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A study was conducted with 26 college students with the purpose of gaining insight into students' conceptual understanding of parameters in algebra. Participants contributed to a whole-class discussion, solved problems with parameters, and identified the parameters in each problem. About one third of the students had difficulty identifying parameters. Even when successful at identifying parameters, students had great difficulty solving the problems with parameters. The difficulty was even greater when the mathematical object was a family of quadratic equations. This suggests that the true difficulty lies not with identifying parameters, but with parameters in action, that is to say when solving problems with parameters.

Keywords: Parameter, algebra, college students.

INTRODUCTION

At the time Šedivý (1976) wrote "A note on the role of parameters in mathematics teaching," the New Math era in the United States was replaced by the Back-to-Basic movement. The New Math secondary algebra curriculum, characterized by a structural approach and deductive reasoning, was replaced by a collection of "basic" algorithms to solve simple equations (Kilpatrick & Izsák, 2008). Solving equations with parameters like, $x + \sqrt{x^2 - 2ax} = b$ (Šedivý, 1976), using symbolic manipulation, disappeared from the secondary algebra curriculum. During the Back-to-Basic movement, the research on parameters was very scarce (Bloedy-Vinner, 1994). According to Furinghetti and Paola (1994), in the journals *For the Learning of Mathematics*, and *Educational Studies in Mathematics* only one article with the word "parameter" in the title was published, the one written by Šedivý (1976). The secondary algebra curriculum of the standards-based era that followed the Back-to-Basic movement departed from the static equation solving and gradually introduced the dynamic functional approach and the use of graphing technologies (Kilpatrick, Mesa, & Sloane, 2007). The research on parameters remained scarce (Ursini & Trigueros, 2004) and only in the last decade gained some momentum by focusing on ways in which graphing technologies may contribute to student understanding of parameters (Abramovich & Norton, 2006; Green, 2008). The purpose of the study reported here is to revisit the students' conceptual understanding of parameters in algebra, given the curriculum shifts from the past five decades.

THE CONCEPT OF PARAMETER

There is a consensus that the concept of variable is multi-facetted and context-dependent (Kuchemann, 1978; Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1988). Variables as parameters have the role to stand for values or numbers "on which other numbers depend" (Usiskin, 1988). Parameters are "general constants" (Philipp, 1992) or "general numbers, but of second order, that is, required when generalizing first order general statements." (Ursini & Trigueros, 2004) Discriminating between parameters and other variables implies both a variable hierarchy (Bloedy-Vinner, 1994; Philipp, 1992; Šedivý, 1976), and a reification of the mathematical objects defined with the help of parameters (Sfard & Linchevski, 1994). With respect to the mathematical objects with parameters, in the secondary algebra curriculum in the United States, the students encounter mostly families of linear and quadratic equations and functions in which one particular value of the parameter generates one specific equation or function.

One example given by Philipp (1992) is conceiving of the parameter k in the family of linear functions

$C = kg$, where C is the cost per gas (in \$), k is the price of gas per gallon (in \$ per gallon), and g is the quantity of gas (in gallons). First, one has to imagine selecting the price of gas, i.e., instantiating k with posible values (e.g., \$2.49, \$2.69, \$2.99), and only then can one construct the specific mathematical object, i.e., the specific linear function that describes the quantitity of gas, g, and the cost, C, varying together with a constant rate of change, k. Thus, discriminating between the parameter k, the dependent variable C, and the independent variable g, implies both a variable hierarchy and a reification of the family of linear functions. With respect to the family of functions $C = kg$ or $C(k, g) = kg$, from the student point of view, it may be difficult to conceive of k as the literal coefficient of the variable g and instantiate it with one value at a time to obtain particular linear functions $C(g) = kg$. If we use the subscript notation for the parameter k in the family of functions $C_k(g) = kg$ as suggested by Šedivý (1976), we fulfill the need to specify explicitly which variable is considered a parameter, and which ones are considered the independent and dependent variables. As well, it may be even more difficult for students to conceive of k as a variable with a specific domain, attend to all its values simultaneously, and conceive of the whole family of linear functions $C(k, g) = kg$ as a mathematical object.

