Using variation theory to design tasks to support students' understanding of logarithms

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In this paper, we discuss three implementations of a task in which students were asked to generate examples of logarithm expressions equal to a given value. We situate the design of the task in variation theory and in research on learner generated examples, which describe learning as developing students' ways of seeing, particularly in regards to the dimensions of variation and the range of permissible change. The analysis of the three implementations reveals students' understanding of logarithms, as well as what is possible to learn given the task-as-implemented, or the enacted object of learning. We claim that using variation theory in task design can support students in developing important capabilities for reasoning about logarithms in powerful ways.

Keywords: Logarithms, variation theory, learner generated examples.

While exponential functions have been emphasized as a key mathematical understanding in secondary school (Confrey & Smith, 1995), their inverses, logarithmic functions, have received very little attention in the research literature. Besides the significance of logarithms for their relationship to exponential functions, the applications of logarithms to various phenomena, such as sound, earthquakes, and human growth (Wood, 2005), are important in their own right.

According to the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), recently adopted in the United States, the emphasis on the algebra of logarithms in a typical Algebra II course, as well as subsequent courses in mathematics (e.g., precalculus), means that upwards of 2.5 million American students will be expected to engage with and develop an understanding of these ideas each year (National Center for Education Statistics, 2014; National Science Board, 2010). Despite this, there is a dearth of research literature involving student learning of logarithmic functions, in general, and the algebra of logarithms, in particular. Of the literature available, much of the focus is on suggested mathematical and pedagogical approaches to logarithms with no empirical data related to students' learning using these approaches. Weber (2002) concluded that many students do not have a process understanding of exponentiation and logarithms and suggested numerical approaches to encourage the development of these understandings, but not for logarithms directly. Wood (2005) suggested verbal explanations from students about the meaning of logarithmic expressions such as in order to build an understanding of this expression as a numerical value. Confrey and Smith (1995) and Panagiotou (2011) both suggest drawing on the historical development of logarithms as a basis for teaching about logarithms. Confrey and Smith further claim that building the isomorphism between the *counting*, or additive, world and the *splitting*, or multiplicative, world is building the rules of logarithms. The purpose of this study is to address this research gap by exploring a task design that develops students' understanding of logarithms. In the recent ICMI 22 Study, Margolinas (2013) indicated the importance of tasks for generating mathematical activity that afford students the opportunity to encounter concepts and strategies. This study focuses on three iterations of a task designed to elicit students' current understanding of logarithms, as well as lead them to generalizations related to the properties of logarithms.

The task design and analysis is situated within both variation theory, developed by Marton and colleagues (Marton, Runesson, & Tsui, 2004), and research related to learner generated examples, LGEs, (e.g. Watson & Mason, 2005). Variation theory is most concerned with the object of learning, comprised of three aspects: (1) the intended, (2) the enacted, and (3) the lived (Marton et al., 2004). The intended object of learning is what the teacher intends the students to learn at the outset or in the planning of a lesson. The enacted object of learning is what was actually made possible for students to learn in the implementation of a lesson. The lived object of learning is what the students actually did learn at the completion of the lesson, and beyond. Marton and colleagues (2004) defines learning as the development of capabilities, where a capability is described as seeing, experiencing, or understanding something in a certain way. In order to develop a particular capability (way of seeing, experiencing, or understanding), one must simultaneously focus on the critical features of the particular object of learning. A variation theory perspective claims that we can only focus on that which we discern; we can only discern what we experience to vary; we can only experience variation if we have experienced different instances previously and can juxtapose our previous experiences with our current experience simultaneously.

