Pre-service teachers' mathematical reasoning

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The focus of this study is the mathematical reasoning of pre-service teachers. One class of pre-service teachers preparing to teach from grade 5 to 10 was organized in small groups where they worked on certain mathematical exercises. While working on these exercises the students were video and audio recorded. The dialogues of each group constitute the unit of analysis. The research framework used in this study distinguishes between imitative and creative reasoning. This distinction is based on the idea that rote learning is imitative, while the opposite kind of reasoning is creative. However, this study reveals some of the variety of the students' mathematical reasoning and indicates that if their reasoning is not imitative it is perhaps not creative either.

Keywords: Pre-service teachers, imitative reasoning, non-imitative reasoning, creative mathematical reasoning, creativity.

INTRODUCTION

To foster creativity among students in general, it would probably be useful to have teachers that could engage students in creative mathematical work. Therefore it might be of interest to study the mathematical work of pre-service teachers. Thus knowledge about pre-service teachers' mathematical reasoning would be useful. To know if some of their reasoning could be characterized as creative or not would be valuable. Such knowledge would perhaps make it possible to prepare pre-service teachers better for their future work. This is what motivates this study.

Skovsmose (2001) distinguishes between the exercise paradigm and landscapes of investigations. Within the exercise paradigm the textbook is central for classroom practice. The relevance of the exercises is not part of the mathematics lesson, and there is one and only one correct answer to each exercise. In contrast to the exercise paradigm, landscapes of investigations is an investigative approach where students are involved in processes of exploration and explanation. Skovsmose makes the point that traditional mathematics education often falls within the exercise paradigm. If students' work with mathematical exercises falls within the exercise paradigm and essentially involves copying the solutions they find in the textbook, their reasoning can hardly be characterized as creative. To copy or imitate solutions from the textbook would be what Lithner (2008) labels as imitative reasoning. He denotes the opposite kind of reasoning as creative reasoning. Haylock (1987) is concerned with creativity in school mathematics and makes the point that overcoming certain kinds of fixations is essential. He calls overcoming fixations "flexibility". Sriraman (2009) has investigated the work of research mathematicians and defines creativity to be the ability to produce novel or original work. The novelty of students would normally be at a personal level only, which is called relative creativity by Leikin and Pitta-Pantazi (2013). To copy the solutions in the textbook might be a normal procedure for students of mathematics except perhaps for students at an advanced or graduate level (Lithner, 2004). An undergraduate textbook giving several examples of solutions to the mathematical exercises in the book, is perhaps asking the students to reason imitatively (Lithner, 2008) rather than to engage them in an investigative approach (Skovsmose, 2001). However if the students cannot find a solution in the textbook to copy, the situation is different. To find a solution to an exercise would then probably require relative creativity (Leikin & Pitta-Pantazi, 2013).

One class of pre-service teachers, preparing to teach students from grade 5 to 10, participated in a study of creative mathematical reasoning. The participating students were not selected for any kind of mathematical giftedness or excellence. The class was organized in small groups and given some mathematical exercises to work on. The number of students in each group varied from two to four. The topic was basically number theory and the exercises were part of a course. The students were recorded on video and audio, and transcripts of the students' dialogues were prepared and analyzed. It was hoped that the dialogues of the students would reflect the actual mathematical reasoning of the students, and perhaps reveal more than what their written works only would have done. This led to the following research question:

Is Lithner's (2008) distinction between imitative and creative reasoning sufficient to analyze pre-service teachers' mathematical reasoning, or can some of their reasoning be neither imitative nor creative?

REVIEW OF THE LITERATURE

In a review of the literature mainly from English speaking countries, Haylock (1987) is concerned with creative thinking in school mathematics. The review indicates that both overcoming fixations and the ability for divergent production are essential components in any assessment of mathematical creativity. One aspect of creative reasoning that would be relevant for mathematical work would thus be to overcome fixations or rigidity. Haylock suggests two key aspects of fixations or rigidity in mathematical reasoning. One is called content-universe fixation where the reasoning is unnecessarily restricted to an insufficient range of elements that may be used or related to the mathematical situation. The other kind of fixation is called algorithmic fixation, where the reasoning continually adheres to an initially successful algorithm even when this becomes less than optimal. The counterpart of fixation or rigidity is called flexibility. In divergent production tests the common element is that the subject is given a mathematical situation with many responses. The creativity of the responses in such tests is conventionally assessed by evaluating them in terms of the number of responses (fluency), the number of categories of responses (flexibility) and originality (the statistical infrequency of the responses). The opposite of divergent thinking is convergent thinking where the subject is supposed to find a single solution to a given problem.