Another example used in this study (see Problem 1, in the Method chapter) is conceiving of the parameter m in the family of quadratic functions $E_m(x) = mx^2 + 2x + 3$. Within this context, we can pose the problem:

> Find the values of m, where $m \in R - \{0\}$, such that the equation $E_m(x) = 0$ has only real solutions.

Unfortunately, in our secondary algebra curriculum we rarely use the subscript notation. Therefore we formulate Problem 1 this way:

> Find the values of m, where $m \in R - \{0\}$, such that the equation $mx^2 + 2x + 3$ has only real solutions.

Assuming that a student with "symbol sense" – a "feel for symbols," "at the heart of competency in algebra", as described by Arcavi (1994), identifies the parameter m, and conceives of the quadratic equation $mx^2 + 2x + 3$, there remains the hardest part yet, requiring both "symbol sense" and knowledge of quadratic equations, linear inequalites, and intersection of sets. The student should interpret the paramenter m as a literal constant, impose the condition of positivity for the discriminant of the quadratic equation, accept the role change for the symbol m (now a variable for the linear inequality), solve the linear inequality $4 - 12m \geq 0$, and intersect the solution set with the domain of the parameter m. The final solution, $A = (-\infty, \frac{1}{3}) - \{0\}$, should have meaning for the student – every value of the parameter m from A instantiates a quadratic equation $E_m(x) = 0$ that has two real solutions. In short, the student should conceive of the family of quadratic equations $E_m(x) = 0$ ($m \in R - \{0\}$) as a mathematical object (Sfard, 1991) at the beginning of the problem, unpack the mathematical processes the mathematical object entails, and reify the new mathematical object $E_m(x) = 0$ ($m \in (-\infty, \frac{1}{3}) - \{0\}$) that satifies the condition imposed by the problem.

The literature on student understanding of parameters points to student difficulty to discriminate between parameters and other variables, reify mathematical objects (Sfard, 1991), and succefully solve problems with parameters especially when the context is unfamiliar (Bloedy-Vinner, 1994; Furinghetti & Paola, 1994; Šedivý, 1976; Ursini & Trigueros, 2004).

The theoretical framework we propose for analyzing student understanding of the concept of parameter has two levels: I) *identifying parameters*; and II) *parameters in action*.

Each level has three categories, corresponding to student actions as observed. When asked to *identify parameters*, students: i) correctly identify parameters; ii) identify "actual" parameters; or iii) identify other variables as parameters, like the independent and dependent variable in a function.

The term "actual" is borrowed from computer science and represents constants or expressions used in places where parameters might have been used in another context. For example, a student may reason this way: a, b, c are parameters in $ax^2 + ax + c = 0$, therefore "m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$."

Across the level *parameters in action*, we have three categories for student actions. When asked to solve the problem with parameters, students: i) solve the problem and check/discuss the solution given the constraints of the problem; ii) apply the right algorithm to solve the problem without considering the constraints of the problem; or iii) cannot solve the problem.

This theoretical framework is a departure from the dichotomy "algebraic-analgebraic" used by Bloedy-Vinner (1994) to analyze students' difficulties with parameters. By treating all the incorrect answers as "analgebraic" we lose access to valuable information with respect to students' difficulties and concequently to ways of overcoming those difficulties. In our framework, the two levels capture those two reifications needed to solve problems with mathematical objects defined with the help of parameters, the second level posing more difficulties to students than the first one. Morevoer, the intermediate categories capture the students' over-reliance on pseudo-empirical abstractions (Piaget, 2001). Piaget (2001) discriminates between pseudo-empirical knowledge abstracted from individual actions on objects, and reflective knowledge abstracted from coordinated actions on objects. For example, in Problem 1, when the student identifies "m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$," we may infer that the parameters are recognized as the coefficients of x^2, x, and x^0 in the mathematical object $ax^2 + ax + c = 0$. As such, the student performs individual actions on the quadratic equation $mx^2 + 2x + 3 = 0$, and therefore relies on pseudo-empirical abstractions. Reflective knowledge may be inferred if the student identifies m as the parameter after conceiving of the family of quadratic equations $mx^2 + 2x + 3 = 0$, where $m \in \mathrm{R} - \{0\}$, poses the condition of positivity of the discriminant to ensure real solutions for quadratic equation, solves the inequality that results from posing the condition of positivity, and intersects its solution with the domain of the parameter. Thus, the student performs coordinated actions on the family of quadratic equations. Moreover, after solving Problem 1, the student conceives of a new mathematical object-the family of quadratic equations with real solutions only, $mx^2 + 2x + 3 = 0$, where $m \in (-\infty, \frac{1}{3}) - \{0\}$. The coordination of actions can be interrupted at any time, for example if a student stops solving Problem 1 after substituting m, 2 and 3 the quadratic formula for solving $mx^2 + 2x + 3 = 0$. We may consider this latter case as another example of generalization via pseudo-empirical abstractions, as the student fails to link it to a new action, at a higher level (Piaget, 2001), in this case the reification of the family of quadratic equations with real solutions only.

