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This study analyses the results obtained from comparing the paths shown by expert mathematicians on the one hand and mathematics teachers on the other, when addressing a hypothetical problem that requires the construction of a mathematical model. The research was conducted with a qualitative approach, applying a case study which involved a group of mathematics teachers and three experts from different mathematical areas. The results show that the process of constructing a mathematical model differs between these two groups mainly by the type of cognitive processes developed. It was observed that the modelling routes depended on the graphical representations used by the individuals to address the problem. The mental model related to the situation plays an important role in the externalisations of the participants.

Keywords: Modelling, representations, mathematics teachers, paths of modelling.

INTRODUCTION

In the context of mathematical modelling (MM), there are two approaches: applications and modelling for mathematics learning, and learning mathematics to develop skills in the construction of mathematical models. The first considers the use of modelling activities as a vehicle for the construction of mathematical concepts; the second involves the application of mathematics to build models (Niss, Blum, & Galbraith, 2007). These two approaches may occur simultaneously or in isolation, and involve a difference in the emphasis put in the development of teaching strategies.

Speaking of teaching activities necessarily brings to mind the teacher-student-task triad, where it is unarguable that the mathematics teacher has great importance, both fulfilling the role of modeller and participating in the preparation and/or selection of tasks in order to support the students to understand the mathematical concepts involved in the activity. Hence, this study focuses on the observation of modelling strategies developed by the teachers when they model a hypothetical situation. Our initial hypothesis is that mathematicians and mathematics teachers develop different ways of modelling. Therefore, we discuss some modelling strategies implemented by our two study groups: mathematicians responsible for the training of mathematics teachers, and in-service mathematics teachers receiving training.

In the literature related to strategies activated by expert and novice modellers when solving a problem, it has been observed that novice modellers do not take enough time to understand the situation in question, thus experiencing difficulties in selecting the relevant information and using it in the construction of a suitable mathematical model (Crouch & Haines, 2004; Crouch & Haines, 2007, Bransford et al., 2000). Therefore, it becomes clear that some of the obstacles in the construction and interpretation of mathematical models are related to difficulties in accessing appropriate mathematical concepts and procedures in order to find a solution.

The modelling process of novices shows a trend toward the use of linear rather than cyclical modelling strategies, while the validation of the obtained models is not a relevant element to them. As a result, they have difficulties recognising the model associated to a given situation and fail to relate their results to the situation that gives rise to it. On the other hand, expert modellers access relevant knowledge more efficiently. Moreover, some authors estimate that it is likely that many years of practical experience are needed to become to be an expert modeller (Haines & Crouch, 2010; Crouch & Haines, 2007). However, there is little research that focuses on the processes and strategies that novices and experts develop when involved in modelling activities. Therefore, we aim to contribute to previous research by showing some aspects of the construction of a model that both groups (experts and novices) use. We identify some elements that permit them to move more effectively through the various stages or nodes of modelling.

In the context of modelling as a vehicle for learning mathematics, a general question arises: Are teachers able to convey to their students the strategies that they themselves develop to model a situation? This is under the assumption that some of the steps that comprise a modelling path take place in the mind. Therefore, if the teachers are not aware of what they are doing mentally when modelling a situation, they will be incapable of conveying their thought process to the students. Therefore, the objective of this research is to identify some of the processes that mathematicians and mathematics teachers in training develop in order to help them to reflect on their own process.

Our research was conducted with two groups. One was a group of expert mathematicians who conduct research in pure mathematics and also participate in in-service teacher training. They are not necessarily involved in modelling activities as part of their research. The other group consisted of mathematics teachers who work at secondary or college level, they received a solid background in mathematics as part of their undergraduate training, but have little experience in the development of modelling tasks in the classroom. The intention of the study is not to make comparisons of the modelling competence between the two groups; rather, we aimed to identify the tools and representations that the members of each group use. This knowledge may serve as a guide to identifying those aspects that would be appropriate to develop both in in-service teachers and in pedagogy students. We argue that one way to address the problems related to the transition of students through the modelling cycle is to identify the strategies used by mathematicians (experts) and try to have students (novices) develop the same tools and skills. This becomes especially relevant since so many mathematicians are involved in training future mathematics teachers. It is clear that knowledge generated in the interaction between these two groups is what is ultimately passed on to students; so it is desirable to identify and analyse the strategies developed in each of our groups. Therefore, the question that drives our research is: What differences are there between the strategies mobilized by mathematicians to model a situation and those mobilized by mathematics teachers?