Sriraman (2009) investigated the work of five creative mathematicians. The study indicated that in general, the mathematicians' creative process followed the four stage Gestalt model of preparation-incubation-illumination-verification (Wallas, 1926) as indicated by Hadamard (1954). According to this model the creative process starts with a period of preparation where in spite of hard work over a period of time apparently no results are achieved. In a period of incubation the problem is left aside and partly forgotten. However, even though the problem is put aside for some time, it is not completely forgotten and it is thought that the mind is occupied with the problem, but in a subconscious way. Later in a moment of illumination an idea comes up which possibly solves the problem. This idea may seem to come more or less out of the blue. Therefore the moment of illumination is also seen as a result of the work of the subconscious mind. Finally, it is necessary to verify the solution. Incubation and illumination may be the work of the subconscious mind. However, both preparation and verification obviously take place in a fully conscious way.

Leikin and Pitta-Pantazi (2013) reveal that the relationship between creativity and giftedness is complex. Some researchers claim that creativity is one form of giftedness, whereas others feel that creativity is an essential part of giftedness, and still other researchers suggest that creativity and giftedness are two independent characteristics of human beings. A distinction is made between relative and absolute creativity. Creativity is relative if the creativity is at a personal level only, as opposed to absolute creativity where creativity is regarded as novel to the professional community. Students' ability to produce solutions to mathematical exercises that are new to the students only would typically be relative creativity, whereas new mathematical discoveries such as those awarded the Abel Prize would be seen as the result of absolute creativity. Researchers have different focuses on where the creativity lies. The focus is either on the creative person, the creative process, the creative product or the creative environment. Research studies that focus on the creative person deal with individuals' cognitive and personality traits. Other research studies focus on the way creative work is produced such as the four stage Gestalt model of Wallas (1926). Research studies that focus on product concentrate on ideas translated into tangible forms. Researchers that focus on environment concentrate on where the creative person acts. In educational settings this could be the educational environment where the creative activity takes place and where the creativity is studied.

Lithner (2004) gives a detailed description of how exercises in undergraduate calculus textbooks may be solved by mathematically superficial strategies. A distinction is made between intrinsic and surface mathematical properties of the components involved in the reasoning. An intrinsic mathematical property is central to the problem as opposed to a surface property which has little or no relevance to the problem (Haavold, 2013). When considering surface mathematical properties it is not necessary to understand the central mathematical ideas and analyze the consequences of their properties. Bergqvist (2007) has classified tasks and task solutions from all introductory calculus courses at four Swedish universities during the academic year of 2003/2004. The analysis shows that about 70% of the tasks do not require creative reasoning. All exams except one were possible to pass without the use of creative reasoning of any kind. In one quarter of the cases it was possible to pass exams with distinction without using creative reasoning of any kind. Lithner (2008) has introduced a research framework for creative and imitative reasoning. The basic idea behind this framework is that rote learning reasoning is imitative while the opposite type of reasoning is creative. The characteristic for imitative reasoning is that the reasoning individual is copying solutions e.g. by looking at a textbook example or remembering a textbook algorithm. The opposite kind of reasoning is called creative mathematically-founded reasoning. This kind of reasoning is characterized by novelty, plausibility and that the reasoning is mathematically founded.

The research framework introduced by Lithner based on empirical data is applied in this study. This framework identifies two types of mathematical reasoning called imitative reasoning (IR) and creative mathematically-founded reasoning (CMR). More precisely imitative reasoning is based on imitating or copying a line of reasoning laid out step by step for the student. On the other hand, the characteristics of creative mathematically-founded reasoning are (Lithner, 2008, p. 266):

1) Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.

2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

The components involved in the reasoning could be objects such as numbers, functions and matrices, (Haavold, 2013). According to Haavold the notion of plausibility in this framework is inspired by Polya (1954). Polya makes the point that there are two kinds of mathematical reasoning. There is demonstrative reasoning and there is plausible reasoning. Demonstrative reasoning is what mathematical proofs are made of, whereas the reasoning used to solve a mathematical problem or find a proof is plausible. According to Polya, the result of the mathematician's creative work is demonstrative reasoning. However, to find a solution to a mathematical problem plausible reasoning is used.

METHODOLOGY

The design of this study was to divide one class of pre-service teachers into small groups and give them some mathematical exercises. The video and audio recordings of the work of each group constitute the data of the study. The exercises were selected from the course. Thus the mathematical work of the teaching experiment would also be relevant for the students. The students were in fact preparing for their exam doing the exercises. It was not expected that the students would solve the exercises using imitative reasoning only. The reason for this was that although the students were given an idea on how to get started, they were not given a complete solution. The episode from the video recording was chosen because it indicates that when pre-service teachers' reasoning is not imitative, it is perhaps not creative either. Similar mathematical reasoning was found in many of the groups involved in the study. However, one example was chosen for this paper to show what was found.

