Mathematization and modelling of physical phenomena: Analysis of two proposals
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The manifold factors involved in mathematical modelling render this activity a complex process. The teaching of mathematical modelling assumes, at least in part, such complexity, though adding some elements inherent to a didactic environment. This paper presents two proposals for the classroom which have led to interesting conclusions, both from the point of view of the construction of mathematical knowledge (in this case focused on the content of the curricular topic of functions) and from the process itself of drafting the models, as well as suggesting some reflections on our own teaching practice.

Keywords: Modelling, functions, GeoGebra, physical phenomena, upper secondary.

INTRODUCTION

In a synthetic way, Blum and Niss (1991, p. 39) characterize a mathematical model as a triple:

\[ (S, M, R) \]

To be a bit more precise, a mathematical model can be viewed as a triple \((S, M, R)\), consisting of some real problem situation \(S\), some collection \(M\) of mathematical entities and some relation \(R\) by which objects and relations of \(S\) are related to objects and relations of \(M\).

Likewise, they make a distinction between mathematization and modelling:

While mathematization is the process from the real model into mathematics, we use modelling or model building to mean the entire process leading from the original real problem situation to a mathematical model.

Thus, mathematical modelling presents a situation in which the real world (identified with the extra-mathematical) relates to something that is mathematical (identified with the intra-mathematical). Hence, work takes place at the heart of mathematics that is closely connected to the initial extra-mathematical situation or problem, where all such elements are intensely interrelated by means of complex processes.

In line with the degree of importance attributed to each one of the present elements (the processes that take place and the aims intended and assigned to such processes), different approaches and perspectives appear in the way that modelling is taught and learnt (Borromeo, 2006; Kaiser & Sriraman, 2006; Kaiser, Sriraman, Blomhøj et al., 2007). A broad variety of schemes result from here (e.g., Perrenet & Zwaneveld, 2012), which describe the elements present in modelling and the relationships established among them, sometimes distinguishing between which elements are most appropriate for researchers, teachers, and students (see, e.g., Blum & Borromeo, 2009). Such schemes undertake the characteristics of a modelling cycle.

We use the cycle of Blum and Leiss (2007) which is divided in seven steps: 1 Constructing; 2 Simplifying/Structuring; 3 Mathematising; 4 Working mathematically; 5 Interpreting; 6 Validating; and 7 Exposing.

The aim of this paper is to illustrate the complexity of the relations established by the students between the extra-mathematical and the intra-mathematical worlds, and between the mathematical model itself and the mathematical model interpreted in the context of the situation from which the model issued (Blum & Leiss, 2007).

Our work shows an example of modelling that prompts students to obtain data that is subsequently processed with a computer, enabling to obtain the
model, which take the form of a function in one variable (see, e.g., Lingefjärd, 2011). Once such a function is obtained, the aim will focus on analysing the students’ answers to questions about the interpretation of the mathematical model against the original real context, the relevant concepts, notions and basic knowledge on functions, and about knowledge that is not directly related to functions.

**METHODOLOGY AND DESCRIPTION OF THE ACTIVITIES**

The two activities we describe below were proposed during two consecutive years (academic years 2010–2011 and 2011–2012) to students of the course Mathematics I in the 1st year of Science and Technology Baccalaureate (aged 16–17 years), as a voluntary practice to be performed after the regular class hours. Participating students of the first year are designated as A1,…, A13, and participating students of the second year as B1,…, B12 (total of 25 students). The activities were performed one after the other, with a two-week interval in between. Students were distributed into working groups G1, G2, etc., (3–4 students per group) and their work was recorded in audio and video.

As a previous step to the proposed activities (two weeks before), the students learnt how to use GeoGebra to obtain the analytical expression of an adjustment function starting by dumping the data from a table as points in a Cartesian plane (adjustment of a function from a set of points on the plane using sliders).

We divide the development of both activities (herein after Spring and Oil and water) in three different phases: (Phase 1) data collection at the laboratory, (Phase 2) data dumping and obtaining the adjustment function using a computer and (Phase 3) asking questions about the obtained model. At the end, the students answered two questions about their opinions on the activities and the phases into which these had been divided.

In the following, we describe the different phases.

**Phase 1.** The first phase of the activities was proposed with the following statements:

*Spring:* “Let’s study how a spring stretches when we hang a weight from it. Take all the data you consider necessary, and do it the way you think is best.”

*Oil and water:* “Let’s study how the diameter of an oil slick on water varies when we add more oil. Take all the data you consider necessary, and do it the way you think is best.”

Each group had their own materials to collect the data: weights, weight stand, spring (different in each working group), tub, detergent, oil, 2-ml, 5-ml and 10-ml syringes, rulers, tape measure. They had to decide how many data should be collected and the way to do that.

