Modelling: From theory to practice

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University of Copenhagen, Department of Science Education, Copenhagen, Denmark, britta.jessen@ind.ku.dk

Mathematical Competence Theory and the Anthropological Theory of the Didactic each offer different frameworks for the analysis and design of "modelling" as a central component of mathematics teaching. Based on two comparable cases from each research programme, we investigate how these differences appear in concrete design work, and what their practical consequences may be.

Keywords: Mathematical modelling, mathematical competence theory, anthropological theory of didactics, bidisciplinarity.

WHAT IS MODELLING AND DOES IT MATTER?

The fact that primary and secondary school students all over the world study a subject called "mathematics", with relatively similar contents and methods, is intrinsically linked to certain assumptions about the *relevance* if not *necessity* of this subject for every citizen in modern society. The formulation of these assumptions change over time and they are of course the object of constant debates, but an invariant common contention seems to be the *utility* of what is taught *in the actual or future lives of students*, or at least *its roles outside of school mathematics*.

One formulation, which has gained importance in the mathematics education community over the past 30 years, is based on the notion of *mathematical modelling*, defined roughly as "translations between reality and mathematics" (Blum & Borromeo-Ferri, 2009, p. 45). More complex descriptions of the modelling process, usually in the form of a *modelling cycle* (e.g., ibid, p. 46) have become commonly known and used in research into the ways in which these translations can appear in the school subject. It is a common assumption among researchers within this line of research that students' experience with all steps of the modelling cycle is essential to the justification of school mathematics in society (ibid, p. 46). In particular, Niss and colleagues (2002) proposed to consider *modelling competence* – the students' capacity to carry out mathematical modelling – as one of eight universal competence goals for the teaching of mathematics, linked to other goals equally defined in terms of competences. Their mathematical competence theory (MCT) thus integrates and develops earlier work on mathematical modelling, as an educational activity and goal, in a comprehensive framework for the analysis and design of school mathematics in a broad sense.

Another perspective on modelling stems from inquiries into the nature of mathematics as a school subject: how it is related to the science called mathematics, and more generally to "mathematical practices" appearing in society outside school? The anthropological theory of the didactical (ATD) emerged from the notion of *didactic transposition* (Chevallard, 1985) according to which school mathematics is a cultural set of practices and knowings which are inseparable from the institutions (schools) in which they are taught and learnt. In this theoretical framework, "mathematics" and "reality" are not *a priori* defined or distinguished; all human activity and knowledge is described in terms of *praxeologies* (Chevallard, 1999). Modelling has a wider meaning in this framework, as the elaboration of praxeologies in one domain in view of studying one or more *questions* in another domain. The school institution refers to this as "intra-mathematical modelling" when both domains are recognized as belonging to school mathematics, e.g. if school algebraic praxeologies are elaborated to study a question from school geometry (García, Gascón, Ruíz-Higueras, & Bosch, 2006). In ATD, modelling thus serves to create meaningful links between otherwise separate praxeological domains, whether or not these are considered as belonging to school mathematics or not.

The two theories are related to specific *design formats* which are often used for the design of teaching that involves modelling (cf. Miyakawa & Winsløw, 2009, for the distinction of theory and design format). In

MCT, it is *problem oriented project work* (PPW), in which students are to develop their competences while experiencing some or all steps in a modelling cycle (Blomhøj & Kjeldsen, 2006). In ATD, it is *study and research paths* (SRP), departing from one or more questions; the further development is sometimes represented with a tree like "map" of derived questions and praxeologies which students did construct while working with the questions (Barquero, Bosch, & Gascón, 2008; Jessen, 2014).

At this point, we have only hinted at some of the differences between two perspectives on modelling. The research question which interests us is a theoretical, but also quite practical, one: *What differences, if any, does it make for the design of new teaching practices, whether the theoretical control apparatus comes from MCT or ATD? In particular, are there differences between uses of the design formats PPW and SRP which can be related directly to the different theoretical notions of modelling found within MCT and ATD?*

We shall take an inductive approach to this question: we first present two cases of design of modelling activities for students in Danish upper secondary school, constructed from each of the two perspectives but otherwise similar in contents. Then we analyse the differences in view of providing tentative answers to the research question. To prepare that analysis, the presentations of the cases focus on the following variable features of modelling activities:

(V1) *Practical meaning of "modelling"* in the activity, as described by the authors

(V2) *Goals for the activity* (e.g. for student learning) *and their assessment*

(V3) *Organisation in time* of the activity

(V4) *Distribution of roles among students and teacher(s)*, in particular the way in which student autonomy is controlled (limited, furthered, differentiated, etc.)