METHOD

This study is part of an ongoing study on student understanding of parameters. We report here only the exploratory phase, used to inform our teaching intervention. Participants in this study were 26 college students, enrolled in an Introduction to Proofs class at a university in the United States. All students completed at least the Calculus I course, and at the time of the study they have just began an introduction to logical quantifiers and elementary methods of proof. There was no lesson taught on the topic of parameters, therefore students' knowledge on parameters was acquired prior to this study. To answer the question "What is a parameter?" all students were asked to come prepared to class with written examples of problems with parameters. The task was to identify the parameters, and justify the choice. A whole-class discussion on the concept of parameter was conducted, and every student attempted to answer the question "What is a parameter?" and commented on the other students' previous answers. The instructor (the first author) wrote the students' examples and comments on the whiteboard, took pictures for analysis (see Figure 1, in the Analysis chapter), and collected the students' written answers for analysis. The discussion lasted the whole class period, 50 minutes. The consensus was reached by the students, without the instructor's validation. The next day, a 30-minute questionnaire was administered to all students. Students were asked to solve four problems, and identify the parameters in those problems. We report here on only two of those problems. Problem 1 was inspired from the research literature on parameters (Šedivý, 1976), and Problem 2 from the current mathematics curriculum at the secondary level. We wanted to minimize the role of context in students' difficulties with parameters, therefore we proposed only problems with familiar contexts (e.g., linear and quadratic equations and functions) and familiar tasks (e.g., solving a quadratic equation, graphing a linear function). At the same time, we wanted to compare our students' difficulties with those reported in literature (Problem 1), and to gain insight into our students' understanding of parameters, given their exposure to graphing technologies (Problem 2):

> Problem 1. Find the values of m, where $m \in \mathrm{R} - \{0\}$, such that the equation $mx^2 + 2x + 3 = 0$ has only real solutions.

> Problem 2. Graph the function $f\colon \mathrm{R} \to \mathrm{R}$, $f(x) = kx$, where $k \in \mathrm{R}$.

Students' answers were scored, first using two rubrics ("0" for correct, "1" for incorrect, and re-scored using three rubrics ("0" for correct, "1" for partial correct, "-1" for "incorrect") by two raters, with high inter-rater agreement, measured using Cohen's k statistic (Cohen, 1960), $k = .90$ ($p < .05$). Open coding techniques and procedures described by Strauss and Corbin (1998) were used to develop the theoretical framework used for analysis. A conceptual analysis (Postelnicu & Postelnicu, 2013; Steffe & Thompson, 2000) of the data was performed, with the goal of answering the following research question: What is the students' conceptual understanding of parameters? We tried to infer students' conceptual understanding of parameters by analysing their constructions and actions. The data collected was subjected to repeated linkage processes, and systematic inferences, both inductive and deductive. We adjusted our working hypothesis. i.e., our proposed models to account for students' conceptual understanding of parameters, until the data no longer contradicted our hypothesis. The last viable hypothesis was reported as the students' conceptual understanding of parameters.