The term commognition has been coined by Sfard (2008) meaning a combination of communication and cognition. This means that interpersonal communication and individual thinking are two facets of the same phenomenon. Thinking is defined by Sfard as the individualized version of interpersonal communication. In this paper we view reasoning as a form of thinking. Thus we study the mathematical reasoning of each individual by studying the interpersonal communication of each group. Therefore the unit of analysis is the dialogue of each group. The idea behind this approach is that the dialogue of each group would perhaps reveal more mathematical reasoning than the written works only would have done.

The analysis was based on the research framework of Lithner (2008), which makes the distinction between imitative and creative reasoning. However, the analysis of the dialogues indicated that some of the students' reasoning was not compatible with Lithner's distinction. The analysis indicated that if the students' reasoning was not imitative it was perhaps not creative either. Therefore it became interesting to analyze the students' reasoning if it was neither imitative nor creative and find a way to characterize it. Hence, in order to analyze pre-service teachers' mathematical reasoning a new distinction of reasoning was introduced. In addition, new categories were introduced.

ANALYSIS

One of the problems the students worked on was the sequence (a_n) starting with the terms 0,4,10,18,28,40... The students were asked to find an expression for a_n. Let us start by looking at the dialogue of one of the groups with three students. The instructor made the students familiar with the idea that they could write down the differences between consecutive terms of the sequence to get a set of equations that could be added. The video shows that when the dialogue begins the students have written down the following equations:

$$\begin{gathered} a_2 - a_1 = 4 = 2 \times 2 \\ a_3 - a_2 = 6 = 2 \times 6 \\ a_4 - a_3 = 8 = 2 \times 4 \\ \dots \\ a_n - a_{n-1} = 2n \end{gathered}$$

Adding these equations, the students arrived at the equation:

$$a_n - a_1 = 2 \times 2 + 2 \times 3 + 2 \times 4 + \dots + 2n$$

Thus the students did what one would expect if they were to follow the line of reasoning given to them by the instructor. Therefore their reasoning was probably imitative so far (Lithner, 2008). We enter the dialogue with the following episode from the video. The numbered transcription of the episode is given with some comments.

Episode:

1. Katherine: And then we can write 2 outside,
2. Elizabeth: And a_1 is zero, we don't need it, we can simply skip it.

Following this dialogue the students have written down the equation:

$$a_n = 2(2 + 3 + 4 + \dots + n)$$

In lines 1 and 2 Katherine and Elizabeth use formulations such as "we can write 2 outside" or "we can simply skip it" as opposed to formulations such as "what do we have to do here?" or "what are we supposed to do here?" This could mean that the students make their own choices about what to do, rather than asking themselves what they are supposed to do which would be characteristic for imitative reasoning. If the students make their own choices then their reasoning would not be imitative. The dialogue of the episode continues as follows:

3. Jennifer: We are now looking for the triangular numbers.
4. Elizabeth: Hm.
5. Katherine: No, plus n, the sum of the n first positive integers, so it is really the triangular numbers we are looking for, but can we...?
6. Jennifer: We are missing 1.
7. Katherine: We are missing 1, yes if we add...
8. Elizabeth: Add 1 to each side.
9. Katherine: We have to add 2...2...2 times 1... to both sides, because we have the number 2...Yes, if we try that, add 2 times 1, then you get a_n plus 2 times 1 equals 2, and then we get 1 plus 2 plus 3 plus 4...plus n.
10. Elizabeth: Yes, we do.
11. Katherine: Yes.
12. Jennifer: Can we just add like that?
13. Katherine: Yes we may add to both sides.
14. Jennifer: Yes, and then we just have to...

Finally, the video recording shows that the students arrive at the following equation:

$$a_n + 2 \times 1 = 2(1 + 2 + 3 + \dots + n)$$

As the students are familiar with triangular numbers, the exercise is now resolved.

Jennifer continues the dialogue in line 3 by making the point that they are looking for the triangular numbers. After having agreed that they should look for the triangular numbers, Jennifer begins a line of reasoning in line 6 by observing that they are missing the number 1. She is obviously referring to the fact that the sum within the brackets should have started with the number 1. Katherine suggests in line 7 that they should add something. This is followed up in line 8 by Elizabeth saying that they should add 1 to each side of the equation. However, in line 9 Katherine introduces a different idea saying that they should add 2 times 1 to each side of the equation. Thus the students change their point of view while reasoning, which would not be typical for imitative reasoning. Instead, changing point of view characterizes flexible reasoning (Haylock, 1997). The formulation "Yes, if we try that" used by Katherine in line 9 further indicates that the students are trying out their own ideas rather than following step by step a given line of reasoning, and thus that their reasoning is not imitative (Lithner, 2008). Hence, their reasoning should not be characterized as imitative but, rather, as flexible.