We present the results of analysing the modelling paths (Borromeo-Ferri, 2007) developed by the participant groups, taking into account the strategies and mathematical tools they rely on.

CONCEPTUAL FRAMEWORK

The main challenges that arise when modelling a situation are related to the transition from reality to the world of mathematics and to the reinterpretation of a Mathematical Model (MM) in terms of reality. For an expert, reinterpretation may be trivial, but not so for novices (Crouch & Haines, 2004). In connection with the transition from reality to the mathematical model, Borromeo-Ferri (2006) identified some stages in a modelling process, focusing mainly on the analysis of the cognitive processes involved, where she distinguishes four phases: Real Situation (RS), Mental Representation of the Situation (MRS), Real Model (RM) and Mathematical Model (MM).

The Real Situation (RS) represents the situation described in the problem; it can be an image or a text. When going from the RS to the Mental Representation of the Situation (MRS) the individual somehow understands the problem more, and she or he mentally reconstructs the situation; even if they do not fully understand the problem, they can start working on it.

MRS may be different in each individual, depending on his or her mathematical thinking style[1]. It can be visual in relation to the experience, or attention can focus on numerical data and relationships given in the problem, depending on the associations that the individual chooses while understanding the task. In addition, the MRS can differ depending on the role

1 Mathematical thinking style is defined as: "The way through which an individual prefers to present, understand and think mathematical facts and make connections between certain internal imaginations and/or outsourced representations" (Borromeo-Ferri, 2012). In individuals aged between 15 and 16 years three different thinking styles have been identified: visual, analytical and integrated thinking.

mathematical activity has for the individual professionally speaking. Borromeo-Ferri (2006) identifies two aspects that mark the difference between RS and MRS: 1) unconscious simplifications of the task, and 2) personal choice on how to deal with the problem.

In the passage from the MRS to the real model (RM), more conscious simplifications and idealisations take place in the individual, since in MRS phase the individual has already made decisions that influence the filtering of information. The transition process may require extra-mathematical knowledge, depending on the type of task.

The RM phase is related to MRS, as RM is practically built internally and external representations are the Real Model depending on the statements that the individual makes when externalising the model. When in transit from RM to the Mathematical Model (MM), an individual's mathematisation progress appears, in which, according to the task in hand, it may be necessary to also use extra-mathematical knowledge.

The MM phase consists of external representations in mathematical expressions or drawings. The expressions of the individual are more related to mathematical facts and to a lesser degree to reality.

The transitions that occur between these phases are crucial since the externalisation that the individual expresses through images, mathematical language and statements are representations of mental activities, which also depend on previous experience and knowledge.

In the transition from MM to the mathematical results (labelled 4 in Figure 1), an individual's mathematics skills are put into play, such as mathematical resources and strategies to analyse and explore the model and to obtain results or conclusions. The mathematical results consist of writing the results of the model. The transition (labelled 5) from mathematical results to real results is given by the reinterpretation of the solution in terms of the problem. Borromeo-Ferri (2006) notes that individuals often make this transition unconsciously.

In the Real Results phase, the mathematical results are discussed concerning their correspondence to the situation. During the validation of the results, the individual seeks relationships between his or her results and MRS, depending on what kind of validation he or she chooses, either intuitive validation or knowledge-based validation.