Marton and colleagues (2004) contend that we can only learn that which we experience to vary. In order to ascertain the enacted object of learning, it is necessary to be concerned with what varies and what remains invariant in a learning situation. Marton et al. describe what varies and what remains invariant as a pattern of variation and identify four of these: (1) contrast, (2) generalization, (3) separation, and (4) fusion. The first pattern of variation, contrast, refers to the comparison between what something is and what it is not. For example, in the context of logarithms, $\log 100 = 2$ can be contrasted with $\log_2 8 = 3$ in order to discern which aspects of a logarithm statement can vary. In order to understand what it means for a logarithm to have base ten, students need to experience logarithms that are not base ten. The second pattern of variation is generalization, which refers to experiencing the varying appearances of an object of learning in order to separate it from irrelevant features. For instance, seeing $\log_2 8$, $\log 1000$, $\log_3 27$, and $\ln e^3$ as equivalent log expressions that all equal three, can help students to generalize what it means for a logarithm to be equal to three. The base of the logarithm is an irrelevant aspect here; but the power of the input in terms of the base is significant. The third pattern of variation, separation, involves varying a particular aspect of an object of learning while holding the other aspects invariant. This draws attention to the particular aspect that is allowed to change. The example above held the value of the logarithm invariant while changing the base, which then determined the input. Marton et al. contend that systematically varying certain aspects, while keeping other aspects invariant, can prepare students for various other situations related to the capability in question. Fusion, the last pattern of variation, is the experiencing of all of the critical aspects simultaneously. Through fusion, learners develop the ability to make generalizations that link the critical aspects of an object of learning (Holmqvist, 2011). For instance, discerning the relationship between the base of a logarithm and the input, in order to hold the value of the logarithm invariant, is a result of fusion and experiencing simultaneous changes in both the base and the input of the logarithm.

Drawing on Marton's work, Watson and Mason (2005) suggest that LGEs are an appropriate way to introduce new concepts in mathematics. The use of LGEs, however, may appear to be in conflict with variation theory, as the task designer/instructor concedes control of the presentation of specific examples to the students. Variation theory, however, does not suggest particular ways of arranging for learning, only that variation must be present for discernment. Rather, Marton and colleagues (2004) claim that the particular way of arranging for learning is dependent upon the thing to be learned, and research can be undertaken to determine the most conducive arrangement for student learning of that particular thing. In this sense, then, there is no tension between variation theory and the use of LGEs, as LGEs, when used in conjunction with collaboration, can create the variation in features necessary for discernment.

Watson and Shipman (2008) found that the use of LGEs, in a supportive classroom atmosphere, can successfully introduce new concepts for both advanced and low-achieving learners. Watson and Shipman suggest that while the discernment of critical features of a concept (Marton's *dimensions of variation*) through a set of examples may reveal the structure of the concept, learning through exemplification occurs through discerning the generalization of relationships across the dimensions of variation. Drawing on this work, the task used in this study was developed with the intended object of learning as generalizations related to the properties of logarithms, such as $\log_b(xy) = \log_b x + \log_b y$, $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, and $\log_b x^n = n \cdot \log_b x$, as well $\log_b b^x = x$, which proceeds from the equivalence relationship between logarithmic and exponential statements: $\log_b a = x \leftrightarrow b^x = a$. As will be discussed in the analysis below, over the course of the three implementations of this task, the intended object of learning was shifted based on the insights gained in the first two implementations of the task.

METHODOLOGY

A task that involved learner-generation of examples of logarithm statements was enacted by the first author as the teacher/researcher with three groups of students: two sections of pre-freshman engineering students enrolled in a summer mathematics course, one of which was composed of students who had previously studied calculus (n=13) and the other of which was composed of students who had previously studied precalculus (n=12), and a small problem-solving session comprised of two pre-freshman students who were recruited from a summer precalculus course. The task was enacted with groups of students of varying achievement levels in order to expand on the teacher/researcher's understanding of the variation in students' reasoning about logarithms as revealed by the task. The first enactment was with the group of engineering students who had previously studied calculus; the second enactment was in a problem-solving session with two pre-calculus students; the third enactment was with the group of engineering students who had previously studied precalculus. Each enactment of the task was carried out by the teacher/researcher, and each of the enactments was video-recorded. Students' written work was also collected.

By the time students encounter logarithms in a typical precalculus course in the United States, students have already been introduced to logarithmic functions, the properties of logarithms, and have used logarithms to solve exponential equations (in a typical Algebra II course). Hence, logarithms were not a new concept for these students, but rather a concept that many of the students still struggled with in terms of recalling, applying, and reasoning with and about the properties of logarithms. Prior research also suggests that students tend to struggle with the relationship between logarithms and exponents, as well as the properties of logarithms (Weber, 2002; Wood, 2005). This study, then, expands on previous work related to the use of LGEs to introduce new concepts (Watson & Shipman, 2008) by using LGEs to deepen students' understanding of and reasoning about a previously introduced concept. This research study addressed the following questions:

1) What does the task, as implemented, reveal about students' understanding of logarithms?

