a mixture of pupils that can be classified as “non-gifted”.

“Experts”: The so-called pupil experts were successful participants of mathematical competitions, namely of the final round of the German Mathematical Olympiad in 2009/10 (8 pupils of grade 5 and 6) and price winners of the Mathematical Kangaroo in 2009 (2 pupils of grade 6). As very successful problem solvers at a young age, these pupils are considered to be...

1 The use of the terms „non-gifted“ and “gifted” is mostly avoided in this context because of possible negative connotations of “non-gifted” and because there was no official test to ensure the “gifted” status of the pupil experts – however, the second group meets the criterion of “reproducible superior performance” for expertise.
2 Mathematik AG an der Leibniz Universität which means Mathematics Working Group at Leibniz University
3 The German Mathematical Olympiad consists of four rounds: (1) tasks to be solved at home to qualify for (2) a written tests at schools (180 minutes for grades 5 and 6). The best 200 students of all grades qualify for (3) the final round of the federal state which takes place at a central place. And the 12 winners of those state finals are invited to the final round of all German states – but this round is only for students of grade 8 and higher.
4 An international competition which consists of 30 tasks (75 min); 5 to 6 % of the German participants receive prices.
“gifted”. They were asked to take part in this study at the venue of the Olympiad and the school of the Kangaroo winners; the pupils did so voluntarily and have not worked with their video-partners beforehand.

To explore the pupils’ problem solving behavior, their processes were videotaped. To ensure uninfluenced problem solving attempts, the pupils worked without interruptions or hints from the researchers. For the same reason the pupils were not trained to think-aloud or interrupted by interview questions; instead, they were asked to work in pairs to enable an insight into their thoughts through their natural communication. Three problems were selected for the comparison (see Figure 1).

**METHODOLOGY**

To analyze the problem solving processes as well as the products (everything that had been written down or sketched during those processes) of both groups of pupils, a mixed methods design has been chosen. This allows for detailed analyses of singular processes as well as an overview of all the data.

**Product Coding:** The pupils’ products were graded in four categories of success: (1) *No access*, when they showed no signs of understanding the task properly or did not work on it meaningfully. (2) *Basic access*, when the pupils mainly understood the problem and showed basic approach. (3) *Advanced access*, when they understood the problem properly and solved it for the most part. And (4) *full access*, when the pupils solved the task properly and presented appropriate reasons, if necessary.

This grading system was customized for each task with examples for each category. Then, all the products were rated independently by the author and research assistants. After calculating Cohen’s kappa (k > 0.85 for each task), the few products with differing ratings were discussed and recoded, reaching consensus every time.

**Process Coding – Heuristics:** Occurrences of heuristic techniques (like *drawing a figure* or *examining special cases*) were coded using a manual that was developed for analyzing videotaped problem solving processes (see Table 1); that development was based on empirical processes as well as on related research literature (Koichu et al., 2007, being the most noteworthy influence; see Rott, 2012, for details). The coding procedure is a sort of qualitative content analysis (cf. Mayring, 2000) which helps ensuring its reliability and objectivity and makes it suitable for qualitative as well as quantitative analyses.

All videos were coded independently by several researchers, who first identified points of time in the processes where heuristics were used and then characterized the heuristics afterwards. In accordance with the TIMSS 1999 video study (cf. Jacobs et al., 2003, p. 103 f.), the “percentage of agreement” approach

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**Figure 1:** The three problems selected for the study
(Bakeman & Gottman, 1997, p. 59) was used to compute the interrater-reliability of randomly chosen videos (40 % of the processes). More than $P_{\alpha} = 0.7$ for identifying points of time in the videos with heuristics and more than $P_{\alpha} = 0.85$ for characterizing the heuristics was achieved. After calculating the reliability, all differing codes were analyzed and recoding consensually (100 % of the processes).

Pupils’ products were coded individually with the result that in 6 of the 55 processes discussed here the two members of a pair obtained results with differing product ratings. To manage the data, for each pair the better result was chosen to further work with. The heuristics in the pupils’ processes have also been coded individually. The numbers given here represent the number of different heuristics noticed for each pair; for example, when one member of a pair drew a figure while the other one didn’t do so, this heuristic was counted for the pair.