The items described play a fundamental role in the constitution of our conceptual framework. Mathematicians and mathematics teachers do not follow the same modelling route, since this is determined by their experience and expertise in the subject area and their thinking style, among others things. Blum and Borromeo-Ferri (2009) have shown how the individuals pass through different phases focusing on some phases and ignoring others. We add that visual imagery (Aspinwall, Kenneth, & Presmeg, 1997) also influences the way in which the individual moves through the modelling cycle, this is because some images may persist in the mind limiting other ways of thinking. Our interest is to document the mathemat-

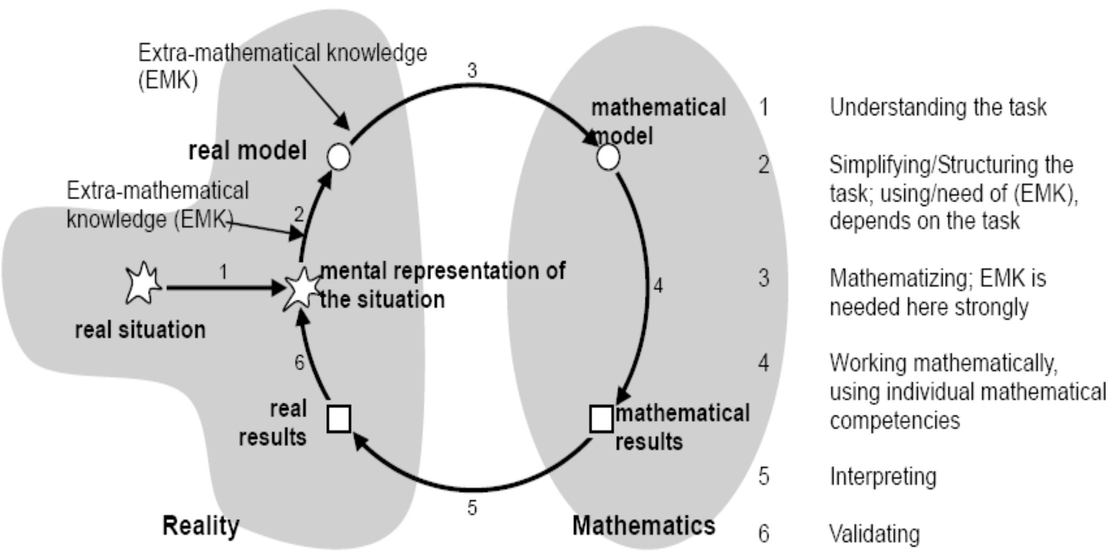

Figure 1: Modelling circle (Borromeo-Ferri, 2006)

ical processes that take place when our observation groups address the modelling of a situation.

METHODOLOGY

This is a qualitative research study (Miles & Huberman, 1994), aimed to analyse the behaviour of participants when they model a hypothetical situation. The data analysis is based on observation, interviews and written material of the individual work of the participants and their arguments while solving the proposed task.

Due to fact that cognitive processes are generally not directly expressed by the individuals, efforts were made to document the mental processes of the participants by analysing individual modelling routes (Borromeo-Ferri, 2010, p. 112). We analysed verbal expressions and representations externalised by the participants when building a mathematical model by asking questions such as: What are you thinking? Why did you do this? How can the situation be related to the representation?

Participants: Three expert mathematicians (Roger, Hugo and Evan) and 20 teachers in training on a Masters' program in Mathematics Education. All were given the same mathematical task. The mathematicians solved the task individually. They were asked to verbally express their thought processes and were interviewed regarding procedures that were unclear to the researchers. The teachers solved the task individually and expressed their reasoning and solving processes to their classmates.

The task was an adapted version of a Vasilyev and Gutenmájer's (1980) task, which can be solved by using knowledge of synthetic geometry and/or analytic Cartesian geometry:

A ladder rests against a wall; on the ladder, there is a cat. The ladder begins to slide on the ground, always touching the wall. What is the path that the cat describes? What would the path be if the cat were not sitting in the middle of the ladder?

DISCUSSION OF RESULTS

The analysis is organised into 4 sections: 1) Understanding the problem, which includes first ideas and impressions, 2) Searching for strategies to address the problem, 3) Model building, 4) Obtaining results and conclusions. The structure of the analysis is consistent with the phases of the cycle of modelling. The first block groups the transition from RS to RM. Block two analyses the externalisations made by the individuals, derived from the transition from MRS to RM. The third block considers the process of mathematisation. The fourth block includes transitions 4, 5 and 6 (Figure 1).

