A pattern-based approach to elementary algebra

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With a focus on epistemology, this paper discusses what pattern generalisation as an algebraic activity involves. Further, it presents a review of empirical studies where a pattern-based approach is used to teach algebra. This shows that students' problems with establishing algebraic rules from patterns and tables can be explained by: 1) difficulties caused by students' use of invalid methods to identify explicit formulae; 2) difficulties caused by students' tendency to focus on recurrence relations; and 3) institutional constraints. As an alternative to a traditional task on a shape pattern, the paper presents an epistemological model designed to implement the equivalence statement: $1 + 3 + 5 + \ldots + 2n - 1 = n^2$.

Keywords: Algebraic activity, pattern generalisation, epistemological model, milieu.

INTRODUCTION

According to Reed (1972), humans have a natural inclination to observe patterns, and to impose patterns on different experiences. Inspired by Steen (1988), Devlin (1994), and others, I consider mathematics as the science of patterns. Mathematicians seek patterns in different areas, including numbers (arithmetic and number theory), form (geometry), motion (calculus), reasoning (logic), possibilities (probability theory), and position (topology). In the development of mathematical knowledge, generalisation is an essential process. This is asserted by for instance Krutetskii (1976), who classifies generalisation as one of the higher cognitive abilities demonstrated by mathematics learners.

Generalisation of shape patterns and numerical sequences is part of the elementary and secondary curriculum in many countries, for example England (Department for Education, 2014); the United States (National Council of Teachers of Mathematics, 2000); Canada (Ontario Ministry of Education and Training, 2005); and, Norway (Directorate for Education and Training, 2013). A purpose of students' engagement with patterns is to provide a reference context (physical, iconic or numerical) for generalisation and algebraic thinking.

A shape pattern is usually instantiated by some consecutive geometrical configurations in an alignment imagined as continuing until infinity. In this paper, geometrical configurations will be referred to as *elements*, and the constituents of an element will be referred to as *components*. Generalising a pattern algebraically rests on noticing a commonality (a structure) of the components of some elements of the pattern, and using it to provide an expression of an arbitrary member of the number sequence mapped from the pattern. This will be explained in more detail below.

PATTERN GENERALISATION AS AN ALGEBRAIC ACTIVITY

A model for conceptualising algebraic activity is proposed by Kieran (2004), where she introduces three interrelated principal activities of school algebra: generational activity, transformational activity, and global/meta-level activity. The *generational activities* involve the creation of algebraic expressions and equations like (i) equations that represent quantitive problem situations; (ii) expressions of generality arising from shape patterns or numerical sequences; and (iii) expressions of the rules that determine numerical relationships (Kieran, 2004). I interpret the letters used in the three examples as having the role as unknowns, variables and parameters, respectively. The *transformational activities* involve syntactically-guided manipulation of formalisms including: collecting like terms; factoring; expanding brackets; simplifying expressions; exponentiation with polynomials; and, solving equations (Kieran, 2004). These are the activities with which school algebra has traditionally been associated. The *global/meta-level activities* involve activities for which algebra is used as a tool, and include:

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problem solving; modelling and predicting; studying structure and change; analysing relationships; and, generalising and proving (Kieran, 2004).

The elements of a shape pattern are carriers of multiple structures which have to be interpreted by the students. The process of interpreting and representing these structures algebraically, involves generalisation of arithmetical relationships in members of the sequence mapped from the shape pattern. A shape pattern can be generalised either through an indirect approach, where the result is a recursive formula (a relationship between consecutive elements), or through a direct approach, where the result is an explicit formula (a functional relationship between position and numerical value of an element).

Måsøval (2011) distinguishes between two types of shape patterns: *arbitrary patterns* (Figure 1), and *conjectural patterns* (Figure 2).