### RESULTS

#### Quantitative results

The evaluation presented here starts quantitatively by comparing some statistical results regarding the product and process codings of both groups. Looking at the novice group (i.e., the regular, non-gifted pupils), the results are distributed widely among the four product categories in all three problems indicating no ground or ceiling effects. As expected, the pupil experts are significantly more successful ($c^2 = 14.54; p < 0.001$), scoring exclusively in categories 3 or 4.

For the novices, the number of coded heuristics is related to the success in solving the problems with mean scores of 1–3 heuristics in less and 3–5 heuristics in more successful processes (see Table 2 for details). There are significant Spearman rank-order correlations$^7$ for the Coasters ($r_s = 0.69; p < 0.01$), the Number Series ($r_s = 0.78; p < 0.001$), and the Chessboard ($r_s = 0.99; p < 0.05$) problems as well as for all three problems ($r_s = 0.72; p < 0.001$) combined. This finding is in accordance with research on the topic and the values mostly match the correlations reported by Komorek and colleagues (2007); this result meets the expectations as the use of heuristics should be helpful in solving problems.

However, there are successful processes with only one or two heuristics as there are unsuccessful ones with three or more. Some pairs picked a heuristic and used it to solve the problem outright; on the other hand, heuristics did not help every time. As expected, there is no straight “the-more-the-better” rule for the use of heuristics.

For the pupil experts, there is no such correlation between the number of heuristic techniques and success.

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5 Chance-corrected measures like Cohen’s kappa are not suitable for this calculation, as there is no model to calculate the agreement by chance for a random number of heuristics distributed randomly over the course of the process.

6 Please note that in previous publications from the MALU data pool (e.g., Rott, 2012), the individually coded results have been reported. This time, the pair data is reported, therefore the numbers do not match those from previous articles.

7 As the product categories yield only ordinally scaled data, no Pearson correlation coefficient was calculated. The web-tool by R. Lowry (http://vassarstats.net/corr_rank.html) also provides a way to calculate the significance level for $n < 10$.  

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Table 1: An extract of the heuristics coding manual

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawing a figure</td>
<td>Drawing a figure, a graph, or a diagram.</td>
<td>Coasters: a drawing of possible positions of the two squares. Number Series: drawing a diagram of numbers with possible neighbors.</td>
</tr>
<tr>
<td>Auxiliary elements</td>
<td>Introducing auxiliary elements like auxiliary lines or additional variables.</td>
<td>Coasters: drawing auxiliary lines to indicate area. Chessboard: drawing borders of squares to illustrate their size or count them.</td>
</tr>
<tr>
<td>Special cases</td>
<td>Assigning special values (like 0 or 1) to algebraic problems or examining special positions in geometric problems.</td>
<td>Coasters: positions of the two squares which make it evident that the marked area amounts to one fourth of a square.</td>
</tr>
<tr>
<td>Mental flexibility</td>
<td>&quot;Thinking outside the box&quot;: special ideas and activities that are not captured by other heuristic categories</td>
<td>Coasters: (mental) rotation of the squares. Number Series: flexible way of adding numbers to both sides of the series. Chessboard: easily identifying the possible overlap of squares bigger than 1x1.</td>
</tr>
</tbody>
</table>
cess \((r_s = -0.06; p = 0.84)\), but this can be explained by ceiling effects. Surprisingly, opposed to the vague “the-more-the-better” rule, the experts use significantly less heuristics compared to equally successful novices (i.e., pairs that reached product categories 3 or 4) \((t = 2.73; p = 0.01)\).

Table 2 summarizes some statistical data of both groups. Column “count (%)” shows the number of pairs for each of the four product categories. Column “heu” shows the mean numbers of heuristics coded in the processes for each product category. The experts use less heuristics for each problem in the product categories 3 and 4.

### Qualitative results

To further explore this surprising result—better problem solvers use less heuristics—the processes are analyzed qualitatively in the following paragraphs. Coded heuristic actions are indicated with *italics*.