**Phase 2.** The second phase was proposed in the following manner in both activities:

“Dump the data you have obtained at the laboratory into the computer and try to obtain a function by adjusting the data using the GeoGebra program.”

Each group was provided with a computer and a photocopy containing the graphs for the fundamental functions: affine, quadratic, inverse proportionality, square root \( f(x)=k \sqrt{x}, k \in \mathbb{R} \), exponential with bases greater than and less than 1, logarithmic with bases greater than and less than 1, sine, cosine and tangent. As in the precedent phase, students did their work in an autonomous and open way. The adequate function for the case of Spring is the linear function \( f(x)=ax \) or affine function \( f(x)=ax+b \), \( a,b \in \mathbb{R}^+ \), depending on whether only the length of the stretched spring is measured or the full length. In the case of Oil and water, the adequate function is \( f(x)=k \cdot \sqrt{x}, k \in \mathbb{R}^+ \).

**Phase 3.** In the third phase, the questions asked about both models were proposed so that each student would reply individually and in writing, except for the last three questions about the model Oil and water. These three questions represented the previous necessary step to apply the model to a hypothetical real situation, i.e. the last phase of modelling the behaviour of oil on water. Below we reproduce the questions asked to students. When analysing their answers, we will only refer to some of them.

**Spring**

1) What type of function have you obtained? Interpret the result. 2) Which variable is dependent and which is independent in the function? 3) In the function you
have deduced, is there any parameter? If the answer is yes, what does it mean in the experiment you are conducting? 4) How much does the spring stretch with 370 g of weight? 5) Which weight corresponds to a length of 21 cm of the spring? 6) What length of the spring do you obtain from the function if you do not put any weight on the stand? Interpret your result. 7) According to the function you have obtained, is it possible to stretch the spring indefinitely? Interpret this result, though taking into account the specific experiment you have conducted. 8) Try to deduce how to calculate the weight applied if you know the length of the spring. 9) Do you think the function you have obtained describes well the behaviour of a spring to which a weight has been attached? 10) The obtained functions are different. What do you think the reason for this is?

**Oil and water**

1) The function describing the data is always expressed as \( f(x) = k \cdot \sqrt{x} \), \( k \) being a constant, \( x \) the amount of oil in ml, and \( f(x) \) the diameter in mm. \( k \) is different depending on each case. In this respect, \( k \) varies. Would it be appropriate to call it a ‘variable’? Why? Would you use another name? Why do you think \( k \) varies in each case studied? 2) One data is the diameter. How would the function change if we had used the radius instead of the diameter? 3) One data is the diameter. How would the function change if we had used the area of oil on water instead of the diameter? 4) What function would we obtain if we represent the diameter on the \( x \)-axis and the amount of oil on the \( y \)-axis? (Or the area and the amount of oil).

Application of the **Oil and water** model to a hypothetical real situation:

![Figure 1: Hypothetical image of an oil spill (its original size has been reduced)](image)

The photograph below is a satellite view of an oil spill off the coast of Cambados.

1) Determine the scale of the photograph. 2) Determine the area of the polluted surface. Use any instruments and knowledge that you consider necessary or suitable. 3) Apply the model you have obtained to determine the amount of fuel in the spill.

In order to determine the scale of the photograph, the students were provided with a photocopy of a nautical chart on a scale of 1:30,000 of the same coastal area, and with material for technical drawing (ruler, compass, set-square, etc.).

**ANALYSIS OF STUDENTS’ ANSWERS**

Here we highlight some general considerations about the development of the activities:

The working environment was relaxed in all the phases, the available time was enough and there were no delays. During the phase of data collection and obtaining the adjustment function all the students participated actively, deciding the distribution of tasks by consensus and assuming the relevance of performing correct measurements and adjustments. The students called the teacher on very few occasions, and most of the questions were related to mere technical issues, which led us to assume that they were committed to perform the work autonomously.

The time employed is one of the obstacles mentioned about the introduction of modelling either in school or university (see, e.g., Blum & Niss, 1991). Regarding this issue, no time limit was fixed to perform the successive phases. The time used by each group to obtain the data table was in no case longer than 18 minutes, in the case of **Spring**. In the case of **Oil and water** it varied to a greater extent, ranging from 30 to 55 minutes. Some groups took longer due to the difficulty of pouring oil on water using a syringe. The time employed for adjusting the function using the computer did not exceed 15 minutes in any group. Most of the time was consumed entering the data from their tables into the computer. The time used to answer the questions posed about the obtained model was approximately the same: 35 minutes in the case of the spring, 40 minutes in the case of the first question and the discussion about **Oil and water**, and 70 minutes in the case of the application of the oil and water model.
Data table and adjustment function

The number of data and the titles of columns each group assigned in their data table varied considerably: in the case of Spring the number of data varied from 9 to 32; in the case of Oil and water they obtained between 9 and 23 data pairs. The groups referred the measurements they made in the first row of the table as Weight and Length (groups G1, G2, G3 and G6) or x and y (group G5). Group G4 did not write anything to describe the columns. So, students identified the real-life variables (Weight, Length, etc.) with mathematical variables (x and y) because they wrote data using the usual representation of a data table of a function. So, we may consider that this phase includes the steps 1 Constructing, 2 Simplifying/Structuring and 3 Mathematising (Blum & Leiss, 2007).