These patterns correspond respectively to two different mathematical objects aimed at in the process of generalising: *formula* (for the general member of the sequence mapped from the shape pattern; e.g. $a_n = 3n + 1$ in Figure 1), and *theorem* (in terms of a general numerical statement; e.g., $1 + 3 + 5 + ... + 2n - 1 = n^2$ in Figure 2). Institutionalisation of the knowledge in the case when algebraic generalisation aims at a formula (for an *arbitrary* pattern) is not institutionalisation of the formula *per se*. It is institutionalisation of how the formula can be derived through identification of an invariant structure in the elements of the pattern. Further, it is institutionalisation of how the invariant structure is interpreted into arithmetical relationships and how these in turn are generalised algebraically in terms of a formula. The cultural, reusable knowledge in this case is the nature of the relationship between the algebraic expression and its referent (a generic element of the pattern). On the other hand, institutionalisation of the knowledge in the case when algebraic generalisation aims at a theorem (illustrated by a *conjectural* pattern) involves decontextualisation of the general numerical statement from the shape pattern on the basis of which it is developed. The cultural, reusable knowledge in this case is a general relationship between sequences of numbers (in Figure 2, between odd and square numbers).

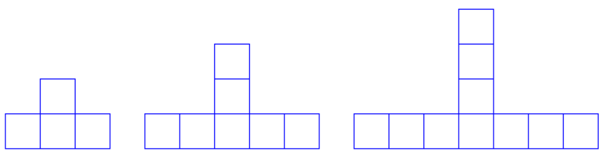


Figure 1: Example of the first three elements of an arbitrary pattern

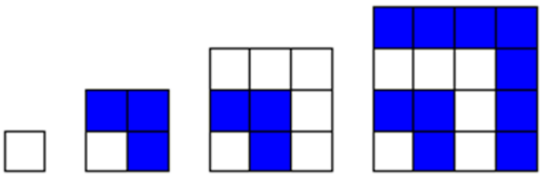


Figure 2: Example of the first four elements of a conjectural pattern

In the following, I illustrate briefly a strategy for generalisation of an *arbitrary pattern*, where I focus on the connection between the iconic, the arithmetical, and the algebraic representation of the pattern. For a detailed epistemological analysis of shape pattern generalisation, see (Måsøval, 2011, Chapter 5). The target knowledge is the nature of the relationship between the sought generalisation (an algebraic expression) and its referent (a generic element of the pattern). A direct approach to generality is employed. The invariant structure of a shape pattern provides the possibility to decompose the elements into different repetitive parts. Decomposition refers to diagrammatic isolation (encircling, painting with different colours, or other techniques) of various parts of the elements in order to visualise the invariant structure of the pattern. The point is to express the number of components of each partition of an element as a *function* of the element's position in the shape pattern. These arithmetical expressions are used to express the total number of components of the element. Generalisation of the sequence of arithmetical expressions will lead to a formula where letters are placeholders for positions. An example of the first three elements of a quadratic pattern is presented in Figure 3.

Figure 4 presents a possible decomposition of this pattern, with corresponding arithmetical expressions

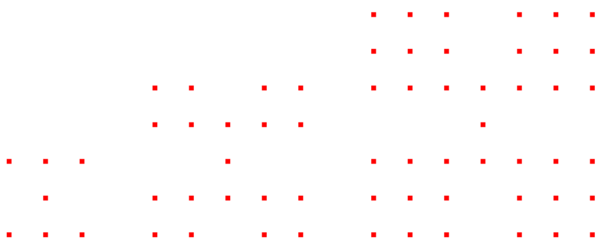


Figure 3: The first three elements of a shape pattern

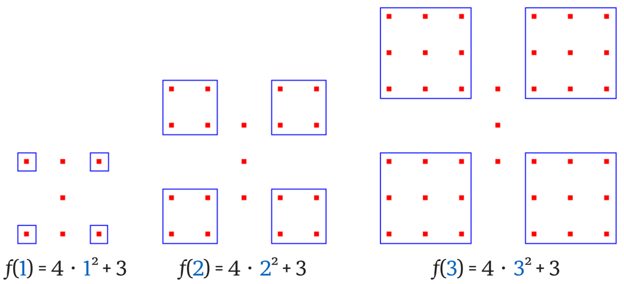


Figure 4: A possible decomposition of elements of the pattern in Figure 3

representing the number of components in the partitions of the first three elements.

