# Creativity and expertise: The chicken or the egg? Discovering properties of geometry figures in DGE

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*The relationship between mathematical creativity, knowledge and expertise is a phenomenon which can be seen as a "creativity-knowledge dilemma": Having knowledge is a necessary condition for a person to be creative; on the other hand, creativity is an important condition for knowledge construction. In this paper, we analyze mathematical activity that is directed at both the development of problem solving expertise and creativity in geometry. Creativity in this study is connected to the discovery of new properties of the given geometrical objects through investigation in DGE. We introduce a framework for the analysis of the complexity of a discovered property. Based on the analysis performed, we hypothesize that (1) discovery skills can be developed in people with different levels of problem solving expertise while the range of this development depends on the expertise; (2) the discovery process is rooted in the problem-solving expertise of a person.*

**Keywords:** Mathematical creativity, problem-solving expertise, geometry investigations, complexity of discovery.

## BACKGROUND

In his arguments about the importance of creativity for child development, Vygotsky ("Imagination and creativity in childhood," 1930, and "Imagination and creativity in Adolescents," 1931) maintains that imagination is goal-directed, culturally mediated, and emerges from the interweaving of fantasy and conceptual thinking. In his view, the development of creativity is one of the essential elements of the child's mental and social development. The educational system should pay attention to this construct.

Vygotsky (1982/1930) argues that imagination (creativity in our terms) is the central mechanism in the development of children's knowledge, since imagination allows them to construct connections between their existing knowledge and the new pieces that they study. Going a step further, we argue that creative activities in mathematics allow students to design mathematical connections and use their mathematical knowledge in an unlearned fashion. In this sense, creativity is a necessary condition for knowledge construction. To the contrary, creative processes in mathematics presume discovery of new mathematical constructs, properties and regularities to expand mathematical knowledge to new territory. This requires previous knowledge and the ability to critically evaluate that the discovered facts are new. In this sense, knowledge is a necessary condition for a creative process. Thus we consider the knowledge-creativity dilemma an intriguing phenomenon and try to explore it.

### Relative and absolute creativity

The rationale for investigating creativity in school children lies in the shift from a static view on mathematical creativity to a dynamic characterization of personal development (Leikin, 2013). Rather than looking at creativity as a personal characteristic given at birth ("a gift"), we consider it a personal creative potential that can be developed if appropriate opportunities are provided for the learner. This position requires a distinction between absolute creativity as associated with discovery or invention at a universal level and relative creativity which is considered with respect to a specific person acting in a creative way within a specific reference group (cf., objective vs. subjective creativity in Lytton (1971), and that of Big C vs. Little C creativity in Csikszentmihalyi (1988)). For example, the distinction between absolute and relative creativity is obvious in Yerushalmy (2009) who provided analysis of curricular design, mainly based on mathematical investigations, which implies the development of mathematical creativity in all students.

Our study explores connections between relative creativity and expertise associated with geometry investigations. We compare their creative activity by prospective mathematics teachers (PMTs) with the creative activity of an expert in mathematical problem solving (Sharon – pseudonym).

**Problem solving expertise**

An expert in mathematics has been described as having a more robust mental imagery, more numerous images, the ability to switch efficiently and effectively between different images, the ability to focus attention on appropriate features of problems, and having more cognizance of their thought and of how others may think (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994). Individuals with a coherent understanding of a particular mathematical topic have a complex system of internal and external representations that are joined together by numerous strong connections to form a network of knowledge. In contrast to experts, a student's system of representations of a mathematical concept may be deficient in number and deficient in connections to form an adequate network of knowledge (Hiebert & Carpenter, 1992; Lester 1994).

Whereas a novice uses a conventional means-ends analysis to solve problems, an expert categorizes problems according to solution principles and applies those principles in a forward-working manner to the givens of the problem. The expert's knowledge-based strategy is dominated by previous experience. He or she "knows into which category the problem should be placed and knows which moves are most appropriate, given that particular type of problem" (Sweller, Mawer, & Ward, 1983, p. 640).

Expert knowledge is also likely to be organised as hierarchical schemas (Chi, Feltovich, & Glaser, 1981). Problem-solving schemas are knowledge structures that consist of prototypical aspects of the problem type including declarative information about the features, facts, principles, and strategies associated with the problem. Experts have been shown to spend more time on features designated as critical to the problem (Morrow et al., 2009; Shanteau, 1992) and to rapidly encode features of problems based on goal-relevant representations. In the study on which this paper is based we were interested in examining how the investigation procedures performed by novice and expert problem solvers differ.

