HAL
archives-ouvertes.fr

CONJUGACY GROWTH SERIES OF SOME INFINITELY GENERATED GROUPS

Roland Bacher, Pierre de la Harpe

▶ To cite this version:

Roland Bacher, Pierre de la Harpe. CONJUGACY GROWTH SERIES OF SOME INFINITELY GENERATED GROUPS. International Mathematics Research Notices, Oxford University Press (OUP), 2016, pp.1-53. hal-01285685v2

HAL Id: hal-01285685
https://hal.archives-ouvertes.fr/hal-01285685v2

Submitted on 15 Jun 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CONJUGACY GROWTH SERIES OF SOME INFINITELY GENERATED GROUPS

ROLAND BACHER AND PIERRE DE LA HARPE

ABSTRACT. It is observed that the conjugacy growth series of the infinite finitary symmetric group with respect to the generating set of transpositions is the generating series of the partition function. Other conjugacy growth series are computed, for other generating sets, for restricted permutational wreath products of finite groups by the finitary symmetric group, and for alternating groups. Similar methods are used to compute usual growth polynomials and conjugacy growth polynomials for finite symmetric groups and alternating groups, with respect to various generating sets of transpositions.

Computations suggest a class of finite graphs, that we call partition-complete, which generalizes the class of semi-hamiltonian graphs, and which is of independent interest.

The coefficients of a series related to the finitary alternating group satisfy congruence relations analogous to Ramanujan congruences for the partition function. They follow from partly conjectural "generalized Ramanujan congruences", as we call them, for which we give numerical evidence in Appendix C.

Pour le parfait flâneur, pour l'observateur passionné,
c'est une immense jouissance que d'élire domicile
dans le nombre, dans l'ondoyant dans le mouvement,
dans le fugitif et l'infini.
(Baudelaire, in *Le peintre de la vie moderne* [Baud–63].)

1. Explicit conjugation growth series

Let G be a group generated by a set S. For $g \in G$, the **word length** $\ell_{G,S}(g)$ is defined to be the smallest non-negative integer n for which there are $s_1, s_2, \dots, s_n \in S \cup S^{-1}$ such that $g = s_1 s_2 \cdots s_n$, and the **conjugacy length** $\kappa_{G,S}(g)$ is the smallest integer n for which there exists h in the conjugacy class of g such that $\ell_{G,S}(h) = n$. For $n \in \mathbf{N}$, denote by $\gamma_{G,S}(n) \in \mathbf{N} \cup \{\infty\}$ the number of conjugacy classes in G consisting of elements g with $\kappa_{G,S}(g) = n$ (we agree that

Date: June 15, 2016.

2000 *Mathematics Subject Classification.* 20F69, 20F65.

Key words and phrases. Conjugacy growth series, finitary symmetric group, wreath product, alternating group, partition function, Ramanujan congruences.

The visit of the first Author in Geneva, where much of this work was done, was supported by the Swiss National Science Foundation.

$0 \in \mathbf{N}$). Assuming that the pair (G, S) satisfies the condition

$$\text{(Fin)} \qquad \gamma_{G,S}(n) \ \text{ is finite for all } \ n \in \mathbf{N},$$

we define the **conjugacy growth series**

$$C_{G,S}(q) \ = \ \sum_{n=0}^{\infty} \gamma_{G,S}(n) q^n \ = \ \sum_{g \in \mathrm{Conj}(G)} q^{\kappa_{G,S}(g)} \ \in \ \mathbf{N}[[q]] \, .$$

Here $\sum_{g \in \mathrm{Conj}(G)}$ indicates a summation over a set of representatives in G of the set of conjugacy classes of G. The **exponential rate of conjugacy growth** is $H^{\mathrm{conj}}_{G,S} \ = \ \limsup_{n \to \infty} \frac{\log \gamma_{G,S}(n)}{n}$; note that $\exp(-H^{\mathrm{conj}}_{G,S})$ is the radius of convergence of the series $C_{G,S}(q)$.

In case G is generated by a finite set S, Condition (Fin) is obviously satisfied, so that the formal series $C_{G,S}(q)$ and the number $H^{\mathrm{conj}}_{G,S}$ are well defined; they have recently been given some attention, see e.g. [AnCi], [BCLM–13], [Fink–14], [GuSa–10], [HuOs–13], [Mann–12, Chap. 17], [PaPa–15], [Rivi–10]. The subject is related to that of counting closed geodesics in compact Riemannian manifolds [Babe–88], [CoKn–04], [Hube–56], [Knie–83], [Marg–69].