2) What is the *enacted object of learning*, or what is possible to learn, given the task-as-implemented?

Task design and data analysis

As the task itself cannot be separated from its enactment, the way in which the task was implemented in each iteration varied. The variation in the task-as-implemented was influenced by the teacher/researcher's insights garnered from the previous implementation(s), as well as the particular *space of variation* opened in that implementation. Despite the differences in the implementation of the task among the three groups of students, commonalities between the *spaces of variation* opened in each of the implementations revealed much about students' understanding of logarithms and served to focus the teacher/researcher's *intended object of learning* for later implementations, as well as to refine the task.

The basic structure of the task in all three implementations involved (1) individual student generated examples, (2) group assessment and group generation of examples, (3) collective class organization/categorization of the student generated examples, and (4) generalization. In each iteration of the task, the teacher asked the students to write a log expression that was equal to three. Then the students were asked to write another. The student generation of examples was meant to draw on students' prior knowledge of logarithms and elicit students' understanding of logarithms, in general, and the value of a logarithm, in particular. The students were then arranged into groups of two to four students to share what they wrote with each other. The teacher distributed a set of index cards to each group, asking them to write a different log expression that equalled three on each card. This potentially required that the group generate additional examples as their original examples may have been duplicative. As groups of students finished writing their examples on the cards, the teacher asked them to tape their cards on the board. The teacher then gathered the students around the cards on the board and asked them to collectively categorize the cards. The collective student sorting of the log statements allowed the teacher to gain insight into what critical features of logarithms the students were attending to, as well as the structure of logarithms, as perceived collectively by the students. A discussion of the students' categories followed, along with additional student generation of examples of logarithm expressions equal to two and five, and finally, student generalizations.

Each of the three implementations were analysed using the framework of variation theory in an attempt to understand *the intended object of learning* and *the enacted object of learning* (or *the space of variation*). While this framework served to answer the second research question most directly, the analysis also addressed what was revealed about students' understanding of logarithms. The variation in the student generated examples, as well as what was revealed about students' prior understanding of logarithms, served to answer the first research question.

RESULTS

This section describes the insights garnered by the teacher/researcher primarily during the first two implementations of the task and the refinement of the task through the third implementation. The intended object of learning shifted from the properties of logarithms to the equivalence relationship between logarithms and exponents and the generalization $\log_b b^x = x$ to generate additional examples, equal to any given number, x. In all three implementations of the task, the categorization of the LGEs, largely by the base, showed that the base of the logarithm and the input of the logarithm, were brought to the fore as aspects of a logarithm that were possible to change, and the range of permissible change for each of these aspects was explored, to some extent. While all students were able to generalize the process of writing a log statement equal to a given number, higher achieving students were more readily able to recognize a generalization comprised of a single statement, such as $\log_b b^x = x$. Extending the task to include combinations of logarithm expressions may create an opportunity for students to verify the properties of logarithms and explore the relationship between the properties of logarithms and the properties of exponents.

The intended object of learning

The intended object of learning in the first implementation of this task was the generalizations related to the properties of logarithms, such as $\log_b(xy) = \log_b x + \log_b y$, $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, and $\log_b x^n = n \cdot \log_b x$, as well as the equivalence relation: $\log_b a = x \leftrightarrow b^x = a$. We had anticipated LGEs of the form $\log_2 8 = 3$, but had also anticipated that some students would extend their thinking to include an example such as $\log_{10} 10 + \log_{10} 100 = 3$. Goldenberg and Mason (2008) discuss how example generation is not just a memory lookup, but rather found that students often start with some known example(s) and through combinatorial approaches can construct new examples. As students began generating examples in the first part of the task, the teacher/researcher quickly realized that while the combinatorial approach was not appearing, other examples that had not been anticipated were being generated by the students, such as examples with fractional bases.

In the second implementation the generalization $\log_b b^x = x$ was the intended object of learning, with b and x as the *dimensions of variation*. We were also interested in how to extend the task in such a way that allowed students to gain access to examples that included a combinatorial approach, as they did not appear in either the first implementation, or in the first portion of this problem solving session. The students were asked to use the logarithm statements that they had already generated and combine them in some ways (using addition, subtraction, multiplication, or division) to generate other logarithm statements equal to three. While this allowed students to recall and verify the properties of logarithms that they had previously learned, students could not discern why these properties were valid and they were unclear about the relationship between the properties of logarithms and the properties of exponents.