In the case of Spring, group G1 (Figure 2) obtained the equation of a straight line determined from two data from their table, and another group (G5) modified the parameters a and b in the equation of a straight line on the plane, \( y = ax + b \). The remaining groups performed a similar procedure using the functional expression \( f(x) = ax + b \). In the case of Oil and water all the groups provided a function of the kind \( f(x) = k \cdot \sqrt{x} \) as the solution, with different values for \( k \) in each group. So, we could say that, when each group inserted the appropriate function, they had previously decided the functional variables \( x \) and \( y \) (identified with weight and length and volume and diameter, respectively) and the necessary parameters they had to use. Therefore, this second phase should include the steps of 3 Mathematising and 4 Working mathematically (Blum & Leiss, 2007). Since they had previously identified real-life variables with mathematical variables, the mathematical model and result (the obtained function) should be interpreted as a real model and result (5 Interpreting).

Answers to the questions about the obtained model

63.6% of the students did not identify the variables correctly (question 2 about Spring; Figure 3a). Also, 95.5% of the students did not recognise the parameters (question 3 about Spring).

In the first question about the model for Oil and water, 52.2% of the students characterized \( k \) as a variable (Figure 3b) and 34.8% as a constant. 16% of the students initially asserted it was a variable and subsequently contradicted themselves and sustained it was a constant.

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**Figure 2:** Data and functions obtained by group G1 \[ y = 0.02x + 9.7; f(x) = 2.3 \sqrt{x} \]

**Figure 3a:** Variables and parameters

Translation: Dependend variable is 0.11x the independent one is 10.8.

**Figure 3b:** Variables and parameters

Translation: \( k \) is a variable, so I would call it x.
So, regarding the variables and parameters, students did not identify the variables correctly and made confusion between the dependent and independent variables and the parameters.

In Spring they had obtained a function that enabled to obtain the length given the weight (mass); however, surprisingly, in questions 4 and 5 half of the students used the rule of three in one or both questions (Figure 4). Only 27.3% of the students used the function in both questions and another one used a rule of three and the function, without questioning themselves why the obtained results were different. Therefore, they associated the graph of the function (straight line) with direct proportionality, and direct proportionality with the constant rate of change.

Sometimes, students designated the function as an equation and the variables as unknowns (Fig 5).

In questions 5 and 8 for Spring, the students did not mention the inverse function in their answers, and they performed the calculations with the function expression as if it were an equation which they must solve to obtain the unknown. In the discussion about oil, the first-year students almost immediately reached the conclusion that the function requested in the fourth question was the inverse function. One of the second-year students, after a few minutes, provided the way to calculate this function without realising it was the inverse function. Another student (B8) reached the conclusion that it was the algorithm they learnt to obtain the inverse function:

Student B8: It’s the inverse of the function. That is the inverse. When we look at the composition. Change x for y and solve for y.

Regarding the discussion about the adequate expression for the radius, there are differences from one academic year to the other. For example, the first-year students discussed for six minutes whether the adequate expression is \( \frac{f(x)}{2} \) and about the consequences of dividing only one member of an equality by a number. The second-year students discussed for 10 minutes whether the expression of the radius should be \( \frac{1}{2} \cdot \sqrt{x} \), \( 2.1 \cdot \frac{1}{2} x \) or \( 2.1 \cdot \sqrt{x} \). Neither the first-year nor the second-year students denoted the new functions they obtained in the discussion about Oil and water in a different way, and they always designated the results as \( f(x) \).

Regarding the application of the Oil and water model to a hypothetical case of a pollution spill, we highlight the following results:

- One fourth of the students determined a value close to the right one for the scale. The remainder either used an inadequate system (55%), or failed to determine the scale (20%).

- Only two students achieved a correct calculation method to estimate the value of the spill area on the photograph. 55% of the students drew a circle on the picture and measured the radius or the diameter, in a clear relation to their observations of the behaviour of oil during the first phase.

- The students had many difficulties in the application of the model: they did not remember basic formulae to calculate areas, they confused the names of geometric figures, they misused the units of area or volume, etc.

- In no case they interpreted both images (map and photograph) as a similarity, so they failed to use the relationship between areas and volumes of similar figures.

- From all the students, only two achieved to provide a volume for the spill following a calculation system that may be considered acceptable.