When S is finite, denote for $n \in \mathbf{N}$ by $\sigma_{G,S}(n) \in \mathbf{N}$ the number of elements $g \in G$ with $\ell_{G,S}(g) = n$. In this situation, it is tempting to compare the series $C_{G,S}$ to the **growth series**

$$L_{G,S}(q) \ = \ \sum_{n=0}^{\infty} \sigma_{G,S}(n) q^n \ = \ \sum_{g \in G} q^{\ell_{G,S}(g)} \ \in \ \mathbf{N}[[q]] \, .$$

For finite series, e.g. for finite groups, we rather write "conjugacy growth polynomial" and "growth polynomial".

The first purpose of the present article is to observe that there are groups G which are not finitely generated, and yet have interesting series $C_{G,S}(q)$ for appropriate infinite generating sets S. Groups of concern here are locally finite infinite symmetric groups, some of their wreath products, and infinite alternating groups. We are also led to compute and compare polynomials $C_{G,S}$ and $L_{G,S}$ for finite symmetric and alternating groups, for various generating sets S.

For a non-empty set X, we denote by $\mathrm{Sym}(X)$ the **finitary symmetric group** of X, i.e. the group of permutations of X with finite support. The **support** of a permutation g of X is the subset $\sup(g) = \{x \in X | g(x) \neq x\}$ of X. Two permutations of X are **disjoint** if their supports are disjoint (below, this will be used mainly for cycles). It is convenient to agree that, for $g, h \in \mathrm{Sym}(X)$,

we denote by gh the result of the permutation h *followed by* g,

such that $(gh)(x) = g(h(x))$ for all $x \in X$. For example, for $X = \mathbf{N}$, we have $(1, 2)(2, 3) = (1, 2, 3)$, and not $(1, 3, 2)$ as with the other convention.

The conjugacy class

$$T_X = \{(x,y) \in \mathrm{Sym}(X) \mid x, y \in X \text{ are distinct}\} \subset \mathrm{Sym}(X)$$

of all transpositions in $\mathrm{Sym}(X)$ is a generating set of $\mathrm{Sym}(X)$. We consider also other generating sets, in particular for $X = \mathbf{N}$

$$S_{\mathbf{N}}^{\mathrm{Cox}} = \{(i, i+1) \mid i \in \mathbf{N}\},$$

which makes $\mathrm{Sym}(\mathbf{N})$ look like an infinitely generated irreducible Coxeter group of type A.

When X is finite, $\mathrm{Sym}(X)$ is the usual symmetric group of X. For $n \geq 1$ and $X = \{1, 2, \dots, n\}$, we write $\mathrm{Sym}(n)$. The sets of transpositions

$$S_n^{\mathrm{Cox}} = \{(1,2), (2,3), \cdots, (n-1, n)\}, \quad T_n = \{(i,j) \mid 1 \leq i, j \leq n, \ i < j\}$$

are particular cases for $\mathrm{Sym}(n)$ of generating sets which are standard for finite Coxeter groups.

In the following proposition, we collect a sample of equalities that appear again in Proposition 8 in a more general situation.

Proposition 1. *Let $S \subset \mathrm{Sym}(\mathbf{N})$ be a generating set such that $S_{\mathbf{N}}^{Cox} \subset S \subset T_{\mathbf{N}}$. For every $n \geq 1$, let $S_n \subset \mathrm{Sym}(n)$ be a generating set such that $S_n^{Cox} \subset S_n \subset T_n$. Then*

(i) $$C_{\mathrm{Sym}(\mathbf{N}),S}(q) = \sum_{m=0}^{\infty} p(m) q^m = \prod_{j=1}^{\infty} \frac{1}{1-q^j} ,$$

in particular the sequence of coefficients of $C_{\mathrm{Sym}(\mathbf{N}),S}(q)$ if of intermediate growth,

(ii) $$C_{\mathrm{Sym}(n),S_n}(q) = \sum_{k=0}^{n-1} p_{n-k}(n) q^k ,$$

(iii) $$\sum_{n=0}^{\infty} C_{\mathrm{Sym}(n),S_n}(q) t^n = \prod_{j=1}^{\infty} \frac{1}{1-q^{j-1}t^j} ,$$

where the partition function $p(n)$ and the second equality of (i) are as recalled in Appendix B.a, and the number $p_{n-k}(n)$ of partitions of n with $n-k$ positive parts as in Appendix B.b. Moreover:

(iv) when $n \to \infty$, the polynomials $C_{\mathrm{Sym}(n),S_n}(q)$ of (ii) converge coefficientwise towards the series $C_{\mathrm{Sym}(\mathbf{N}),S}(q)$ of (i).

