Connections between algebraic thinking and reasoning processes

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The aim of the present study is to investigate the relationship of algebraic thinking with different types of reasoning processes. Using regression analyses techniques to analyze data of 348 students between the ages of 10 to 13 years old, this study examined the associations between algebraic thinking and achievement in two tests, the Naglieri Non-Verbal Ability Test and a deductive reasoning test. The data provide support to the hypothesis that a corpus of reasoning processes, such as reasoning by analogy, serial reasoning, and deductive reasoning, significantly predict students' algebraic thinking.

Keywords: Algebraic thinking, cognitive mechanisms, reasoning processes.

INTRODUCTION

In recent years, researchers, policy makers and curriculum designers have recommended that algebraic thinking should become central to all students' mathematical experiences across K-12 grades (e.g., NCTM, 2000; RAND Mathematics Study Panel, 2003). The realization of this need stems primarily from the fact that algebra and algebraic thinking are closely linked to the development, establishment and communication of knowledge in all areas of mathematics, including arithmetic, geometry and, statistics (NCTM, 2000). Secondly, students' abrupt and isolated introduction to algebra in the middle school has led them to experience difficulties in understanding core algebraic concepts (Cai & Knuth, 2005).Thirdly, it has been argued that the mere focus of elementary mathematics on arithmetic and computational fluency deprives the conceptual development of mathematical ideas (Blanton & Kaput, 2005). Fourthly, the call for reconceptualising the nature of school algebra across all grades is underlined by the belief that algebraic thinking is within the conceptual reach of all students. This belief is supported by several research findings which offer evidences that as early as the elementary grades students are able to develop algebraic thinking in supportive classroom environments (e.g., Radford, 2008). Moreover, available research provide insights into appropriate pedagogical factors, such as curriculum materials, technological tools, and instructional strategies that facilitate this development (e.g., Blanton & Kaput, 2005).

Despite the considerable advances in the field, still realizing and achieving the goal for developing algebraic thinking as early as the elementary grades is challenging. The NCTM's research agenda (Arbaugh et al., 2010) highlighted that a main topic of focus is the identification of mathematical concepts and reasoning processes which facilitate the learning of algebra. English (2010) also stressed that one of the main priorities in the field of mathematics education is to define the key mathematical understandings, skills, and reasoning processes that students need, in order to succeed in mathematics. In this context, the present study aims to unfold the relationship between algebraic thinking and specific reasoning processes. This analysis might provide useful insights onto the skills which enable younger students to think algebraically. Students of the 4^{th}, 5^{th}, 6^{th} and 7^{th} grades were selected, in order to illuminate the ways by which abilities involved in reasoning processes might facilitate or restrict algebraic thinking within this age range.

THEORETICAL FRAMEWORK

The notion of algebraic thinking

Several researchers made efforts to analyze the nature and content of algebraic thinking, focusing on what individuals do and the way in which their abilities for generalizing and using symbols develop. One of the most influential developments of the past decades in respect to conceptualizing the notion of algebra as a multidimensional activity is Kaput's theoretical model. Kaput (2008) specified that there are two core aspects of algebraic thinking: (i) making generalizations and expressing those generalizations in increasingly, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In the case of the first aspect, generalizations are produced, justified and expressed in various ways. The second aspect refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. Kaput (2008) asserted that these two aspects of algebraic thinking denote reasoning processes that are considered to flow through varying degrees throughout three strands of algebraic activity: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages for describing generalizations.

This conceptualization breaks down the wide field of algebraic thinking into major components of mathematical activity. Furthermore, Kaput's (2008) ideas articulate ways in which algebraic activities might be applied both in early and secondary school algebra contexts.

Algebraic thinking and reasoning processes

The view of algebraic thinking reported above focuses on the establishment of generalizations, taken to mean the detection and expression of structure and a growing understanding of symbolization. This approach raises the question: "which are the cognitive mechanisms that regulate this process?". English and Sharry (1996) show that analogical reasoning constitutes an essential mechanism when students resolve algebraic tasks. Specifically, they describe analogical reasoning as the mental source for extracting commonalities between relations and constructing mental representations for expressing generalizations. For example, the action of noticing differences and commonalities among different expressions of equations is considered as cognitive in nature and includes the formulation of a generalized concept that does not completely coincide with any of its particular cases.