(V5) *Adaptation to local conditions and constraints* (features of the activity which result from these adaptations, including choices made for (2)–(4)).

The case presentations given below are based on more extensive studies (Jessen, 2014; Blomhøj & Kjeldsen, 2006). V5 is further treated in these papers.

CASE 1: A STUDY AND RESEARCH PATH

The first case we will present comes from an experiment with study and research paths (SRP) in the context of Danish high school students' study line reports written in the second year of high school (a study line report is a bidisciplinary report students write in the second year as a preparation for the bidisciplinary "study line project", which is a high stake final exam in upper secondary school in Denmark, cf. Jessen, 2014, p. 2). The reports are about 15 pages long accounts of an autonomous work done by one or two students, within 6 weeks and with very limited access to help from teachers (V3). The study line of the students determines what disciplines are to be involved in the report. Before the 6 week period, the teachers formulate a set of questions for the students to work on (V4). For the study line of the experiment, the theme should combine the disciplines *mathematics* and *biology* with equal weight. These circumstances are constraints (V5) which affect the concrete design and in particular the variables V1-V4.

The aim for the study and research path (V2) was for students to develop new praxeologies in the domains of nervous physiology and differential equations by working with a certain generating question, given by the teacher together with some supplementary questions to ensure the involvement of both disciplines:

Q_0: How can a patient be relieved from his pain by painkillers like paracetamol – how does deposit medication work and how can we model this mathematically? Q_1: Explain the biological functioning and consequences of taking paracetamol orally versus taking it intravenously. Q_2: Create a mathematical model using differential equations that illustrates the two processes and solve the equations in the general case. Q_3: Give a concrete example, where the patient is relieved from pain and estimate from your own model how often paracetamol has to be dosed – which parameters (absorption, elimination factor, bioavailability) are important to be aware of? $Q_{3,1}$: Does it make any difference whether the dose is given oral or intravenously? Use your models while giving your answer. (*Translation of the original questions in Danish*)

Notice that in ATD, *modelling* means the elaboration of praxeologies in the two domains – done by students in view of answering the generating question (V1). However, in the assignment, "mathematical model" refers to a more restricted sense, which is closer to the notion of model found in MCT and, at least in outline, is the one found in official documents and text books for Danish high school.

The above assignment is based on a generating question Q_0 which the students can immediately understand, but not answer. In general, a generating question should be so strong, that it is necessary for the student to formulate derived questions Q_i, each representing a branch of inquiry, in order to answer Q_0. The answers R_i to the derived questions adds up to a final answer of Q_0 (Chevallard, 2012, p. 6). At the same time it is purposed that the generating question must be "alive" in the sense that students should be able to relate the question to things they perceive as interesting and real. These aims were deliberately pursued by the teaching design, knowing that several students in the class wanted to study medicine or similar after graduating.

The derived questions formulated by the teachers serve as supports for the students' study process (V4). In general, it is crucial that the students are not left with "big questions" that are unrelated to their praxeological equipment (Chevallard, 2012, p. 11); the relation to praxeologies from specific disciplines must be ensured. This was even more crucial in our context since no teaching activity was accompanying the SRP work of the students. Some students met after classes and formed their own working groups discussing strategies for answering the questions. The teachers were allowed to answer questions during the six weeks, and in order to keep track of the students working progress, the exchange of questions and answers was only permitted in writing (V4). For the same reason, students were asked to provide their immediate answer to the generating question Q_0 when it was handed out (without the derived questions Q_1–$Q_{3,1}$). After that, the entire assignment was given to them. After two weeks, and again two weeks later, the students were asked to answer the following questions in writing:

> What is your answer to the generating question right now? What have you done to answer the question? What are you planning to do next in order to come up with more fulfilled answers?