The decomposition shown in Figure 4 corresponds to an interpretation which means that each element consists of three components plus four times the square of the position of the element. The corresponding arithmetical expressions suggest a generalisation in terms of the formula, $f(n) = 4n^2 + 3$. An alternative decomposition (discussed without illustration) involves filling in components at "empty places" to make each element into a square; this is compensated by subtraction in the arithmetical expression. In order to build a square, the n-th element would need to get an extra of $4n - 2$ components. The resulting formula would then be given by $g(n) = (2n + 1)^2 - (4n - 2)$.

The example in Figure 3 conceptualised through Kieran's (2004) framework

The process of interpreting and decomposing the pattern is a *global/meta-level activity*. It is a modelling process that involves studying and representing the quadratic relationship between the position and the corresponding member of the number sequence mapped from the pattern. Establishment of a formula (here, a function) is a *generational activity*, where the variable n is a placeholder for an element in the domain (natural numbers). The two different decompositions presented above result in different formulae, where *transformational activity* can be used to justify that they are equivalent.

Justification of the new knowledge (the formula), which is a global/meta-level activity, can be done through decomposition of a *generic example*. This is done to illustrate references between the partitions of the generic element, on the one hand, and the mathematical symbols of the formula, on the other.

EMPIRICAL STUDIES OF STUDENTS' PATTERN GENERALISATION

Students' difficulties in establishing algebraic rules from patterns and tables

Several studies have documented students' difficulties in establishing algebraic rules from patterns and tables. Stacey (1989) reports responses to linear generalising problems of 140 students aged between 9 and 13. Generalisation of the given problems was of the type $f(x) = ax + b$ with $b \neq 0$. It turned out that mainly two ideas were used. Stacey refers to these as the *difference method* and the *whole-object method*. The difference method involves multiplying the common difference between members of a sequence by the rank of a member to calculate its numerical value. The whole-object method involves taking a multiple of the numerical value of a member of a sequence to calculate the numerical value of a member with a higher rank; that is, implicitly assuming that $f(mn) = mf(n)$. The two methods will be applicable only when the linear problems are direct proportionalities. Because the problems used in Stacey's study were not of this type, the difference method and the whole-object method were invalid. The erroneous generalisations were not discovered by the students because they failed to check the validity of the rules they produced.

Another finding from Stacey's (1989) study was that students showed a tendency to focus on recurrence relations in one variable rather than on functional relationships between two variables. The same conclusion about students' tendency to focus on recurrence relations was reached by MacGregor and Stacey (1995). They tested approximately 1200 students in Years 7 to 10 in ten schools on recognising, using, and describing rules relating two variables; 14 students were interviewed. The results showed that the students had difficulties in perceiving functional relationships and expressing them in words and as equations. The students' tendency to find recurrence relations in patterns and tables were in most cases counter-productive to identification of a relationship between two variables. Hence, MacGregor and Stacey recommend teachers to use examples where it is not possible to find differences between consecutive members of a sequence.

Orton and Orton (1996) conducted a study in which 1040 students from Years 6, 7 and 8 (ages 10 to 13) completed a written test on different pattern questions;

30 of the students were interviewed about their responses. Results from this study are consistent with findings from Stacey (1989) and MacGregor and Stacey (1995): students have a clear tendency to use *differencing methods* and identify a recursive pattern.

Lannin, Barker and Townsend (2006) explored students' use of recursive and explicit relationships by examining the reasoning of 25 sixth-grade students, including a focus on four target students, as they approached three generalisation tasks while using computer spreadsheets as an instructional tool. Their results demonstrate students' difficulty in moving from successful recursive formulae towards explicit formulae. One obstacle to students' ability to connect recursive and explicit formulae was their limited understanding of connections between mathematical operations, such as addition and multiplication.

Måsøval (2011) reports from a case study of six (two groups of three) student teachers' collaborative engagement with four tasks on shape pattern generalisation (with some teacher involvement). Three categories of constraints to students' generalisation processes emerged from a process of open coding of transcripts of video recordings, conceptualised through the theory of didactical situations (Brousseau, 1997). The constraints are explained in terms of: 1) an inadequate *adidactical milieu*, in particular caused by unfavourable design of tasks (where the focus is on number of components rather than on the multiple structures inherent in the pattern); 2) complexity of transforming observations and conjectures represented in informal language into algebraic symbolism (from *action* to *formulation*); and 3) complexity of justifying proposed generalisations (*validation*), in particular caused by students' use of empirical reasoning instead of rigorous mathematical reasoning (Måsøval, 2011).