For example:

$$\begin{aligned}
C_{\mathrm{Sym}(2),S_2}(q) &= 1 + q, \\
C_{\mathrm{Sym}(3),S_3}(q) &= 1 + q + q^2, \\
C_{\mathrm{Sym}(4),S_4}(q) &= 1 + q + 2q^2 + q^3, \\
C_{\mathrm{Sym}(5),S_5}(q) &= 1 + q + 2q^2 + 2q^3 + q^4, \\
C_{\mathrm{Sym}(6),S_6}(q) &= 1 + q + 2q^2 + 3q^3 + 3q^4 + q^5 , \\
C_{\mathrm{Sym}(n),S_n}(q) &= 1 + q + 2q^2 + \cdots + \lfloor n/2 \rfloor q^{n-2} + q^{n-1} \ \ (n \geq 5).
\end{aligned}$$

The main ingredients for the proof of Proposition 1 are the classical Observation 2 and Lemma 3. We use the following standard notation: for an integer $n \geq 0$, we denote by $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$ a *partition* of *weight* $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, with $k \geq 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

In Proposition 18 of Section 2, we come back to the convergence of $C_{\mathrm{Sym}(n),S_n}(q)$ to $C_{\mathrm{Sym}(\mathbf{N}),S}(q)$.

Observation 2. Let X be a non-empty set, finite or infinite. Denote by $|X|$ its cardinality. Conjugacy classes in $\mathrm{Sym}(X)$ are in natural bijection with appropriate sets of partitions. More precisely, for each pair (L, k) of non-negative integers with $L + k \leq |X|$, there is a bijection between the set of partitions of the form

$$\lambda \,=\, (\lambda_1, \ldots, \lambda_k) \vdash L \tag{2.a}$$

on the one hand, and conjugacy classes in $\mathrm{Sym}(X)$ of elements of the form

$$\begin{aligned} g \,=\, c_1 c_2 \cdots c_k \in \mathrm{Sym}(X) \ \text{ where} \\ & c_i \ \text{ is a cycle of some length } \lambda_i + 1 \geq 2 \text{ for } i = 1, 2, \ldots, k, \\ & c_i \text{ and } c_{i'} \text{ are disjoint for } i \neq i', \\ & \text{and therefore } |\mathrm{sup}(g)| - k = L = \sum_{i=1}^{k} \lambda_i, \end{aligned} \tag{2.b}$$

on the other hand. In this article,

the length of a *cycle* is at least 2, unless otherwise stated;

we always make it explicit when we want to consider *fixed points* as cycles of length 1.

Lemma 3. *Consider two integers $L, k \geq 0$, a set X of cardinal at least $L + k$ (possibly infinite), an element $g \in \mathrm{Sym}(X)$ product of k disjoint cycles with $|\mathrm{sup}(g)| = L + k$, and the corresponding partition $\lambda \vdash L$ in k parts, as in Observation 2.*

(i) There exist transpositions $s_1, \ldots, s_L \in \mathrm{Sym}(X)$ such that $g = s_1 \cdots s_L$ and $\mathrm{sup}(s_l) \subset \mathrm{sup}(g)$ for all $l \in \{1, \ldots, L\}$.

(ii) There exist transpositions $t_1, \ldots, t_M \in \mathrm{Sym}(X)$ such that $g = t_1 \cdots t_M$ if and only if $M \geq L$ and $M - L$ is even.

Suppose moreover that X is given together with trees $T_1, \ldots, T_k$ with the following properties: for $i \in \{1, \ldots, k\}$, the vertex set of T_i is a subset of X of cardinality $\lambda_i + 1$, and these subsets are disjoint from each other. Let $\big\{\{x_1, x'_1\}, \ldots, \{x_L, x'_L\}\big\}$ be an enumeration of the edges of the forest $\bigcup_{i=1}^{k} T_i$.

(iii) The product $h = (x_1, x'_1) \cdots (x_L, x'_L)$ is conjugate to g in $\mathrm{Sym}(X)$.

We postpone until Section 2 the proofs of these, and of further propositions in the present section. Before we can state more general cases of some of the equalities of Proposition 1, we introduce two definitions and provide examples.

Definition 4. For a set S of transposition of a set X, the **transposition graph** $\Gamma(S)$ has vertex set X and edge set those pairs $\{x, y\} \subset X$ for which the transposition (x, y) is in S.