We cannot go into all the details of the analysis of this SRP, neither before nor after the experience (the latter being analyses of students' reports, cf. Jessen, 2014). However we notice that to construct the "mathematical model" asked for in Q_0, students must somehow examine the relationship between the amount of drug given, and the distribution of the drug in the body. How the pain is cured and how the drug is eliminated must be answered by praxeologies from the domain of physiology. The latter leads to consider that the pain is relieved in relation to how often the drug is given, the size of the body and the pain perception. Thus, the progressive development of a mathematical praxeology (involving tasks, which can be solved using techniques available to the students, e.g., CAS-based solution methods for differential equations) is closely articulated to the development of a biological praxeology. The modelling process in terms of ATD is not a question of following certain steps, it is an individual process where the students uses their praxeological equipment to investigate domains, form new questions, answer them with existing or new praxeologies unfolding the disciplinary organisation at stake (V1).

The intermediate answers from the students showed a variety in their working progress, which reflected different praxeological equipment among the students. Some students responded the first time, that they needed to know the half-life of the painkillers this indicates, that the students suspect, that there is a time dependence in the model, and that the model includes an exponential function. During mathematic classes they have seen that exponential equations are part of the solution to many differential equations. This implied, that they were trying to relate the generating question to the newly developed praxeologies in mathematics. Also they studied relevant medias since they were able to formulate relevant search topics. The students formulated derived questions such as the following: Q_1: How is pain registered? Q_2: How does paracetamol relieve pain (pharmaco dynamic)? Q_4: How can the dosing be modelled mathematically based on the biological knowledge? (Jessen, 2014, p. 11). The entire analysis shows that the students are constantly narrowing down their inquiry, by alternately studying the questions through physiology and differential equations.

The teacher involved was sure that for some students the generating question would not suffice to develop a reasonable model. It was for this reason that a part of the derived questions was handed out before the independent work of the students. Some of the students would otherwise not have been able to develop new praxeologies in the intended domains. With these more precise questions, they were able to identify relevant media (web-pages etc.) and although some of them uncritically adopted models constructed by others, they were all able to make use of them for simple calculations (e.g. of the amount of drug in the vein of a patient) (Jessen, 2014). Thus their modelling of the intended praxeologies was not as richly developed as in the previous case.

CASE 2: A PROBLEM ORIENTED PROJECT ON ASTHMA MEDICINE

Our second case presents a PPW on mathematical modelling related to the administration of asthma medicine. In MCT modelling competency is defined (V1) as

> A person's insightful readiness to autonomously carry through all aspects of a mathematical modelling process in a certain context and to reflect on the modelling process and the use of the model (Blomhøj & Jensen, 2003, p. 127).

The key words are *autonomy*, *modelling process*, *reflections*. PPW is particularly well suited to foster students' autonomous participation in the modelling process (Blomhøj & Kjeldsen, 2011). The goal for students' learning (V2) in MCT is to develop and/or enhance their competency.

A mathematical modelling process can be depicted analytically as a cycle consisting of six sub-processes (ibid., p. 387). Concrete modelling activities, like the case presented here, may have a variety of more specific goals for students' learning (V2) in order to adapt to local conditions and constraints (V5).

In a PPW, students work in teams with a problem for a longer period of time to produce a product representing the team's solution (V2+V3). The central idea is that the problem should function as the "guiding star" for all decisions made by the students in the sense, that all decisions should be justified by their contribution to the solution of the problem. This provides the students with (parts of) the responsibility of directing the project. It is crucial that the students are involved in (most of) the decisions taken in the modelling process and become involved in reflections upon the different steps in the modelling cycle. PPW opens for a distribution of roles among students and teacher(s) that makes it possible to direct the students' autonomy e.g. through specific requirements to the product of the project (V2+V4). PPW has the potential to foster in the students all the key elements in developing modelling competency which makes this format an obvious pedagogical choice in MCT.