**Mathematical investigations**

We believe that mathematical investigations should become an core component of any mathematical course whether the participants in the course are teachers or students. While mathematical investigations are central to the activity of any research mathematician who is ultimately creative in the field of mathematics, mathematical discovery is always accompanied by enjoyment, it raises self-esteem and leads to curiosity and courage to make a new discovery. And as such it is necessary and should be in curricula for all learners.

Leikin (2014) analysed interplay between Mathematical Investigations (MIs) and Multiple Solution Tasks (MSTs). She argues that MSTs and MIs are effective instructional tools for balancing the level of mathematical challenge in the mathematics classroom and, thus, for realizing students' mathematical potential at different levels. Particular emphasis is placed on varying levels of mathematical challenge in school mathematical classroom by employing MSTs and MIs.

In this paper we provide an additional view on MIs with DGE used as a basic element of a teacher education course. In this context, MI is an integrative mathematical activity that includes experimenting, discov-

**Given:** AB – diameter in circle (O, R)
AD, BC, DC – tangent segments

**Prove**:

a. Prove the DOC = 90° in at least 2 different ways
b. Find at least 3 additional properties of the given object
c. Prove each discovered property in at least 2 different ways.

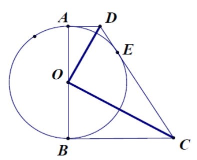



**Figure 1:** Problem 1

ering, conjecturing, verifying and proving. Problem 1 illustrates an investigation problem in our study.

## THE PROPOSED FRAMEWORK FOR THE EVALUATION OF INVESTIGATIONS

The goal of the study presented in this paper was to design the criteria for the evaluation of MIs from the point of view of investigation processes and investigation products.

We analyse MIs using the construct of ***spaces of discovered properties*** as analogous to the notions of example spaces (Watson & Mason, 2001) and solution spaces (Leikin, 2007). We distinguish between *individual spaces of discovered properties* which are collections of properties discovered by an individual based on a particular problem and *collective spaces of discovered properties* which are a combination of the properties discovered by a group of individuals.

The analysis presented in this paper is based on two case studies (CS):

Case study-1 (CS-1) is focused on the collective space of properties discovered by Prospective Mathematics Teachers (PMTs) who are considered (in this study) as non-experts in geometry problem solving. The PMTs participated in the 56 hours courses directed at the development of their problem-solving expertise and the ability to create new geometry problems through investigations in DGE (see also Leikin, in press). The sessions with PMTs were videotaped and artefacts of their works were collected. Additionally the PMTs presented their investigations to the whole group of PMTs and these presentations were also video-recorded.

Case study-2 (CS-2) is focused on Sharon's (expert's) individual space of discovered properties. The investigation in this case was performed in the form of a thought experiment. Sharon was interviewed after the thought experiment to better understand the ways in which he arrived at the discoveries.

## SPACES OF DISCOVERED PROPERTIES

In this section, Figure 2 depicts the non-expert collective space of discovered properties for Problem 1 which was achieved at the end of the 56 hours courses (CS-1). Note that the investigation activities were new for the PMTs at the beginning of the courses and the discovery skills were developed through the course. Figure 3 presents Sharon's expert space of discovered properties for Problem 1 (CS-2).

## THE FRAMEWORK FOR THE ANALYSIS OF DISCOVERED PROPERTIES

The framework for the analysis of discovered properties is based on the analysis of the expert spaces of the discovered properties in CS-1. The distinctions between the *discovered properties* was defined as a complex function of