Likewise, Radford (2008) developed a definition of the process of generalizing a pattern which unfolds the involvement of various forms of reasoning:

> Generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some particulars (say $p_1, p_2, p_3, \ldots, p_k$); extending or generalizing this commonality to all subsequent terms ($p_{k+1}, p_{k+2}, p_{k+3}, \ldots$), and being able to use the commonality to provide a direct expression of any term of the sequence. (p. 84)

As the quotation suggests, this process first involves the identification of differences and similarities between the parts of the sequence – described as analogical reasoning by English and Sharry (1996). Then the commonality founded is generalized through predicting a plausible generalization. This stage is considered by Rivera and Becker (2007) as abductive in nature since it is abductive reasoning that boosts conjecturing and adopting a hypothesis that is considered testable. Finally, the tested commonality becomes the basis for inducing the generalized concept of the sequence. Here, the role of inductive reasoning is considered as pivotal (Ellis, 2007).

The role of processes of induction and deduction has also been highlighted by recent literature. Ayalon and Even (2013) emphasized the role of inductive reasoning when students investigate algebraic expressions. Martinez and Pedemonte (2014) have shown that a prerequisite for linking inductive argumentation in arithmetic and deductive proof in algebra is the co-existence of arithmetic and algebra for supporting the arguments developed within an argumentation.

METHODOLOGY

Research Question

The purpose of the present study is to investigate the way specific reasoning processes influence achievement in tasks that examine their algebraic thinking. Specifically, the present study addresses the following question: *Is there a relation between specific reasoning processes and individuals' algebraic thinking abilities?*

Participants

The participants were 348 students that were selected by convenience from four different schools. The students were divided to four age groups: 55 were students of Grade 4 (10 years old), 89 were students of Grade 5 (11 years old), 101 were students of Grade 6 (12 years old) and, 120 were students of Grade 7 (13 years old). Taking into consideration the fact that the data collection instruments would be the same for all of the participants of the study, no younger or older groups of students were selected. On the one hand, third grade students would not be able to manipulate the tasks, probably due to developmental reasons and absence of experience. On the other, eighth grade students were considered as more skillful in solving algebraic tasks due to their intensive involvement in algebra courses.

The tests

The participants were tested with three tests. Forty minutes were allowed to complete each of the three tests.

Algebraic thinking test

The test consisted of 25 tasks that were adapted from previous research studies related to the notions of algebra and algebraic thinking or algebraic proof (e.g., Blanton & Kaput, 2005; Mason et al., 2005) and past studies that evaluated students' mathematical achievement in international or national level (e.g., TIMSS, 2011; NAEP, 2011; MCAS, 2012). These were accordingly categorized into four groups:

(a) The use of arithmetic as a domain for expressing and formalizing generalizations (generalized arithmetic). These tasks involved solving equations and inequalities. The participants had to treat equations as objects that expressed quantitative relationships, without any reference to the meaning of the symbols.

(b) Generalizing numerical patterns to describe functional relationships (functional thinking). These tasks required finding the n^{th} term in patterns and functional relationships and expressing them in a verbal, symbolic or any other form.

(c) Modeling as a domain for expressing and formalizing generalizations: These tasks required the expression and formalization of generalizations by analysing information that are presented verbally, symbolically or in a table.

(d) Algebraic proof: These tasks reflected different activities and associated abilities of algebraic proof. For example, one of these tasks required the use of a generalization that was previously established (what is the sum of two odd numbers) for building a new generalization (what is the sum of three odd numbers).

The first three groups reflected the three strands of algebra as these were described by Kaput's (2008) theoretical framework. The fourth group was added to the test addressing the strand of algebraic proof. The examination of the construct validity of the items in the test to measure the factors of algebraic thinking was assessed through Confirmatory Factor Analysis using the MPLUS statistical package. The results indicated that the data fit the model well (CFI=0.95, x2=103.345 df=131, x2/df=1.19, RMSEA=0.03), verifying the structure of the proposed model. Table 1 presents examples of tasks for each of the four categories.

The Naglieri Non-Verbal Ability Test (NNAT)

The NNAT measures cognitive ability independently of linguistic and cultural background (Naglieri, 1997). There are seven different levels of the test corresponding to different age-groups of students. The test is a matrix reasoning type of exam that contains patterns formed by shapes that are organized into designs. All the tasks are multiple choice and students are asked to choose the answer that best completes the pattern.

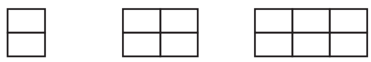
Generalized Arithmetic	The sum ***245676 + 535731*** is odd or even number? Explain your answer.
Functional thinking	Bill is arranging squares in the following way. How many squares there will be in the 16^{th} figure? Figure 1 Figure 2 Figure 3 
Modeling as a domain of expressing and formalizing generalizations	Joanna will take computers lesson twice a week. Which is the best offer? **OFFER A:** €8 for each lesson **OFFER B:** €50 for the first 5 lessons of the month and then €4 for every additional lesson
Algebraic Proof	What is the sum of three odd numbers?