The asthma project was designed by two teachers for first year students in mathematics in high school. The students were to: 1) work more independently than usually over a longer period (ten mathematics lessons of 1.5 hour each and a similar amount of homework); 2) develop new theory by working with modelling within a subject area (exponential growth) they hadn't worked with before; 3) work with a more complex and authentic problem for which they did not possess a standard method or technique such that the modelling, the mathematization, the interpretation of the results and the reflections about the modelling process and the use of the model became part of the project; 4) analyse a set of data in order to build a mathematical model; 5) use familiar concepts such as graphs and equations for functions in a concrete context; 6) develop their mathematical communication skills; 7) use ICT throughout the project. (V2)

These aims were achieved through a strict organization of time (V3) and a setup that allowed for and supported the students' autonomy (V4). The teachers divided the project into four phases (Figure 1). The teachers controlled phase 1–3, and the students controlled phase 4. The aim of the first three phases was to prepare the students for their independent work in phase 4. In phase 4, the teachers took on the role of

1. (1.5 module) Presentation of the problem (see Fig 2). Excel course, IT for project management, social contract. Crash course in the modeling process.
2. (1 module) Problem formulation Phase and decision.
3. (3 modules) Work with a set of four exercises related to the project in phase 4.
4. (4 modules) Working with the actual project.

Figure 1: The four phases of the design. 1 module corresponds to a 90-minute lesson (Blomhøj & Kjeldsen, 2006)

consultants (V4) that the students could ask for advice on specific problems.

In phase 1, the teachers introduced the students to a cyclic representation of the modelling process. The teachers used the process to inform the students about the various elements in mathematical modelling within MCT, and they asked the students to be aware of and to explain where in the modelling process they were at any given stage in their work. Hereby, the teachers made sure that the students became engaged in posing the modelling problem, constructing the model, solving the mathematical system and suggesting solutions to the problem (V2, V3 & V4/V5). In phase 2, the teachers trained the students' competence in posing mathematical modelling problems through discussions in the class room guided by the teachers (V1/V5).

The problem from phase 2 was given to all students with some data (Figure 2). The exercises in phase 3 were not included in the students' independent work. They served as inspiration and illustrated the level of mathematics, communication and documentation expected in phase 4. The product of the project work was a report, handed in by each group after phase 4 (V4/V5). The teachers formulated a set of requirements for the report to direct the students' autonomy in phase 4.

ANALYTIC COMPARISON OF THE CASES

A synthetic presentation and comparison of the two cases can be achieved using the five variables identified in the first section and indicated as they are "filled" by the above presentations (see Figure 3).

Despite evident similarities between Q0 in case 1, and the problem (Figure 2) underlying case 2, the contexts and constraints are quite different: in case 1, the students must work independently most of the time, and have to combine the two major disciplines (mathematics and biology) of their study line; while in case 2, the work is done as part of the regular teaching of one discipline (mathematics). In the Danish regulations for high school, *mathematical modelling* more or less understood as in MCT forms part of the competency goals for mathematics as a discipline (Niss et al., 2002); the bidisciplinarity required for study line projects is a more diffuse and general principle for the study line projects while in the case of mathematics, it is also often associated with the same notion of mathematical modelling. Despite these differences coming from the contexts, some more principal differences arising from the theoretical background of the two cases can also be identified.

Differences coming from the design formats

The variables V2–V4 are clearly shaped by the design formats. In PPW, everything begins with a *problem* defined in more or less commonly accessible terms, which should then be sharpened and translated into mathematical terms, in order to allow for applications of relevant mathematical machinery, either known in advance or developed through the project work. The PPW in itself does not suggest explicit structuring

Asthma patients' problems with exhalations may be alleviated medically by increasing the concentration of the drug theophylline in the blood of the patient. If the concentration of theophylline is below 5 mg / L it has hardly any positive effect. If the concentration is above 20 mg / L it has toxic effects. The problem is to administer medication such that the concentration of theophylline stays within a certain range in which the medicine is effective, say a concentration between 5 and 15 mg / L. The substance is excreted from the body through the kidneys; hence the amount of the substance in the blood will drop with time, which means that the patient will suffer, unless you "fill up" periodically. At the hospital where the patient is hospitalized you try, for the sake of the daily organization of work and to reduce the risk of errors, to schedule the medication so the patient is supplemented with an equal dose, D mg, with equal intervals of time, T hours. A doctor is examining how to choose D and T so that the concentration of theophylline remains within the range of 5- 15mg / L. On a patient, he has measured how the concentration of the substance decreases with time following an injection of 60 mg of the drug.