The foregoing discussion of students' difficulties in establishing algebraic rules from patterns and tables can be summarised in three points. First, there are difficulties caused by students' use of invalid or unsuccessful methods to identify explicit formulae (the *difference method* and *differencing*, and the *whole-object method*). Second, there are difficulties caused by students' tendency to focus on recurrence relations which are not easily transformed into explicit formulae. Third, there is an institutional constraint caused by the use of stereotype tasks (focusing on "How many?") and further, by the way pattern generalisation is taught.

Components of a successful pattern-based approach to elementary algebra

Results from Redden's (1996) study demonstrate a significant correlation between natural language descriptions and symbolic notation used by students. On the basis of investigation of how 1435 children aged 10 to 13 responded on requests to generalise shape patterns, he found that natural language descriptions exclusively in terms of functional relationships appear to lead to students' successful use of algebraic notation. This finding points at the importance of relating the independent variable (the position of a member) to the dependent variable (the member itself).

Warren (2000) demonstrates significant correlation between students' ability to reason visually (identify, analyse, and describe patterns) and successful algebraic generalisations from shape patterns and tables of values. Warren's finding is based on responses on two written tests administered to 379 students (aged between 12 and 15 years); 16 of the students were interviewed in groups of four. Warren, Cooper and Lamb (2006) examined the development of students' functional thinking during a teaching experiment that was conducted in two classrooms with a total of 45 Year 4 students (average age nine and a half years). They found that tables with input values not increasing in equal steps assisted students to search for a relationship between two data sets instead of focusing on variation within one. Randomness of the input values encouraged students to think relationally instead of sequentially, a finding consistent with MacGregor and Stacey's (1995) recommendation referred to above.

The results referred to above can be combined to provide a recommendation for students' engagement with shape patterns: Students' should be encouraged to *express functional relationships in natural language*, because this is important for the ability to use symbolic notation. It is relevant here that students remain connected to the iconic representation and *reason visually*, which is a condition for successful algebraic generalisations. Further, visual reasoning can potentially prevent students from senseless pattern spotting in numerical sequences without connection to the original mathematical situation. The strategy presented above (exemplified by the pattern in Figure 3) for generalising a pattern algebraically is favourable in that it has the recommended features: first, it involves analysis of geometrical configurations (decomposition); and second, it involves identification of functional relationships between two sets (position and numerical value of element, respectively).

Several studies suggest that it is not generalisation tasks in themselves that are difficult; the problems that students encounter are rather due to the way tasks are designed and limitations of the teaching approaches employed (Moss & Beatty, 2006; Måsøval, 2011, 2013; Noss, Healy, & Hoyles, 1997). Motivated by this, I designed an epistemological model – a *situation* (Brousseau, 1997) – that involves a problem that can be solved in an optimal manner by using the knowledge aimed at. In the concluding section of the paper, I explain this epistemological model and its devolution to a class of twenty student teachers enrolled on a master programme for primary and lower secondary education. The data from the experiment are students' notes and solutions in addition to my field notes.

AN EPISTEMOLOGICAL MODEL OF A PIECE OF KNOWLEDGE

Inspired by TDS, the theory of didactical situations (Brousseau, 1997), I created an epistemological model of the general numerical statement that "the sum of the first n odd numbers is equal to the n-th square number". The choice of this piece of knowledge was motivated by Måsøval (2011) who reports from students' (less successful) engagement with this equivalence statement through a traditional task, based on a shape pattern similar to the one in Figure 2.

Devolution

I had cut out sets of twelve paper forms, consisting of from 1 to 25 unit squares (a selection of which is shown below). In the classroom I presented the context (below), and gave each pair of students an envelope with a set of cut-outs.