But for their names, these graphs appear in [Serg–93]. It is well-known and easy to check (Lemma 32) that

$$\text{(GC)} \qquad \begin{array}{l} \text{the group } \mathrm{Sym}(X) \text{ is generated by } S \\ \text{if and only if the graph } \Gamma(S) \text{ is connected.} \end{array}$$

Definition 5. For a set X, a set S of transpositions of X is **partition-complete** if it satisfies the following condition:

$$\text{(PC)} \qquad \begin{array}{l} \text{the transposition graph } \Gamma(S) \text{ is connected and,} \\ \text{for every partition } \lambda = (\lambda_1, \dots, \lambda_k) \vdash L \text{ such that } L + k \leq |X|, \\ \Gamma(S) \text{ contains a forest consisting of } k \text{ trees} \\ \text{having respectively } \lambda_1 + 1, \dots, \lambda_k + 1 \text{ vertices.} \end{array}$$

The graph $\Gamma(S)$ itself is partition-complete when S is so.

Example 6. When $X = \{1, \dots, n\}$, sets of transpositions satisfying Condition (PC) include sets S such that $S_n^{\mathrm{Cox}} \subset S \subset T_n$, and also those for which $\Gamma(S)$ is one of the Dynkin graphs D_{2n+1} with $n \geq 2$, or E_7 or E_8.

But if S is such that $\Gamma(S)$ is one of D_{2n} with $n \geq 2$, or E_6, then S does not satisfy Condition (PC), because D_{2n} does not contain n disjoint trees with two vertices each, and E_6 does not contain two disjoint trees with three vertices each.

When X is finite, S is partition-complete as soon as the graph $\Gamma(S)$ is semi-hamiltonian; recall that a graph is semi-hamiltonian [respectively hamiltonian] if it contains a path [respectively a cycle] containing every vertex exactly once. Condition (PC) for a graph can be seen as a weakening of the property of being semi-hamiltonian.

When X is infinite, Condition (PC) is equivalent to (PC_∞):

$$(\mathrm{PC}_\infty) \qquad \begin{array}{l} S \text{ generates } \mathrm{Sym}(X) \text{ and, for all } n \geq 1, \\ \text{the graph } \Gamma(S) \text{ contains a disjoint union} \\ \text{of } n \text{ trees with at least } n \text{ vertices each.} \end{array}$$

When $X = \mathbf{N}$, here are two families of examples of sets S satisfying Condition (PC_∞). The first is that of sets of transpositions of which the transposition graph contains arbitrarily long segments; this family contains sets S such that $S_{\mathbf{N}}^{\mathrm{Cox}} \subset S \subset T_{\mathbf{N}}$. For a set of the second family, choose an increasing sequence $(k_n)_{n \geq 1}$ of positive integers such that $k_{n+2} - k_{n+1} > k_{n+1} - k_n$ for all $n \geq 1$; define then S as the set of transpositions $(0, k_n)$ and (k_n, j) for all $n \geq 1$ and j with $k_n + 1 \leq j \leq k_{n+1} - 1$, so that $\Gamma(S)$ is obtained from a star with centre 0 and infinitely many neighbours k_n by attaching $k_{n+1} - k_n - 1$ vertices to each vertex

k_n; thus $\Gamma(S)$ is a tree of diameter 4, with all vertices but one (the origin) of finite degrees.

On the contrary, the set $S^0_{\mathbf{N}} = \{(0,n) \mid n \geq 1\}$ does not satisfy Condition (PC_∞). Proposition 9 provides the conjugacy growth series for the pair $(\mathrm{Sym}(\mathbf{N}), S^0_{\mathbf{N}})$.

We ignore the existence of a simple criterion for graphs or trees to be partition complete.

Using Definition 5, we reformulate Lemma 3(iii) and generalize Proposition 1 as follows:

Lemma 7. *Let X be a non-empty set and S a partition-complete set of transpositions of X. Let $g = c_1 \cdots c_k \in \mathrm{Sym}(X)$ be a product of disjoint cycles of non-increasing lengths; denote these lengths by $\lambda_1+1, \dots, \lambda_k+1$, and set $L = \sum_{i=1}^k \lambda_i$, so that $|\mathrm{sup}(g)| = L + k$. Then*

$$\kappa_{\mathrm{Sym}(X),S}(g) \ = \ L.$$

Proposition 8. *Let X be an infinite set and $S \subset \mathrm{Sym}(X)$ a partition-complete set of transpositions.*

(a) The equalities of (i) in Proposition 1 hold true. In particular the series $C_{\mathrm{Sym}(X),S}(q)$ does not depend on the cardinality of X, as long as X is infinite.

For every $n \geq 1$, let $S_n \subset \mathrm{Sym}(n)$ be a partition-complete set of transpositions.

(b) Claims (ii), (iii), and (iv) in Proposition 1 hold true.