Table 1: Examples of tasks in the algebraic thinking test

The NNAT was selected among other tests that are used extensively for assessing students' cognitive ability due to the fact that the NNAT includes different categories of questions which reflect different types of reasoning skills. Specifically, it contains four different groups of questions: pattern completion, reasoning by analogy, serial reasoning and spatial visualization.

Based on available mathematics education literature, the reasoning processes that seem to be related to algebraic thinking and at the same time are measured by the NNAT are:

(a) Reasoning by analogy: In this category of items, students have to recognize commonalities between several geometric shapes and determine which answer is correct by focusing on how the objects change as one moves across the rows and down the columns in the design. Correspondingly, English and Sharry (1996) have described as analogical reasoning the process where students map similarities between algebraic expressions.

(b) Serial reasoning: The items in serial reasoning matrices are constructed using a series of shapes that change across the row horizontally and the columns vertically throughout the design. As the design moves down the matrix it also moves one position to the right, creating a series of designs that changes over the matrix. Students have to identify where the sequence finishes and starts again from a different starting point. The strategy that students follow in this kind of items shares common features with inductive reasoning. According to mathematics education literature, inductive reasoning is pivotal when students explore pattern sequences (e.g., Rivera & Becker, 2007). In these tasks students have to make generalizations, based on recognizing that a series of numbers or figures constitute a sequence that follows a specific rule.

The NNAT test' reliability was tested with norms based on a sample of more than 100,000 students (Naglieri, 1997). In this study, the internal consistency of scores measured by Cronbach's alpha was satisfactory for the NNAT test (a=0.84).

Deductive Reasoning Test

A test on deductive reasoning was constructed, guided by existing theory and research on deductive reasoning. In particular, items in this test were adapted from a test that was used by Watters and English (1995). This test was considered as appropriate due to the fact that it was used and validated for measuring deductive reasoning among students that were approximately of the same age as the participants in the current study. In Watters and English's study, students' performance in the deductive reasoning test was related to their performance in scientific problem solving. In our case, students' performance in the deductive reasoning test will be related to their performance in algebraic thinking. The items in this test represented 10 syllogisms which requested the students to reason deductively. This process included the analysis of premises that describe formal truth relationships, without reference to their empirical or practical truth value and the extraction of a logical fact, result or consequence. The internal consistency of scores measured by Cronbach's alpha was satisfactory for this test (a=0.79).

Analysis

The quantitative analysis of the data was carried out using the SPSS statistical package. Pearson correlation analysis and Regression analyses were performed. This study assumes that reasoning processes (as these are indicated by available literature) might predict algebraic thinking abilities. To test this assumption, Regression analysis was selected since this kind of analysis informs on the way one or more independent variables predicts the variance of a dependent variable.

The assumptions of multilinear regression are met since the Tolerance and VIF values were for all of the independent variables close to 1 (.972, .863, .876 and 1.03, 1.16, 1.14). This fact indicates that the multicollinearity and singularity assumptions are met. Moreover, standardized predicted X standardized residuals plot showed that the residuals did not violate the homoscedasticity of residuals and linearity assumptions.

RESULTS

The question of the present study addressed the relationship between algebraic thinking abilities and specific reasoning processes. Therefore, a correlation analysis was conducted in order to find out whether algebraic thinking and abilities involved in reasoning processes are significantly correlated. According to Pearson indicator, there is a statistically significant correlation between the individuals' achievement in the algebraic thinking test and the NNAT test (R=0.510, p=0.000<0.05). Moreover, the results show that there is a statistically significant correlation between the achievement in the algebraic thinking test and the deductive reasoning test (R=0.278, p=0.000<0.05). These results support previous reports which denoted that successful engagement with algebraic tasks involves several types of reasoning processes.

The nature of the relationship between algebraic thinking and specific reasoning processes was further explained by conducting regression analyses. Specifically, the analysis examined the way in which the achievement in the NNAT test and the deductive reasoning test (control variables) predict the achievement in the algebraic thinking test (depended variable).

Table 2 presents the results of the regression analysis. The B is the regression coefficient and represents the change in the outcome resulting from a unit change in the predictor, whereas, the beta coefficient (β) is the standardized version of the B coefficient where all variables have been adjusted to standard score form (Field, 2005). As the R-square shows, a percentage of 54.2% of the variance can be explained by the independent variables NNAT and deductive reasoning. This result indicates that as achievement in the two tests increases, the total achievement in the algebraic thinking test also increases. NNAT categories and deductive reasoning are indicated as predictors of algebraic thinking abilities. In order to further examine this relationship, multiple regression analysis was conducted with criterion (depended variables) the total achievement in the algebraic thinking test and predictors (independent variables) the abilities in three reasoning processes: reasoning by analogy, serial reasoning, and deductive reasoning. The results of the multiple regressions are presented in Table 3. According to these, all of the three reasoning processes exert a significant influence on the prediction of individuals' achievement in algebraic thinking.