-- *the newness of the property* discovered in the courses of investigation,

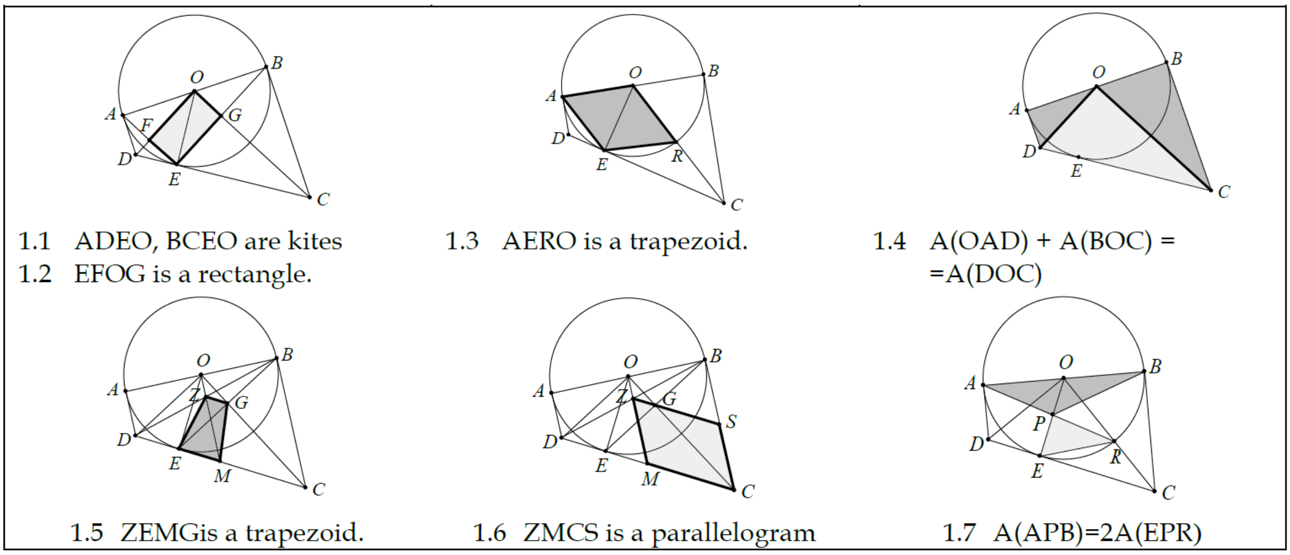
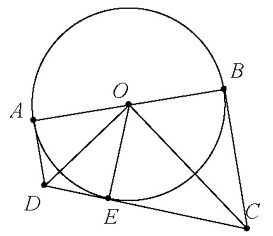
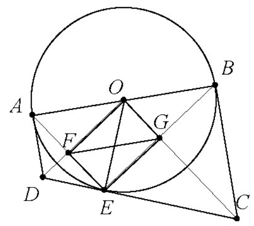


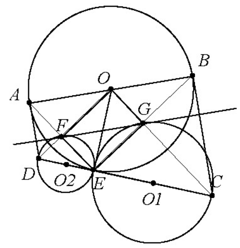

**Figure 2:** The PMTs non-expert collective space of discovered properties



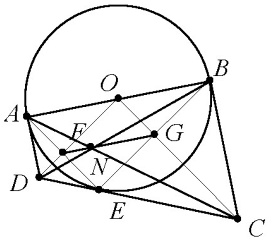

1a $OE \perp DC$
1b $BCEO$ – kite; $ADEO$ – kite
1c $ABCD$ trapezoid
1d $\sphericalangle COD = 90°$



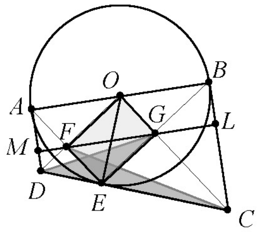

1e $EFOG$ rectangle
1f $AB \parallel FG$; $FG = \frac{1}{2}AB$
1g $BCEO$, $ADEO$ – cyclic quadrilaterals



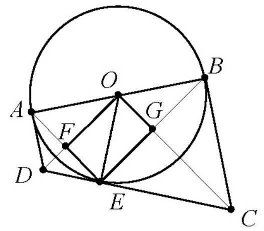

1h Circles $O1$ – diameter CE, $O2$ – diameter ED pass through $F$ and $G$ respectively
1i $FG$ is tangent to circles $O1$ and $O2$
$GO1||BC$; $FO2||AD$; $FG||AB$



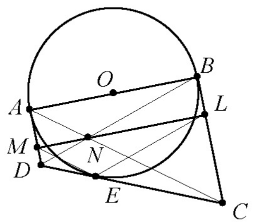

1* If $N = AC \cap BD$ then FG passes through N



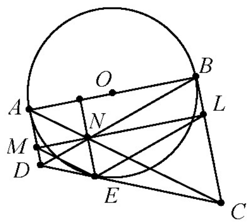

1j $A(CEF) = A(DEG) = \frac{1}{2}A(EFOG)$
1k $ABLM$ - rectangle
$A(ABE) = A(ABLM) = 2 \cdot A(EFOG)$



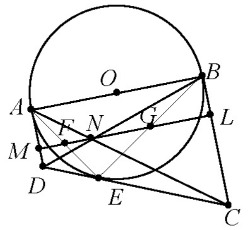

1l $\Delta ADE \sim \Delta BOE$, $\Delta AOE \sim \Delta BCE$
1m $\frac{A(BOE)}{A(ADE)} = \frac{A(BCE)}{A(AOE)}$
1n $\frac{A(ABCD)}{A(CDO)} = 2$