For the next proposition, we consider the generating sets of transpositions

$$\begin{aligned} S^0_{\mathbf{N}} \ &= \ \{(0,n) \in \mathrm{Sym}(\mathbf{N}) \mid n \geq 1\} \ \subset \ \mathrm{Sym}(\mathbf{N}), \\ S^0_n \ &= \ \{(0,i) \mid 1 \leq i \leq n-1\} \ \subset \ \mathrm{Sym}(n) = \mathrm{Sym}(\{0,1,\dots,n-1\}), \end{aligned}$$

which *do not* satisfy Condition (PC).

Proposition 9. *Let $S^0_{\mathbf{N}} \subset \mathrm{Sym}(\mathbf{N})$ and, for every $n \geq 1$, let $S^0_n \subset \mathrm{Sym}(n)$ be as above. Then*

$$(i) \ \ C_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(q) \ = \ 1 + \sum_{k=1}^{\infty} q^{3k-2} \prod_{j=1}^{k} \frac{1}{1-q^j} \ = $$
$$1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 4q^7 + 5q^8 + 6q^9 + 9q^{10} +$$
$$10q^{11} + 13q^{12} + 17q^{13} + 21q^{14} + 25q^{15} + 33q^{16} + 39q^{17} + 49q^{18} +$$
$$60q^{19} + 73q^{20} + 88q^{21} + 110q^{22} + 130q^{23} + 158q^{24} + \cdots \ ,$$

$$(ii) \ \ C_{\mathrm{Sym}(n),S^0_n}(q) \ = \ 1 \ + \ \sum_{k=1}^{\lfloor n/2 \rfloor} q^{2k-2} \sum_{j=k}^{n} p_k(j) q^j \ .$$

Moreover, when $n \to \infty$, the polynomials $C_{\mathrm{Sym}(n),S^0_n}$ of (ii) converge coefficient-wise towards the series $C_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(q)$ of (i).

At the day of writing, the sequence

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
c_n	1	1	1	1	2	2	3	4	5	6	9	10	13	17	21	25	33	39	49

of coefficients of the series $C_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(q) := \sum_{n=0}^{\infty} c_n q^n$ of (i) does not appear in [OEIS]. The equality of (ii) is repeated in Proposition 22 below.

Numerically, the series of Proposition 9(i) converges in the unit disc, and shows two roots of smallest absolute value, near $-0.53 \pm 0.68i$. This makes it unlikely that the series of Proposition 9 has such a nice product expansion like that of Proposition 1(i).

Let X be an infinite set and H a finite group. Let $W = H \wr_X \mathrm{Sym}(X)$ be the corresponding permutational wreath product. Let S be a generating set of W containing a set of transpositions S_X of X generating $\mathrm{Sym}(X)$ and satisfying Condition (PCwr) of Section 3. Denote by M the number of conjugacy classes of H.

Proposition 10 (see Proposition 19 below)**.** *Let $W = H \wr_X \mathrm{Sym}(X)$, S and M be as above. Then*

$$C_{W,S}(q) = \Big(C_{\mathrm{Sym}(X),S_X}(q)\Big)^N = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^M} .$$

The **finitary alternating group** of $\mathbf{N}$ is the subgroup $\mathrm{Alt}(\mathbf{N})$ of $\mathrm{Sym}(\mathbf{N})$ of permutations of even signature. Consider its generating set

$$S^A_{\mathbf{N}} = \{(i, i+1, i+2) \in \mathrm{Alt}(\mathbf{N}) \mid i \in \mathbf{N}\},$$

as well as the subset $T^A_{\mathbf{N}} := \bigcup_{g \in \mathrm{Alt}(\mathbf{N})} g S^A_{\mathbf{N}} g^{-1}$ of all 3-cycles. Proposition 11 is the analogue for the finitary alternating group of $\mathbf{N}$ of Proposition 1(i) for the finitary symmetric group.

Proposition 11. *Let $S \subset \mathrm{Alt}(\mathbf{N})$ be a generating set such that $S^A_{\mathbf{N}} \subset S \subset T^A_N$. Then*

$$\begin{aligned} C_{\mathrm{Alt}(X),S}(q) &= \sum_{u=0}^{\infty} p(u) q^u \sum_{v=0}^{\infty} p_e(v) q^v \\ &= \frac{1}{2} \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^2} + \frac{1}{2} \prod_{j=1}^{\infty} \frac{1}{1-q^{2j}} \\ &= 1 + q + 3q^2 + 5q^3 + 11q^4 + 18q^5 + 34q^6 \\ &\quad + 55q^7 + 95q^8 + 150q^9 + 244q^{10} + \cdots . \end{aligned}$$

where $p_e(v)$ denotes the number of partitions of $v \in \mathbf{N}$ involving an even number of positive parts, as in Appendix B.c.