When Katherine introduces her idea in line 9 she shows little uncertainty. In fact she says: "Yes, if we try that, add 2 times 1, then (...)" thus perhaps showing some confidence. Elizabeth easily accepts the idea of Katherine in line 10. Only Jennifer hesitates a little in line 12 (Birkeland, 2013) but accepts the idea in line 14. Thus the idea they use is introduced rather smoothly. This would hardly be the case if the idea was new to the group. Therefore nothing indicates that the idea has novelty to the group. If the students' reasoning is based on a relational understanding (Skemp, 1978) of the components involved, then it is reasonable to assume that it is mathematically founded (Lithner, 2008). This would indicate that the reasoning of the group is mathematically founded, has flexibility (Haylock, 1987) but no novelty or originality.

DISCUSSION

The first part of the students' reasoning was to write down the differences between consecutive terms of the given sequence to get a set of equations that could be added. The students were made familiar with this idea by the instructor. Therefore the first part of their reasoning was probably imitative (Lithner, 2008).

The second part of the students' reasoning was not laid out step by step for them. The instructor did not give them any hints. The students tried out certain ideas they had and finally chose their own line of reasoning. If their reasoning was based on their own choices then it should not be characterized as imitative. Assuming that their reasoning was based on relational understanding (Skemp, 1978) it was probably both plausible and mathematically founded. Further, nothing indicated that their reasoning had novelty. However, their reasoning was found to have flexibility. Probably the students were working flexibly with familiar lines of reasoning.

One may argue that flexible reasoning is part of creative reasoning. However, if creativity is the ability to produce novel or original work (Sriraman, 2009), then flexibility alone is not sufficient for the reasoning to be creative. Therefore the students' reasoning should perhaps not be characterized as novel or creative. Consequently the first part of the students' reasoning could be said to be imitative, however the last part of their reasoning would be neither imitative nor creative as defined by Lithner (2008).

The episode chosen for this paper was quite typical for the mathematical reasoning of most of the groups involved in the study. It appears that they all followed the line of reasoning given to them by the instructor on how to get started. Therefore, for all groups the first part should be characterized as imitative reasoning. However, having started each group followed their own line of mathematical reasoning, varying slightly from one group to another. The reasoning of all the groups except for one should not be characterized as imitative. Their reasoning had flexibility but nothing indicated novelty. Only one group continued with imitative reasoning. The video shows that this group found an earlier example to compare with. The group appears to have found similarities between the two examples. However, the similarities should be characterized as surface similarities. Therefore their mathematical reasoning should be characterized as superficial reasoning.

Hence, according to this study, an analysis of the mathematical reasoning of pre-service teachers should be based on a distinction between imitative reasoning (IR) and non-imitative reasoning (NIR). Imitative reasoning is defined by Lithner (2008) as the kind of reasoning where each element of the reasoning is laid out step by step for the reasoning subject. Non-imitative reasoning is introduced in this paper as the kind of mathematical reasoning which is not imitative.

It is quite possible that the imitative reasoning of the first part was based on a relational understanding (Skemp, 1978). However, if the reasoning is based on surface property considerations (Lithner, 2008) only, it should be characterized as superficial reasoning (SR).

This study indicates that imitative reasoning may be mathematically founded or plausible (IMR). However, imitative reasoning may also be based on surface property considerations only (SR). Hence, imitative mathematical reasoning may or may not be mathematically founded or plausible.

Non-imitative reasoning (NIR) can hardly be based on surface property considerations only because that kind of reasoning would always involve some elements of copying. If plausible reasoning (Polya, 1954) is about supporting conjectures, then plausible reasoning would not be imitative reasoning. Therefore plausible reasoning could be an example of non-imitative reasoning (NIR). The kind of non-imitative reasoning found in this study is characterized by the fact that it is plausible and mathematically founded (Lithner, 2008), has flexibility (Haylock, 1987) but no novelty. This kind of non-imitative reasoning is called flexible mathematically founded reasoning (FMR) in this study.

FINAL REMARKS

This study shows that to analyze the mathematical reasoning of pre-service teachers, it might be useful to base the analysis on a distinction between imitative and non-imitative mathematical reasoning (NIR) rather than imitative and creative reasoning (Lithner, 2008). Imitative reasoning includes both superficial reasoning and imitative mathematically-founded reasoning (IMR). Non-imitative reasoning (NIR) includes both flexible reasoning (FMR) and creative mathematically-founded reasoning (CMR). Thus non-imitative reasoning may or may not involve novelty. If novelty is essential to the concept of creativity as defined by Sriraman (2009) then non-imitative reasoning would include mathematical reasoning that is neither imitative nor creative. Hence, to analyze pre-service teachers' mathematical reasoning, Lithner's (2008) distinction between imitative and creative reasoning might not be sufficient. Some of the students' reasoning might be neither imitative nor creative.

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