The first (abbreviated and smoothed) example deals with the Coasters problem. After reading the problem formulation, the novices Hannelore and Lucy start to **measure** the length of the sides of the squares to somehow calculate the requested area; Hannelore also **introduces notations** to points in the given figure. They soon notice that their first approach does not work and start to question whether the squares could be arranged in another way. Lucy then **draws figures**, among them a **special case** in which the size of the area is easily identified (see Figure 2). They return to the given figure and start adding **auxiliary lines** to indicate a **decomposition** (see Figure 2). Lucy notices that she could “cut off” a triangle and add it at the other side to regain the special case. She concludes that the area is always as big as in the special case. They then do not write down “a fourth of a square” but calculate the size of the requested area. Overall, they worked nearly 12 minutes on this problem.

Bernd and Tobi, two of the pupil experts, worked on this problem for 2 minutes. After reading, Bernd says: “Wait, this is exactly one fourth. Because you can push it over there, so that you get exactly four parts.” *(mental flexibility)* Tobi quickly agrees and they request the next problem without further justifying their solution. Bernd’s recognition can be interpreted as **Consideration of Aspects** within the framework of intellectual flexibility, because he “recognize[s] the correlation of facts and [is] easily able to vary them.” *(Bruder, 2003, p. 17)*

The second comparison of novice and expert processes deals with Marco’s Number Series. It takes the novices Birk and Janus more than 30 minutes to solve this problem. They start with “1, 3, 6, 10, 15” and are stuck, because there is no number left to add to 15 (they try...
In this period, they occasionally- tool of systematization. In this period, they occasionally use backtracking, i.e. deleting the last number(s) to continue with a different number instead of starting a new row. After more than 20 minutes, Birk introduces a new idea; he creates a table with all possible combinations of two numbers smaller than 16 adding up to a square number. This way, he realizes that the numbers 8 and 9 only have one possible neighbor each and have to start and end the row (looking for patterns). Shortly after this realization, they solve the problem.

It took the experts Robert and Lasse about 5 minutes to solve this problem. They immediately start with a list of numbers to cross out (tool of systematization) and begin their first row with the given example, “3, 6, 10”. Instead of only adding numbers to the right, they work on both sides of the row (mental flexibility). To the right, they add “15, 1, 8”. Finding no neighbor for the “8”, they do not restart, but complete the row on the left side: “13”, “12”, “4”, “5”, “11”, “14”, “2”, “7” and “9”. Within the framework of intellectual flexibility, this idea can be interpreted as Change of Aspects, because “[b]y intuition they consider different aspects of the problem which avoids or overcomes getting stuck.” (Bruder, 2003, p. 17)

Of course, these two examples are more obvious than the majority of the processes; they have been selected to illustrate an argument. However, the appearance of mental flexibility seems to be a distinguishing factor between novice and expert processes as this code appears significantly more often in the experts’ processes: overall, mental flexibility appears in 15 of 43 novice processes compared to 10 of 14 expert processes ($c^2 = 5.73; p = 0.017$); this trend continues at the level of individual problems (Coasters: 8/16 compared to 3/5; Number Series: 6/16 compared to 4/5; Chessboard: 1/9 compared to 3/4).

This has implications for practicing and teaching problem solving. Of course, intuitive problem solvers have an advantage working on problems as “the active intuition might be the most important device for discoveries, e.g. the sudden realization of analogies can only be thought of as an intuitive event.” (Winter, 1989, p. 177, translated by the author) But not-so-intuitive problem solvers might overcome their disadvantage by learning and practicing heuristic techniques.

Of course, there are limitations to this study: Firstly, the group sizes are quite small; especially the group of the pupil experts consists of only ten pupils. A bigger number of both regular and gifted pupils would be desirable to see if the observed patterns can be confirmed. Secondly, the pupils have worked in pairs to enable access to their thoughts through their natural communication. This might have influenced their problem solving behavior and it might be problematic to generalize the findings of this study to individual problem solving behavior of regular and gifted pupils.
Further studies need to evaluate whether the actions of "mental flexibility" are a genuine part of gifted pupils' problem solving behavior or whether being "mentally flexible is a learned (and thus learnable) ability these pupils picked up when participating in and training for mathematical competitions. It might be that the actions coded as "mental flexibility" are a combination of heuristics which are performed mostly cognitive and so elaborate that they are unidentifiable for the raters observing the problem solving processes of the pupil experts.

It would be interesting to see whether a follow up study with older students might provide a clearer distinction between regular and gifted students and similar results.

REFERENCES