Observation 12. For the series of Proposition 11, set

$$C_{\mathrm{Alt}(\mathbf{N}),S}(q) \;=\; \sum_{n=0}^{\infty} p^A(n) q^n.$$

The coefficients $p^A(n)$ satisfy the following congruence relations:

$$\begin{aligned} p^A(5n+3) &\equiv 0 \pmod 5, \\ p^A(10n+7) &\equiv 0 \pmod 5, \\ p^A(10n+9) &\equiv 0 \pmod 5, \\ p^A(25n+23) &\equiv 0 \pmod{25}. \end{aligned}$$

Moreover, *conjecturally*:

$$\begin{aligned} p^A(49n+17) &\equiv 0 \pmod 7, \\ p^A(49n+31) &\equiv 0 \pmod 7, \\ p^A(49n+38) &\equiv 0 \pmod 7, \\ p^A(49n+45) &\equiv 0 \pmod 7, \\ p^A(121n+111) &\equiv 0 \pmod{11}. \end{aligned}$$

See Proposition 30 for the first four relations. The conjectured relations have been verified numerically for $p^A(m)$ when $m \leq 5000$, as discussed in Section 6 and Appendix C.

Remark 13. (i) Let G be a group generated by a subset T. Then

$$(\kappa_T = \ell_T) \qquad\qquad \kappa_{G,T}(g) \;=\; \ell_{G,T}(g) \;\text{ for all }\; g \in G$$

if and only if T is closed by conjugation, as it is straightforward to check.

(ii) Suppose that G is also generated by a subset S, and assume that $T = \bigcup_{h \in G} hSh^{-1}$. Then

$$(\kappa_T \leq \kappa_S) \qquad\qquad \kappa_{G,T}(g) \;\leq\; \kappa_{G,S}(g) \;\text{ for all }\; g \in G,$$

but equality need not hold.

For example, if $G = \mathrm{Sym}(4)$ and $S = \{(1,2),(2,3,4)\}$, then

$$\kappa_{G,T}((1,2)(3,4)) = 2 \;<\; \kappa_{G,S}((1,2)(3,4)) = 4.$$

(iii) It is remarkable that we have

$$(\kappa_T = \kappa_S) \qquad\qquad \kappa_{G,T}(g) \;=\; \kappa_{G,S}(g) \;\text{ for all }\; g \in G,$$

in many cases of interest here, including

– $G = \mathrm{Sym}(\mathbf{N})$ and S as in Proposition 1(i), so that $T = T_{\mathbf{N}}$,
– $G = \mathrm{Sym}(n)$ and $S = S_n$ as in Proposition 1(ii), so that $T = T_n$,
– $G = \mathrm{Alt}(\mathbf{N})$ and S as in Proposition 11, so that $T = T^A_{\mathbf{N}}$.

In these cases, it follows that

$$(C_T = C_S) \qquad\qquad C_{G,T}(q) \, = \, C_{G,S}(q).$$

Note however that, in the case of $G = \mathrm{Sym}(\mathbf{N})$ and $S = S^0_{\mathbf{N}}$, and therefore $T = T_{\mathbf{N}}$, the series $C_{G,S}(q)$ of Proposition 9 and $C_{G,T}(q)$ of Proposition 1(i) are different, so that the equalities $(\kappa_T = \kappa_S)$ and $(C_T = C_S)$ do not hold.

Overview. Section 2 contains proofs of Propositions 1, 8, 9 and Lemmas 3, 7. In Section 3, we write and prove formulas for conjugacy growth series of wreath products, see Propositions 10 and 19.

Suppose that G is a finite symmetric group $\mathrm{Sym}(n)$, and S a system of generators. When S is either S_n^{Cox} or T_n, the polynomial $L_{G,S}(q)$ is well-known, and is recalled in Proposition 20 below. Indeed, these polynomials make sense and are explicitely known for all finite Coxeter systems; they appear in many places, for example [Solo–66] and [Bour–68, exercises of § IV.1], as well as [ShTo–54]. In Section 4, we compute $C_{\mathrm{Sym}(n),S}(q)$, and compare these polynomials with those for another generating set, the set S_n^0 defined above; this uses lemmas of Section 2, as well as some facts on derangements recalled in Appendix B.

In Section 5, we present results of analogous computations for finitary alternating groups, and in particular the proof of Proposition 11. In the final Section 6, we discuss the context of Observation 12.