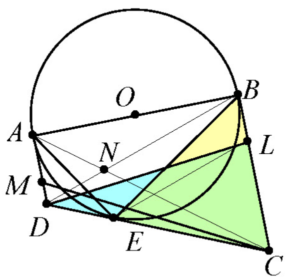

1o $N = AC \cap DB$;
$ML \perp BC \Rightarrow ABLM$ rectangle
1p $EL||BD$
1q $ME||AC$



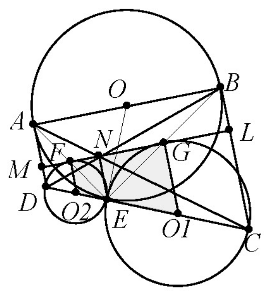

1r $NE||BC$
1s $BLEN$, $AMEN$ parallelograms
1t $JN = NE$




1u $F, G \in ML$
$NG = GL$
$MF = FN$




1v $A(CDL) = A(BCE)$
1w $A(CDM) = A(ADE)$




1x OE is tangent to circles O1 and O2
1y $FGO_1O_2 \sim ABCD$; $\frac{A(ABCD)}{A(FGO_1O_2)} = 4$
1z $FGO_1O_2 \sim ABCD \sim MNED \sim NLCE$

**Figure 3:** The expert individual space of discovered properties

-- *the complexity of the auxiliary construction* performed for the investigation and

-- *the complexity of the proof* of the new property.

**Newness of the discovered property**

The newness of the discovered property is relative to the educational history of the participant. This criterion reflects participants' critical reasoning and the ability to evaluate the property as either discovered in the process of investigation or one that was known previously. As such, we distinguish between three levels of newness of the discovered property: 0-trivial, 1-less trivial, 2-nontrivial.

For example, in Figure 3, properties 1a, b, c, d, g, are categorised as trivial, properties 1e, 1f, 1l, 1m are less trivial, whereas properties 1h, 1i, 1j, 1k, 1o. 1p, 1q, 1r, 1s, 1t, 1*, 1u, 1v, 1w, 1y, 1z are categorised as not trivial since they require complex proofs. Property 1* is a special one since through search for the proof of the property, Sharon discovered additionally several new properties

**Complexity of the auxiliary construction**

We distinguish between three levels of complexity of the auxiliary constructions (Table 1) according to two criteria:

Criterion 1: *The location of the auxiliary construction*. The property is discovered based on the auxiliary construction "within the given figure" vs. auxiliary construction "outside the given figure" (e.g., properties 1x, 1y, 1z Figure 3). Constructions "within/in the given figure" include (but are not restricted to) marking points on the border or in the interior part of the figure, construction of segments within the figure by connecting existing points or the new ones, construction of special lines (medians, bisectors, altitudes, inscribed circles) and more. Constructions "outside" the figure are more complex than those "within" the figure.

Criterion 2: *The number of the auxiliary constructions*. The property can be discovered without any auxiliary construction while the conjecture is raised on the basis of the measurement of segments, angles, areas and perimeters of the existing elements in the given figure and the comparison of these measurements. The number of the auxiliary constructions required for the discovery of a property determines the level of complexity of the property.

Through a combination of Criteria 1 and 2, we determine the complexity of the auxiliary construction as presented in Table 1.

For example, in Figure 3, the levels of complexity of the auxiliary constructions are 1 for properties 1l and 1m; 2 for properties 1e and 1f, 3 for properties 1o, 1r.

**Complexity of the proof of a discovered property**

Complexity of a proof of the discovered property is determined by the length of the logical chain of the proof and its conceptual density (Silver & Zawodjewsky 1997). Table 2 presents the levels of proof difficulty as determined in this study.