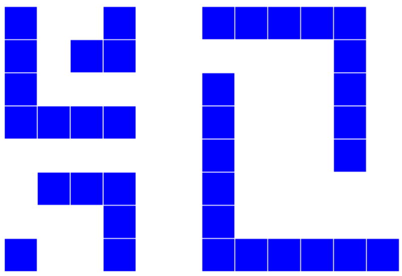
"The company TILEL (in class, represented by the envelopes) sells a special kind of tile formations that can be used to cover *squares*. The tile formations have shapes as Ls, and consist of an *odd number* of unit squares. There is also a degenerated L-form which consists of only one unit square. You and your partner are supposed to construct a quadratic area of tiles, using L-forms from TILEL. You decide on the *size of a square*, and the task is for your partner to go and get a selection of L-forms which *precisely covers* the chosen square. There is a restriction on the L-forms: they shall all be of *different size*.

When your partner returns with the L-forms he/she has bought, the two of you shall arrange them into a square. If you lack some L-forms, or have anyone left over, your partner will have to go back to TILEL for supplements or returns. Each time this is necessary, a *charge* must be paid, so buying the right L-forms at once is important."

Some of the features of the milieu were, due to time constraints, hypothetical (e.g., purchase of tile formations and the fee charged for supplements and returns). The adidactical situation might have been designed as a game, where the winner would be the group that solved the task with least costs, and/or had the best recipe, etc. Figure 5 shows the task given to the students for work in pairs, after the context was presented.

Features of the milieu derived from the knowledge at stake

The target knowledge in this case is the equivalence statement: "the sum of the first n odd numbers is equal to the n-th square number", potentially represented by $1 + 3 + 5 + ... + 2n - 1 = n^2$. A model of the target knowledge is created using a dissection of a square into L-forms consisting of consecutive odd numbers.



1. One of you chooses the size of a square; the other one gets L-forms to cover it. Collaborate to arrange the L-forms into the chosen square. (ACTION)
2. On the basis of the work you have done, make a recipe for how to cover a square of random size with L-forms of different sizes, without having to go back for supplements or returns. Let another group try your recipe and see if it works. (FORMULATION)
3. Explain why your recipe will always work (for a random square). (VALIDATION)

Figure 5: The task given to the students for work in pairs

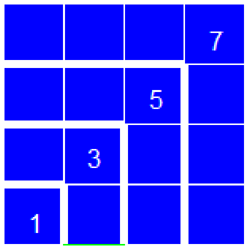


Figure 6: Dissection of the fourth square into the first four odd numbers

of unit squares (1 is represented by one unit square, hence a degenerated L). A generic example is given in Figure 6, illustrating that $1 + 3 + 5 + 7 = 4^2$.

It is important that the L-forms given to the students are of different size; this is what guarantees that the square is made of a sum of *consecutive* odd numbers. The fee charged for not getting the correct L-forms is intended to motivate students' (in the situation of action) to develop a model that relates odd numbers and square numbers. It is expected that the students (in the situation of formulation) express in natural language that it is necessary to add as many odd numbers (from 1 and upward) as the rank of the chosen square number. The situation of validation is intended to motivate reasoning based on the nature of the knowledge at stake, even if algebraic notation might not be used. For this to happen, it is necessary that the students have a technique (prior knowledge) for representing odd numbers in terms of $2 \cdot 1 - 1, 2 \cdot 2 - 1, 2 \cdot 3 - 1$, and so on.

Due to limited space, I comment only on the validation phase (in whole class). Two approaches were used to justify the conjecture: One was a visual proof (cf. Figure 6), where students argued by a generic example that the next square is reached by adding the next odd number $(2(n+1) - 1)$ to the current square. The other approach started with the statement in algebraic notation, $1 + 3 + 5 + \ldots + 2n - 1 = n^2$; students showed that the sum on the left hand side is equal to n^2 by using the Gaussian method (adding the first and last terms, then the second and last but one term, etc.). There was a discussion of implementation of the model in different grades in school.

The adequacy of an epistemological model is based on the quality of the underlying epistemological analysis (EA). An EA should provide a rationale that would make the students' engagement in the problem situation sensible. An EA is however "work in progress"; experiments feed back to, and might strengthen, the EA and hence the model based on it. A relevant direction for future research on pattern generalisation is therefore the design and study of implementation of epistemological models of pieces of algebraic knowledge – in order to improve the models. Due to its epistemological focus, TDS would be a favourable framework to use in this kind of research.

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