There is a short Appendix A with three lemmas on symmetric and alternating groups, and a longer Appendix B that is a reminder of various definitions and identities involving partitions and derangements. Finally, in Appendix C, we define a generalization of Ramanujan congruences and we record a large number of these for the coefficients $p(n)_{(e_1,e_2,e_3,\dots)}$ of the power series

$$\sum_{n=0}^{\infty} p(n)_{(e_1,e_2,e_3,\dots)} q^n \, = \, \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{e_1}(1-q^{2n})^{e_2}(1-q^{3n})^{e_3}\,\cdots} \, ,$$

where $(e_1, e_2, e_3, \dots)$ is a finite sequence of non-negative integers. Some of these congruences are established in the literature, but most are (as far as we know) conjectural only, based on our numerical evidence.

2. Proof of Lemma 3 and 7, and Propositions 1, 8, and 9

We will moreover state and prove a sharpening of Proposition 1(iv), in Proposition 18.

2.a. Proof of Lemmas 3 and 7.

As a preliminary step for the proof, consider a cycle

$$c \, = \, (x_1, \dots, x_{\mu+1}) \in \mathrm{Sym}(X),$$

where $1 \leq \mu \leq |X| - 1$. By Lemma 31 applied $\mu - 1$ times (see Appendix A), the cycle c can be written as a product of μ transpositions with supports in $\sup(c)$.

Let $g \in \mathrm{Sym}(X)$ and $\lambda = (\lambda_1, \dots, \lambda_k) \vdash L + k$ be as in Lemma 3. Write $g = c_1 \cdots c_k$, where $c_1, \dots, c_k$ are disjoint cycles of lengths $\lambda_1 + 1, \dots, \lambda_k + 1$ respectively. For $i \in \{1, \dots, k\}$, it follows from the preliminary step that c_i can be written as a product of λ_i transpositions with supports in $\sup(g)$. Hence g can be written as a product of $L = \sum_{i=1}^k \lambda_i$ transpositions with supports in $\sup(g)$. This proves (i) of Lemma 3.

With the extra ingredient of Lemma 32, this also proves (iii) of Lemma 3 and Lemma 7.

Consider now $g = t_1 \cdots t_M$ as in (ii) of Lemma 3. For $i = 1, \dots, k$, write $c_i = (x^i_1, x^i_2, \dots, x^i_{\lambda_i+1})$. Define a multigraph $G = G(t_1, \dots, t_M)$ as follows: its vertex set is $V_G := \bigcup_{\nu=1}^M \sup(t_\nu)$, and there is one edge between the two vertices of $\sup(t_\nu)$ for each $\nu \in \{1, \dots, M\}$. Observe that $V_G \supset \sup(g) = \bigcup_{i=1}^k \sup(c_i)$.

Erasing from the product $t_1 \cdots t_M$ those t_ν contributing to connected components of G disjoint from $\sup(g)$ does not change this product. We can therefore assume that each connected component of G intersects $\sup(g)$. For each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \lambda_i + 1\}$, the connected component of G containing x^i_j contains $\sup(c_i)$; it follows that each connected component of G contains at least one of the $\sup(c_i)$'s, and therefore that the number of connected components of G, say γ_G, is at most k.

Given any finite multigraph with v vertices, e edges, and γ connected components, $e \ge v - \gamma$, with equality if and only if the multigraph is a forest. For the multigraph G, we have therefore

$$M \;\ge\; |V_G| - \gamma_G \;\ge\; |\sup(g)| - k \;=\; \sum_{i=1}^k \lambda_i\,.$$

Moreover, M and L have the same parity, which is also the signature of g.

Conversely, for every $M \ge L$ with $M - L$ even, g can be written as a product of M transpositions, for example the L transpositions of (i) and $(M-L)/2$ times the product $s_1 s_1$. This proves (ii) of Lemma 3. □

2.b. **Proof of Propositions 1 and 8.** We prove the equalities of Proposition 1 in the more general case of Proposition 8.

(i) Let X be an infinite set and $S \subset \mathrm{Sym}(X)$ a partition-complete set of transpositions. The series $C_{\mathrm{Sym}(X),S}(q)$ is a sum over partitions $\lambda \vdash L$ as in (2.a) of Observation 2, and the contribution of such a partition is q^L by Lemma 7. Hence $C_{\mathrm{Sym}(X),S}(q) = \sum_{L=0}^\infty p(L) q^L$. Equality with $\prod_{j=1}^\infty \frac{1}{1-q^j}$ is Euler's identity (EP_1) recalled in Appendix B.a.