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Figure 4: Use of the rule of three

Figure 5: Function equality as an equation and identifying the expression with its graph
Opinions and valuations of the students

From the analysis of the opinions and valuations by the students, we highlight below the aspects that we consider as the most relevant.

83.3% of students stated that the modelling for Oil and water was the one they liked most or found most interesting, often mentioning the discussion as a very positive way to learn from their peers and exchange ideas among themselves (41.7%).

Regarding the phases, the students considered all of them important, with special mention to the data collection at the laboratory (62.5%) because this makes the model becomes their model. One third of the students highlighted the application of the Oil and water model as interesting and enabling to apply the model to a real, socially problematic situation, and so they valued the model obtained as being useful.

Overall, they considered it very important to perform this kind of activities (that they sometimes referred to as experiments), however, at the same time, they thought that this kind of activities had nothing to do with mathematics teaching. They acknowledged that the kind of teaching they receive is based on learning algorithms.

In order to illustrate the comments above, we present a sample of the interviews made to the students:

Student B6: (…) I didn’t know how to measure an oil slick or spill on water and, for example, that was one of the things I enjoyed most, wasn’t it? Because you ask yourself: how is it possible to measure that on water? And then, you even see how the shape is formed and everything.

Student B3: (…) working on a problem is finding the solution and the same goes for equations. But here it’s all about finding the data and then you have to find out something else; it’s different, these are different steps you have to take with your own observations and collecting data. In class they give them to you, you don’t have to do it for yourself.

Student A1: (…) You can’t spend all the time doing experiments, you must have class.

Student B3: Well, in maths, when you have class it’s basically doing exercises and learning what you have to know. This is something on the side that helps but it has nothing to do with the class, at least for me it has nothing to do with maths, with the class, I mean, because it’s something different.

Student A1: (…) Because the problem with maths is that some people see it as something they’re never going to use in their lives. A lot of people say: why would I want to know how to solve an equation? Why do I want to know about what a function is? I don’t know, if you actually see it in real life then at least it should stir your curiosity.

Conclusions and elements for discussion

The issue that we consider essential is that the students succeeded in obtaining a model (a function) at the end of the second phase. However, their answers to the questions in subsequent phases lead us to think that the mathematising process (step 3, modelling cycle, Blum & Leiss, 2007) has been performed in a very limited manner. Consequently, it could be questioned whether the obtained function actually represents a mathematical model and result (Blum & Leiss, 2007). However, if we see only the mathematical result obtained by the students, it is possible to think they completed at least four of the seven steps of the modelling cycle. Therefore, when we propose to students to develop a mathematical model in an autonomous and open way, it is necessary to set questions related to important mathematical concepts and notions during the modelling process and not just at the end of it.

Moreover, for this result to become a solution to the originally proposed problem, students should interpret the mathematical result they obtained as a real result (step 5, Blum & Leiss, 2007). In fact, two functions coexist in both modelling processes with different domains and paths (the mathematical function and the function that relates mass to length and volume to diameter).

We must also add to the above the differences between both academic years regarding the development of
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the discussion as well as the problems that arose in the application of the Oil and water model. Modelling introduces some elements of uncertainty about what the teacher may expect to encounter in the classroom, such as the students’ answers. The difficulties that arise during the process are not entirely predictable and consequently teaching becomes more open (see, e.g., Blum & Borromeo, 2009). Moreover, both models start from extra-mathematical situations inherently linked to physical laws and magnitudes of certain relevance (difference between mass and weight, Hooke’s Law, behaviour of fluids of different densities, Archimedes’ Principle, surface tension, etc.). Therefore, a series of questions arise that the teacher has to consider; the answers to those will determine a different kind of modelling and, consequently, a different modelling cycle. To give just a few examples: one could opt for omitting the data collection phase (by considering it to pertain to experimental science) and focus the modelling on such aspects perceived as more appropriate to mathematics, or we could also opt for using the obtained model to introduce some magnitudes and laws of physics; the degree of difficulty of obtaining the adjustment function could be increased by not providing a photocopy of the graphs for the fundamental functions; the square root may be presented on the photocopy as \( \sqrt{x} \) or as \( k \sqrt{x} \), etc.

Likewise, answers to the questions raised to the students could be approached in a joint discussion; or by distributing the students into smaller, independent discussion groups; or individually in writing.

Finally, if we focus on more general objectives, modelling can be proposed as a means to introduce key concepts and notions about functions, or also to detect difficulties and obstacles in students who have already studied such content previously.

Therefore, modelling introduces some elements that force teachers to make previously meditated and reflected decisions about their teaching methods (see, e.g., Doerr, 2006). Such decisions will affect the kind of modelling proposed, the ways in which it is developed, the answers of students and the steps of the modelling process.

REFERENCES