The third implementation of the task was a refined version, based upon our insights from the first two implementations of the task. In this implementation, the intended object of learning focused on the equivalence relationship between logarithmic and exponential statements: $\log_b a = x \leftrightarrow b^x = a$, as well as the general logarithm statement $\log_b b^x = x$ for generating logarithm statements equal to a given value, x. Students were not asked to combine logarithm expressions to generate new statements equal to a given value in this task. Rather, the combination of logarithms was separated into a distinct, but linked task, for the sake of time and depth.

Categorization of LGEs

After students shared their generated examples in a small group and on the board, the whole class categorized the examples they had generated. The first class separated the examples into four categories, determined largely by the base of the logarithm: (1) whole number, (2) base 10, (3) fractional bases, (4) special, and (5) wrong. The students in the third implementation were not able to use graphing calculators (that can calculate logarithm expressions with various bases), but rather used simple four-function calculators to calculate a larger power of a number. As suspected, perhaps due to the change in the type of calculator available for a tool, fractional bases for the logarithm expressions did not appear in the third implementation of the task. In the first portion of the second implementation, the students still organized their examples according to the base, first separating them into even and odd categories, then deciding to list them out from statements with base two up to statements with base ten. In the second part of the session, the students were asked to generate combinations of logarithm expressions equal to three and reorganize the examples, in consideration of the additional examples. The students chose, then, to categorize the LGEs by operation. The students in the third implementation also categorized the LGEs according to the base of the statement, however, they included four subdivisions or "branches": (1) Odd, (2) Even, (3) Base 10, and (4) Fractional, under the "big tree" of the equivalence relationship $\log_b a = x \leftrightarrow b^x = a$. This is similar to the students in the first implementation who also explained how all of their correct examples "followed the same rule," after they had determined a generalization for logarithms with whole number bases. The students in the third implementation also included a "wrong" category, a "natural log" category, which were actually exponential statements that included natural log in the exponent (e.g. $e^{\ln 3} = 3$), and a "unique" category, which included only the statement $\log_3 (\frac{729}{27}) = 3$.

In all three implementations of the task, the students' categorization of the LGEs showed that the base of the logarithm, as well as the input of the logarithm, was brought to the fore as aspects of a logarithm that were possible to change. The combinatorial approach did not spontaneously appear. In terms of revealing student understanding, however, Watson and Goldenberg (2008), point out that, "the fact that [students] don't display an example does not imply that it is not within their accessible [example] space, just that they have not perceived a reason to express it" (p. 189). As such, it appears that the task as enacted did not cue or trigger students to think of examples using the combinatorial approach. The *enacted object of learning*, in the first and third implementation, as well as the first portion of the second implementation, was restricted to the generalization $\log_b a = x \leftrightarrow b^x = a$, the equivalence relationship between logarithm and exponential statements, and the generalization $\log_b b^x = x$, which expresses the structure of the correct logarithm statements.

Range of permissible change

The LGEs served to reveal students' understanding of the *range of permissible change*, particularly in the base of the logarithm statement. As mentioned above, the combination of logarithm statements did not arise spontaneously, indicating that students did not, in that instance, recognize combinations of statements as within the *range of permissible change* for the structure of a logarithm expression equal to three. Without this range of LGEs, students tended to categorize the examples of logarithm statements according to the value of the base, despite the common structural form. Their choice to separate logarithm statements with a whole number base from those with base ten was perhaps indicative of their greater familiarity with base ten, or their understanding of the common log as somehow more important than logs in other bases. We did not anticipate the use of fractional bases students' generated examples; this may have been related to students' graphing calculator usage while generating their examples, particularly since this did not occur in the subsequent implementations when students were restricted to four function calculators. Students' choice of separating base e and base π logarithm statements as "special" could perhaps be indicative of students' sense of e and π as symbols that represent something other than a specific number. Students' recognition, however, of $\log_1 1 = 3$, $\log_0 0 = 3$, and $\log_{-1} -1 = 3$ as "wrong" logarithm statements served to restrict the *range of permissible change*. Student attempts at both justifying these statements and explaining why these logarithm statements were incorrect created an opportunity to deepen students' understanding of both exponents and logarithms.