h	0	2	4	6	8	10	12	14	16	18
mg/L	10.0	7	5.0	3.5	2.5	1.9	1.3	0.9	0.6	0.5

Figure 2: The problem and the data (Blomhøj & Kjeldsen, 2006)

and requirements regarding the students' work besides the fact that the problem should be formulated in such a way that it can function as a guide. The formulation of the problem is part of PPW. Hence, it is left to the teacher to set the "scene" for the students' work within the given context, depending on his or her learning goals. A SRP begins with a *question* which, like the problem in PPW, is too open to allow for immediate, complete answers. In order to proceed, students need to work with subquestions arising from supplementary assumptions, suggested by the original question or by some first, intuitive hypotheses or answers. Both design formats leave the teachers with tools for *directing* the students work: in PPW, the structuring can allow students more or less autonomy depending on how the teacher choose to structure the work, and through specific requirements for the product – in this case a report - the students should deliver (Blomhøj & Kjeldsen, 2006, p. 168), while in SRP, the teacher may supply students with some derived questions to start with, some specific media to study, etc. (Winsløw, Matheron, & Mercier, 2013, pp. 271–282). In both cases, an initial planning may be adjusted to the work of the students, with the tree diagram of the SRP and the learning goals and (parts of) the modelling cycle as the main tools for control of these adjustments of the initial design.

Differences coming from the theories

MCT assumes a clear and evident boundary between mathematical and extra-mathematical phenomena, which implies (through the processes of problem formulation, demarcation of a domain of inquiry, and systematization), the construction of an object to be modelled. This object is then translated into a mathematical representation, which in daily work is also often referred to as *the model*. The preparation and conduct of the PPW can thus be structured according to the movements from the problem to the mathematical domain, and back – with an explicit notion of being "outside" and "inside" mathematics. ATD, on the other hand, is based on a general theory of human practice and knowledge, in which the organisation of praxeologies into disciplines is merely an institutional construction; the boundaries of what is called "mathematical" are not universal but contingent.

In MCT, it is part and parcel of mathematics teaching to develop students' explicit knowledge and experience of how mathematics (as a universal entity) *applies* to problems outside of that domain. In ATD, praxeologies are simply answers to questions which have been developed sufficiently to allow students to find culturally established answers through media or through research based on praxeologies they are familiar with; the main feature of modelling to experience is the development of praxeologies through this dynamics of study and research, independently of institutional classifications into disciplines of the praxeologies.

These theoretical differences have an impact on practice. In PPW based on MCT the disciplinary contents are in principle subordinate to the problem. The chief purpose is to reach a satisfactory solution to the problem through realisation of (specific features of) the

	Case 1: study and research path	Case 2: problem oriented project work
V1	Starting from a big question Q_0, develop derived questions and praxeologies which can answer these and in the end, at least partially, Q_0. Didactic theory is not taught.	Starting from a problem *P outside* mathematics, reformulate it as a mathematical problem, treat this, and evaluate solution relative to *P*. The modelling process is explicitly taught.
V2	Develop specific bidisciplinary praxeologies as answers to Q_0.	Modelling competency through phases of modelling of data and problem.
V3	Six weeks of independent work (individually or in pairs) based on Q_0 and some derived questions, with encouragement to search for media.	Project team work for ten 90-minute modules and similar amount of homework, structured by phases of modelling as shown in Figure 1.
V4	Teachers deliver Q_0 and some derived questions; students do study and research on these, with very limited access to teachers, to prepare their study line reports.	Teachers structure the work of teams according to the phases, with most autonomy required in the last phase (once mathematical formulation and expectations are established).
V5	Regulations of study line reports (combining math and biology)	Aims for regular mathematics lessons, which include mathematical modelling.

Figure 3: Syntheses of didactic variables as set by the two cases

mathematical modelling process including choosing disciplinary theory relevant for solving the problem. The mathematical content brought into play will depend on the mathematical competencies and knowledge of the modellers and their abilities to expand these. In the ATD approach to modelling, a more or less strongly directed SRP can be planned based on a priori analysis of its potential to realise certain institutionally defined disciplinary praxeologies as answers to the initial question. This could make the ATD approach to modelling implemented through SRP more attractive in institutional contexts where the disciplinary focus is strongly constrained. On the other hand, as we have argued and illustrated, the choice of design has theoretically determined consequences for the kinds and qualities of mathematical modelling activity, which students get to engage in. For further investigation one might analyse the activity students carry out in the classroom (how are answered produced and validated, etc.) and to what extend are the students able to solve other modelling problems in the future.

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