1. Understanding the problem – first ideas and impressions

This task conditions the solution process through the use of pictorial diagrams to explain the situation or to represent the problem, the pictures drawn by individuals bring to life their visual imagery related to the locus plotted by the dynamics of the situation. The mathematicians almost immediately moved the problem to a mathematical representation (Figure 2a) and focused on finding the mathematical formalism. All the teachers supported their solution processes with the use of diagrams to get an idea of the movement described in the situation and remained in this phase for an extended period of time (Figure 2b).

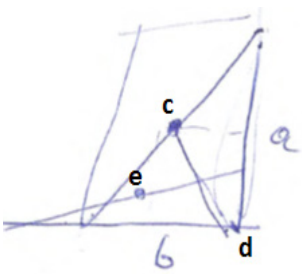

Figure 2a: Diagram made by Roger (mathematician)

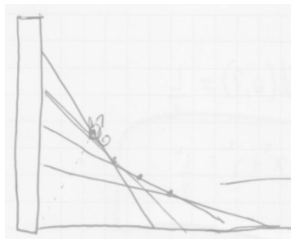

Figure 2b: Diagram made by one of the teachers

The transcriptions provide evidence of the simplification and idealisation that the teachers carried out. For example, they indicate that the cat is considered as a point, which has no weight, that there is no friction between the ladder and the floor, etc. These aspects show the transition between MRS and MR, highlighting the simplifications that the individual consciously performs. The mathematicians do not offer explanations of the characteristics of the phenomena or the elements that are idealised to build the model. They only allow us to observe the transition between the phases of the Real Model (RM) and the Mathematical Model (MM); this may be because the mathematicians immediately look at this situation as a mathematical problem. It appears that the teachers make more connections with reality while mathematicians quickly dissociate themselves from it.

2. Searching for strategies to address the problem

It is difficult to identify the exact moment when the participants begin to search for solution strategies because in many cases this process is determined by the understanding of the situation and the simplifications applied to it. These strategies can be mental operations that cannot be observed until the individual externalises them verbally or in written form. For example, Roger represented the problem geometrically (Figure 2a) and showed some aspects of his strategy only in the interview:

Researcher: What did you think first?

Roger: Of looking at whether the distance of this point [d] to the point that I drew [d and e] remained constant; for example, to see if that was a constant [segment cd],... then I tried to complete this figure like this, to see if the symmetry helped.

On the other hand, after the teachers analysed the problem and got a sense of the trajectory of the midpoint, they expressed different strategies for solving the situation. For example, one of them addressed the problem by placing it in a coordinate system and solved it with the use of tools of analytic geometry (Figure 3a). Another teacher turned to synthetic geometry by observing congruence between triangles (Figure 3b).

In this block teachers already knew the path of the cat so they focused on finding different strategies to determine the algebraic model. Many of their strategies did not help them build the algebraic model and they had to redefine how they achieved the solution. Conversely, we note that two mathematicians apparently reconstructed the locus while they were reading the problem.

3. Model building

The search for solution strategies and model building are closely related. When an individual follows a solution strategy and does not find a model that she/he considers relevant, he/ she tends to propose a new strategy, thus creating a cyclical process between understanding the problem, finding solution strategies and building the algebraic model. This transition was easier to observe in teachers, as the mathematicians went directly to the mathematical representation of the problem.

The teachers initially identified qualitative properties of the situation, and then they used geometric tools and located the problem in a coordinate system, seeking to obtain algebraic expressions that are traditionally associated with a locus (equation of the ellipse and circle); an example of this is seen in Figure 3a.

Two of the mathematicians offered a solution closer to the representation built in grasping the situation. Later they mobilized their mathematical resources for the construction of a mathematical model, showing preference for the use of concepts related to synthetic geometry. Only as a last resource they used analytic geometry tools.

Two moments in the construction of the mathematical model were identified: one related to the visual identification of the locus, and the other after the visualization and related to the construction of the algebraic model. Both moments are manifested in the transition from the mental representation of the situation (MRS) to the Mathematical Model (MM).