Only a single example began to directly confront the *range of permissible change* for the input of the logarithm. The students seemed to recognize that the input of the logarithm would change, dependent on the base of the logarithm, but failed to see the range of possibilities in writing the input. In the third implementation, the example $\log_3(\frac{729}{27}) = 3$ was placed in the "unique" category. The statement $\log_3(\frac{729}{27}) = 3$ is equivalent to the statement $\log_3 27 = 3$. Therefore, it has the same structure as the other LGEs, but the students did not recognize it as such. This indicates that students could use more exposure to variation in the input of the logarithm. This was also the only instance when combinations of logarithm statements arose spontaneously. Two students in the third implementation insisted that the way that you would deal with this statement is to rewrite it as $\log_3 729 - \log_3 27 = 6 - 3 = 3$. Thus, students recalled the logarithm property $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, and applied it, but this appeared to overshadow a more flexible and efficient means of simplifying the statement to show its equivalence to three.

Generalizations

In the last part of the task, the teacher/researcher asked the students to write a logarithm statement that was equal to any number. Some students generated two separate statements that was often a restatement of the equivalence relation: $\log_b a = x \leftrightarrow b^x = a$. Others were able to succinctly describe the generalization as $\log_b b^x = x$. Being able to symbolically write down a generalization did not indicate that students would be successful at verbalizing the generalization. For instance, Evette, in the third class, wrote the symbolic statement $\log_b a = x \leftrightarrow b^x = a$, and the verbal statement: "That 'any number' that you want it to equal, must be raised to the base value." While Evette correctly wrote the symbolic statement, she incorrectly stated that the exponent would be the base value. It is also ambiguous what she is referring to with the use of "it". This is perhaps related to a lack of opportunities to explain and communicate verbally in the mathematics classroom.

Lower achieving students had more difficulty than higher achieving students in discerning a generalization comprised of a single statement, such as $\log_b b^x = x$. This is perhaps related to an underdeveloped sense of variable and equality, and the failure to recognize the substitution of equivalent expressions. This could also be related to a preference of seeing each "part" of the logarithm statement (the base, the input, and the output) as distinct and a failure to fuse these critical aspects together.

Combinations of logarithm expressions

The expansion of the example space to include combinations of logarithm expressions in the second implementation served to broaden the *space of variation* when compared to the first and third implementations. Through opening up the variation of the statement to include combinations of logarithm expressions, it becomes possible to discern both *how* the logarithm expressions can be combined and the ways in which the *dimensions of variation* are related within a given statement. Watson (2000) described these two ways of seeing pattern as 'going with the grain,' indicating a recursive continuation of pattern to generate more instances that may not indicate structure (in this case, *how* the logarithm can be combined), and 'going across the grain', a metaphor that indicates the revelation of the internal structure itself (here, the relationship between the *dimensions of variation*). Thus, this opening of the *space of variation* has the potential to provide students with the opportunity to "see" the relationship between logarithms and exponents in ways that they had, perhaps, not experienced before. One of the students wrote about what he had learned during the problem solving session:

> The exponents in exponential equations are used as the values for logarithmic functions. For example:
>
> $$5^{②} + 5^{①} = 5^{③}$$
>
> $$\log_5(25) + \log_5(5) = 3$$
>
> $$② + 1 = ③$$

While the exponential statement is erroneous, this student is beginning to discern variation 'across the grain', and we would argue, is on his way to developing a certain way of seeing logarithms, and hence developing important capabilities for reasoning about logarithms in powerful ways.

CONCLUSION

Based on the three iterations of this task, using LGEs with students who have had previous exposure to a concept can serve to reveal students' understanding of the *dimensions of variation* of a concept, as well as the *range of permissible change* in those aspects. The use of LGEs, in this particular task, revealed students' understanding of the *range of permissible change* in the base of the logarithm and led to opportunities to connect exponential and logarithmic functions. Holding the value of the logarithm statement invariant (for a time) created an opportunity for students to explore what happens when a single aspect, namely – the base of the logarithm, is varied.

While this task draws students' attention to the relationship among the *dimensions of variation* in a logarithm statement, as well as the *range of permissible change* of those dimensions, further consideration of how tasks can be designed to generate and develop an understanding of the relationship between combinatorial properties of exponents and logarithms is needed to more fully develop students' facility with logarithms.

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