Algebraic thinking	B(SE)	B
NNAT categories	.391 (.040)	.473*
Deductive reasoning	.471 (.122)	.188*

R^2=.542 , *p=.000

Table 2: Regression Analysis of the achievement in NNAT test and the deductive reasoning test with dependent variable the achievement in the algebraic thinking test

The data show that the factor with the greatest effect on the prediction of achievement in algebraic thinking tasks is reasoning by analogy (β=.308). Serial reasoning also seems to be a significant predictor of individuals' total achievement in the algebraic thinking test (β=.238). Serial reasoning addresses the recognition of sequences and finding changes in the sequence. The abilities involved in the serial reasoning tasks share common features with the abilities involved in activities with pattern. According to the model, deductive reasoning (β=.180) explains a respectable proportion of variance in individuals' total achievement in the algebraic thinking test. It is anticipated that the effect of deductive reasoning could be higher if deductive reasoning was measured through non-verbal methods, as in the NNAT. The deductive reasoning test was not a language-free test of ability. Students from different linguistic groups where tested through a test that involved logical premises written in Greek. In contrast, the NNAT is not dependent on verbal abilities.

Algebraic thinking	B(SE)	B
Reasoning by analogy	.883 (.161)	.308*
Serial reasoning	.962 (.229)	.238*
Deductive reasoning	.452 (.123)	.180*

R^2=.544 , *p=.000

Table 3: Regression Analysis of the achievement in each of the three reasoning processes with dependent variable the achievement in algebraic thinking test

DISCUSSION

As English (2010) suggested, a priority for mathematics education research is the definition of fundamental skills and reasoning processes which enhance students' efforts for achieving understanding in mathematical learning. NCTM (Arbaugh et al., 2010) also emphasized the need for coherently defining mathematical concepts and reasoning processes that enable individuals to develop algebraic thinking. Motivated by this argument, the present study aimed at examining students' algebraic thinking abilities with different reasoning processes. The importance of this study also lies in the fact that aims to capture a more holistic view of the algebraic thinking concept, by using Kaput's theoretical model as a referent point.

The findings obtained from the quantitative data indicate that students' achievement in algebraic thinking tasks is influenced by reasoning by analogy, serial reasoning, and deductive reasoning. Reasoning by analogy appears to be the factor with the most significant effect on algebraic thinking abilities. This result lends support to the findings of previous studies which indicated a relationship between algebraic thinking and analogical reasoning. According to English and Sharry (1996), analogical reasoning provides the basis for algebraic abstraction in tasks where students have to identify similarities and differences between a group of algebraic equations. Therefore, it seems reasonable to argue that analogical reasoning constitutes a basic process for succeeding in tasks of identifying structure and relationships.

Our findings also suggest that serial reasoning has a significant role in algebraic thinking. This result might be attributed to the fact that serial reasoning shares common features with inductive reasoning. This ability has been reported by related literature as crucial for the engagement in activities of determining pattern rules, recognizing the part that is repeated, and finding not observable terms (e.g., Rivera & Becker, 2008). Deductive reasoning also seemed to account for some variance in the algebraic thinking test. One plausible explanation for this result might be the fact that deductive reasoning is associated to the notion of proof. According to Blanton and Kaput (2005) activities such as using generalizations to build other generalizations, generalizing mathematical processes, and testing conjectures, and justifying reflect categories of algebraic thinking that are interwoven with proof.

As Kaput (2008) recommended, several reasoning processes run through algebraic activities. The analysis of the data provides empirical validation of these ideas and sheds some light on the crucial issue of which might these processes be and what is their nature. Furthermore, these findings can be used for informing educators about the sources that act as means for mastering different forms of algebraic thinking. Future teaching interventions might support the development of different types of reasoning, in order to test advances in developing algebraic thinking. According to English (2011), the key to reach advanced forms of reasoning is the creation of cognitively demanding learning activities in appropriate contexts. The present study seemed to provide some evidence regarding the associations between students' algebraic thinking and fundamental reasoning processes. Yet, one limitation of the study is the context in which it was conducted. The relationships found in the present study need to be further examined, in other educational systems in which algebraic thinking might be approached through the mathematics curriculum in a different way. In respect to the methodology, a limitation of the study seems to be the fact that analogical reasoning and inductive reasoning were examined with a non-verbal test and deductive reasoning was tested with a verbal test. Future research could examine these associations within tests that follow similar design and features. Also, students of lower primary grades or higher secondary grades could be added to the sample. Finally, future research could identify the associations of algebraic thinking with other core processes and mental operations, in order to approach a wider picture of the algebraic thinking concept.

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