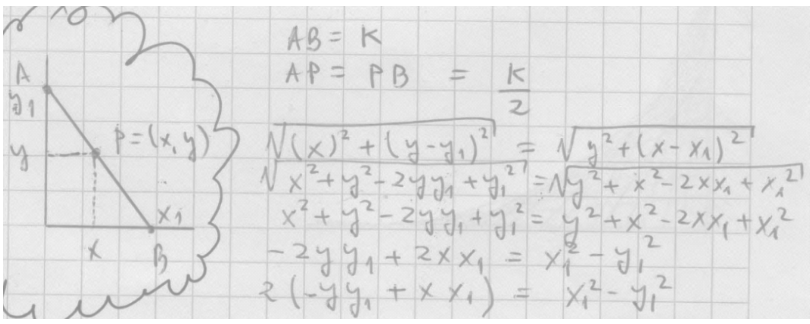

Figure 3a: Solution strategy show by a teacher using analytic geometry

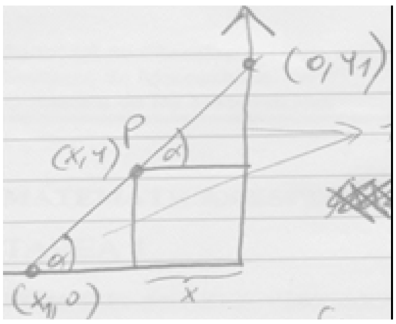

Figure 3b: Solution strategy using congruence between triangles

4. Getting results and conclusions

Finally all participants identified the path of the object. Different strategies were observed. Roger, although he knew the locus, did not refer to it, until he translated the situation to a geometric representation. Hugo did this during the construction of the geometric model. The teachers were aware of the fact that it was the locus of a semi-circle or an ellipse only after plotting different graphic representations and obtaining a mathematical model to represent it.

We summarize our main results in the next table.

CONCLUSIONS

The mathematicians and teachers use diagrams as a strategy to solve the problem. While Hugo and Roger mentally built the path generated by the movement based on its geometric properties and Evan found the locus by using additional geometric and dynamic construction, the teachers were not able to imagine the locus without the support of pictorial representation. For teachers the idea of a locus was also constituted by writing an analytical expression.

We found evidence of some differences in the mental representation of the situation (MRS) and the real model (RM) constructed by each individual. In the case of some of our participants, it can be wrong and largely mediated by their intuition of the behaviour of the situation, sometimes the erroneous visual imagery is persistent which leads to difficulties in obtaining the algebraic model. The fact that the mathematicians and teachers show different approaches to the modelling process can be associated with their experience and their domain of mathematical knowledge, but it can also be related to the way in which everyone is able to abstract the mathematics relationships in a problem. These aspects are links that determine the transitions of the participants in the modelling cycle in an efficient way.

Different strategies were identified when observing how the participants validate their results. For Hugo and Roger validation is associated with their prior knowledge of the locus and to a lesser degree with the verification of the mathematical procedure. For the teachers validation was based on the association of the algebraic model with its graphical representation.

We have shown some differences in the routes followed by mathematicians and mathematics teachers when solving a simple hypothetical problem, of course given a more complicated problem the differences would become more pronounced. In the educational context, we believe it is necessary to make teachers aware of the ability of abstraction possessed by some individuals (as characterised by the corresponding transition between the RS and MM phases), in order that they can make a rational effort to show their students the internal thought processes that occur when modelling a particular situation.

	Mathematicians	Math teachers
1. Understanding the problem -first ideas and impressions	They immediately visualise and construct a mathematical representation. They do not express the simplifications of the situation. It is only possible to observe the transition from RM and MM.	They use pictorial representations to help their understanding of the situation. They externalise the simplifications made to build the model. It is possible to observe the transition between MRS to RM.
2. Searching for strategies to address the problem	The strategies are related to the analysis of the mathematical properties of the situation.	The strategies are directed toward the algebraic solution of the problem.
3. Model building	Use of the Pythagoras and Thales theorem. They showed a preference for concepts associated to synthetic geometry and, as a last resort they used tools from analytical geometry.	The majority of the models were constructed using tools from analytical geometry.
4. Getting results and conclusions	Two of the participants reconstructed the geometric locus parallel to the construction of the MM.	They were aware of the geometric locus after drawing various scenarios. In some occasions they drew it until they determined the corresponding equation.

Table 1: Summary of results

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