ANALYSIS AND RESULTS

Identifying parameters

During the whole-class discussion, the students seemed to agree that parameters are some sort of

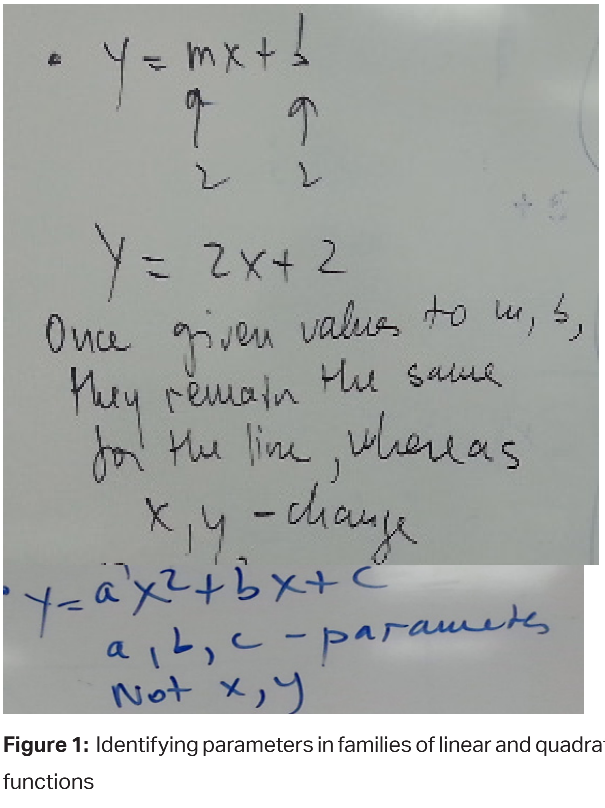

Figure 1: Identifying parameters in families of linear and quadratic functions

"general constants" (Philipp, 1992) or variables that appear in the definition of a mathematical object, but do not affect the structure of the mathematical object. When parameters change their values, the specific mathematical object changes, but keeps its structure (see Figure 1).

As can be seen from Figure 1, during the whole-class discussion, the students could discriminate between parameters and other variables, in the context of linear and quadratic functions.

There was some debate if parameters are variables or constants, illustrated by the following comments:

> "Parameter is the quantity that influences the output of a function and is usually constant. Example 1: m and b are parameters in $y = mx + b$ because they're constants in this case that affect what the function looks like. This is a linear function so it'll always look like a line."

> "[A parameter is] a constant or variable that determines the specific form of the function but not the general nature. E.g., $y = ax^2 + ax + c$ where a, b, and c are all parameters."

> "Parameters are variables that have fixed number. An example is the equation of a line $y = mx + b$, [where] m is a parameter that is equal to the slope of the line, and b is a parameter that is equal to the y-intercept of the line."

The idea of variable hierarchy (i.e., first parameters are instantiated, and only then the mathematical object becomes specific) did not appear explicitly in students' discussions.

After the whole-class discussion, our working hypothesis was that our students conceived of parameters as "general numbers of the second order" (Ursini & Trigueros, 2004), and they could identify parameters in the context of linear and quadratic functions and equations. It was not clear if our students considered the parameters variables or constants. Students' answers on the questionnaires showed that the issue of the nature of parameter – variable or constant that appeared during the whole-class discussion was more problematic (see Table 1).

	Identifies parameters	Identifies "actual" parameters	Identifies other variables as parameters
Problem 1	(N=18) m is the parameter in $mx^2 + 2x + 3 = 0$	(N=3) m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$	(N=5) x is the parameter in $mx^2 + 2x + 3 = 0$
Problem 2	(N=21) k is the parameter in $f(x) = kx$	(N=2) the parameter depends on what value takes k in $f(x) = kx$	(N=3) x is the parameter in $f(x) = kx$