The complexity of a discovery is determined according to the combination of the complexity of auxiliary construction and the complexity of proof (see Table 3).

| The number of constructions / Location | 0 | 1 | 2 | 3 and more |
|---|---|---|---|---|
| Within | N/A | Easy (1) | Medium (2) | Difficult (3) |
| Outside | N/A | Medium (2) | Difficult (3) | Difficult (3) |

**Table 1:** The complexity of the auxiliary construction

| Proof length / Conceptual density (number of concepts and properties used during the proof) | 1–3 steps easy | 4–6 steps medium | 7 and more difficult |
|---|---|---|---|
| 1–2 concepts/properties – easy | Easy (1) | Medium (2) | Difficult (3) |
| 3–4 concepts/properties – medium | Medium (2) | Medium (2) | Difficult (3) |
| 5 and more concepts/properties – difficult | Difficult (3) | Difficult (3) | Difficult (3) |

**Table 2:** The complexity of a proof

## DISCOVERY STRATEGIES

Within the space constrains of this paper we present shortly the discovery strategies identified in this study (CS-2). We identified eight *types of discovery* as associated with the *process of discovery*:

1) Immediate discovery
2) Discovery by chance (through wondering dragging in DGE)
3) Discovery through association with another problem
4) Discovery in the search for a proof
5) Discovery based on the previous knowledge of related properties
6) Discovery in the course of proof
7) Discovery by symmetry considerations
8) Intuitive discovery

## COMPARING EXPERT AND NON-EXPERT SPACES OF DISCOVERED PROPERTIES FOR

Table 3 shows the investigation process performed by the non-experts (properties 1.1.-1.7) and the expert (properties 1.a-1z) in problem solving. It also includes the analysis of the complexity of the discovered properties and describes the type of discovery.

| | Property (in Tables 1 and 2) | Complexity of the discovery (A, B, C)<br>A-newness of discovery<br>B-complexity of auxiliary construction<br>C-complexity of proof | *Type of discovery* |
|---|---|---|---|
| CS-1: Non-Expert space of discovered properties | 1.1 | (0, 0, 1) | *By chance (in DGE)* |
| | 1.2 | (1, 2, 1) | *By chance (in DGE)* |
| | 1.3 | (1, 2, 2) | *By chance (in DGE)* |
| | 1.4 | (0, 0, 2) | *By chance (in DGE)* |
| | 1.5, 1.6, 1.7 | (2, 3, 3) | *By chance (in DGE)* |
| CS-2: Expert space of discovered properties | 1a, 1b, 1c, 1d | (0, 0, 1) | Immediate |
| | 1n | (0, 0, 2) | Immediate |
| | 1g | (0, 0, 1) | By knowledge of properties |
| | 1e | (1, 2, 1) | *Through association with other problem* |
| | 1f | (1, 2, 2) | *By knowledge of properties* |
| | 1l | (1, 1, 2) | *By knowledge of properties* |
| | 1m | (1, 1, 3) | By knowledge of properties |
| | 1o | (2, 3, 1) | Through search of proof |
| | 1r | (2, 3, 2) | By knowledge of properties |
| | 1h, 1t | (2, 3, 2) | Through association with other problem |
| | 1i, 1j, 1k | (2, 3, 3) | Through search of proof |
| | 1*, 1p, 1v, 1y | (2, 3, 3) | *By chance (in DGE)* |
| | 1s, 1u | (2, 3, 3) | Immediate |
| | 1z | (2, 3, 3) | *Intuitively* |
| | 1q, 1w | (2, 3, 3) | Symmetry considerations |

**Table 3:** Complexity and the type of discovery

## DISCUSSION

Based on the framework for the analysis of geometry investigations and the findings presented in this paper, we argue that creativity can be developed in people with different levels of problem solving expertise, but the range of the development depends on the expertise. The differences in such a range are reflected in the amount and the complexity of discoveries depicted in Figure 2 and Figure 3 and Table 3.

While surprisingly both expert and non-expert spaces of discovered properties include trivial and non-trivial discoveries, obviously, the expert space of the discovered properties is richer. Interestingly, non-experts discovered properties that were not discovered by the expert (properties1.5, 1.6, 1.7) and thus we do not consider any expert space of discovered properties as a complete one.

The major difference between expert and non-expert investigations is in the investigation strategies applied. Most of the discoveries by non-experts in problem solving are discovered "by chance" by observation of regularities which are immune to dragging. On the contrary, the discoveries by the expert were based on his mathematical knowledge and problem-solving expertise. Moreover, many of the discoveries "by chance" were based on the experts' attempt to search for a proof with the help of DGE.

Based on the findings presented in this short paper, we argue that problem solving expertise is a core element in the development of investigation skill in geometry that is a development of creativity. This development in PMTs is a rather "painful" process which requires them to overcome multiple difficulties related to geometry construction, grasping the meaning of dragging, proving and refuting multiple conjectures. Through the development of investigation skills, we also develop problem-solving expertise in PMTs and thus hope that creativity will be further developed. The process of this development is an objective of our current new study.

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