Table 1: Identifying parameters

In Table 1, we present the frequencies of student answers per problem and category across the level *identifying parameters*, accompanied by corresponding examples of student answers. We have already commented on students' answers to Problem 1 from the mid-column, when the students rely on pseudo-empirical abstractions, and identify the parameters based on their places in the symbolic representation of the quadratic equation. With respect to students' answers to Problem 2 from the mid-column, we can infer pseudo-empirical knowledge, too. We interpret the observation that the parameter depends of the value taken by k as the students' need for instantiation. We infer that the students operate with instances of the mathematical object defined with the help of parameters, i.e., one specific linear function instantiated by a specific value of k, and not with the whole family of linear functions. Looking at the students' answers from the last column, when the other variables, x or y were identified as parameters that can affect the mathematical object, equation or function, we infer that the parameters might have been considered variables by students if they attended to the domain of the variable x (at least in the linear function in Problem 2), or might have been considered constants by students if they attended to particular instances or values of x, one at a time. In both situations, a failure to identify the parameters implies that the student operates with a particular instance of the mathematical object defined with the help of parameters. This raises the question whether, in the absence of the reification of the mathematical object, one can perform any operations on the object. The obvious answer is no, and the analysis across the second level, *parameters in action*, supports this answer.

Parameters in action

The students who could not identify the parameters in Problem 1 could not solve the problem. Most students (N=21) tried unsuccessfully to solve it by manipulating it symbolically in various ways (e.g., solving the quadratic equation by factoring, completing the square, using the quadratic formula). All students exhibited a weak competency in algebra (Arcavi, 1994), when they tried to solve the equation and write the solutions explicitly, instead of just posing the condition of positivity of the discriminant and solving the linear inequality obtained. From those 18 students who identified the parameter m in Problem 1, none solved the problem completely. Five students almost solved the problem, without completing the last necessary step – checking that the solution they found ($m < \frac{1}{3}$, sometimes $m < \frac{1}{3}$) fulfils the constraints of the problem ($m \neq 0$). In this case, when $m = 0$, we obtain a degenerated equation. Considering the students' schemes of operations (Piaget, 2001), we may infer that their reifications of the family of quadratic equations from Problem 1 are structurally weak (Sfard, 1991). Indeed, when solving Problem 1, about one third of the students failed the first reification the family of equations $mx^2 + 2x + 3 = 0$, where $m \in \mathrm{R} - \{0\}$ and there is no evidence of a successful second reification of the family of equations $mx^2 + 2x + 3 = 0$, where $m \in (-\infty, \frac{1}{3}) - \{0\}$.

Under the assumption that our students have been exposed to graphing technologies, we did not anticipate difficulties with Problem 2, which required graphing a family of linear functions. In Problem 2, only one student provided the correct graphic representation and highlighted the fact that the line $x = 0$ is not part of the solution, fifteen students graphed several lines, while all the other students graphed only one line. We can infer that the students who graphed only one or several lines conceived only of an instance or several instances of the family of linear functions, respectively. With the exception of one student, all the students failed to represent graphically the family of linear functions as the geometric locus of all the lines passing through the origin, except the line with the equation $x = 0$.

DISCUSSION

In retrospect, during the past five decades, the concept of parameter has remained an elusive one, even for college students. Our study supports previous findings with respect to student difficulty identifying parameters. We believe that this difficulty can be addressed by using subscript notation or logical quantifiers when using mathematical objects defined with the help of parameters. This means that the curriculum ought to be augmented by special topics, like connecting symbolic representations and logical quantifiers with ways of defining mathematical objects with the help of parameters. Identifying parameters is only the tip of the iceberg, the true difficulty lies with *parameters in action*. Of note, Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa (2015) also proposed two levels of algebraic thinking involving parameters, the superior level referring to the "treatment of parameters" in problems requiring higher algebraic competency. Indeed, the real issue seems to be the weak algebraic competency (Arcavi, 1994; Sfard, 1991) inferred from the students' pseudo-empirical knowledge that hinders the reification of mathematical objects, or from the lack of active knowledge, like the knowledge about the role the discriminant of a quadratic equation in Problem 1. Our findings suggest that in spite of the use of graphing technologies, students continue to have difficulty connecting symbolic and graphic representations of mathematical objects defined with the help of parameters, and thus they have difficulty conceiving of those mathematical objects as geometric loci, like in the case of the family of linear functions (family of lines passing through the origin, except the line $x = 0$) in Problem 2. To conclude, the students' difficulty when solving problems with parameters – is the reification of the mathematical objects, reification that is dependent on the students' fluency in unpacking and packing the mathematical processes behind the mathematical objects.

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