(ii) Consider a positive integer n and a partition-complete set $S_n \subset \mathrm{Sym}(n)$. Conjugacy classes in $\mathrm{Sym}(n)$ are now in bijection with partitions of n as follows: a partition $(\mu_1, \dots, \mu_k) \vdash n$ with exactly k positive parts corresponds to a permutation $g = c_1 \cdots c_k$ where c_j is a cycle of length μ_j, and "cycles" of length 1, i.e. fixed points of g, are now allowed (this is why we use μ here rather than λ as above). By Lemma 7, the S_n-conjugacy length of such a g is $\kappa_{\mathrm{Sym}(n),S_n}(g) = \sum_{j=1}^{k}(\mu_j - 1) = n - k$. Hence the polynomial $C_{\mathrm{Sym}(n),S_n}(q)$ is a sum over partitions of n (where n is fixed) with exactly k parts (where k ranges from 1 (long cycles) to n (identity)), and each such partition contributes by q^{n-k}. Hence $C_{\mathrm{Sym}(n),S_n}(q) = \sum_{k=1}^{n} p_k(n) q^{n-k} = \sum_{k=0}^{n-1} p_{n-k}(n) q^k$.

(iii) Exchanging product and sum, we have

$$\begin{aligned}\prod_{k=1}^{\infty} \frac{1}{1 - q^{k-1}t^k} &= \prod_{k=1}^{\infty} \sum_{\ell_k=0}^{\infty} q^{\ell_k(k-1)} t^{\ell_k k} = \sum_{\ell_1,\ell_1,\ell_3,\ldots \geq 0} \prod_{k=1}^{\infty} q^{\ell_k(k-1)} t^{\ell_k k} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\substack{\ell_1,\ell_2,\ell_3,\ldots \geq 0 \\ \ell_1+2\ell_2+\cdots+k\ell_k+\cdots=n}} q^{\sum_{k=1}^{\infty} \ell_k(k-1)} \right) t^n.\end{aligned}$$

For $n \geq 0$, there is a contribution to the coefficient of t^n for each sequence $(\ell_1, \ell_2, \ell_3, \ldots)$ of non-negative integers such that $\ell_1 + 2\ell_2 + 3\ell_3 + \cdots = n$, equivalently for each partition $1^{\ell_1}2^{\ell_2}3^{\ell_3}\cdots$ of n, with ℓ_1 parts 1, and ℓ_2 parts 2, and ℓ_3 parts 3, ..., equivalently for each conjugacy class in $\mathrm{Sym}(n)$. Since $0\ell_1 + 1\ell_2 + 2\ell_3 + \cdots$ is the S_n-length of such a conjugacy class, the contributions to the coefficient of t^n add up precisely to $C_{\mathrm{Sym}(n),S_n}(q)$.

(iv) The polynomials of (ii) converge coefficientwise towards the series of (i) because $p_{n-k}(n) = p(k)$ when $2k \leq n$. See (EP$_4'$) in Appendix B.b). ☐

2.c. **A computation of lengths.** For the next two lemmas, we agree that $\mathrm{Sym}(n)$ denotes the group of permutations of $\{0, 1, \ldots, n-1\}$, and we consider the generating set S_n^0 defined just before Proposition 9.

Lemma 14. *Let $g = c_1c_2\ldots c_k \in \mathrm{Sym}(n)$, where $c_1, \ldots, c_k$ are disjoint cycles, each of length at least 2; set $m = |\mathrm{sup}(g)|$.*

$$\ell_{\mathrm{Sym}(n),S_n^0}(g) \leq \begin{cases} m + k & \text{if } g(0) = 0, \\ m + k - 2 & \text{if } g(0) \neq 0. \end{cases}$$

Proof. Choose $i \in \{1, \ldots, k\}$. Let μ_i denote the length of c_i, and write $c_i = (x_1, x_2, \ldots, x_{\mu_i})$.

If $\mathrm{sup}(c_i)$ does not contain 0, then

$$c_i = (0, x_1)(0, x_{\mu_i})(0, x_{\mu_i-1}) \cdots (0, x_2)(0, x_1)$$

and $\ell_{\mathrm{Sym}(n),S_n^0}(c_i) \leq \mu_i + 1$. If $\mathrm{sup}(c_i)$ contains 0, say $x_1 = 0$, (this occurs for at most one value of i), then

$$c_i = (0, x_{\mu_i})(0, x_{\mu_i-1})(0, x_{\mu_i-2}) \cdots (0, x_2)$$

and $\ell_{\mathrm{Sym}(n),S_n^0}(c_i) \leq \mu_i - 1$.

Since $\ell_{\mathrm{Sym}(n),S_n^0}(g) \leq \sum_{i=1}^{k} \ell_{\mathrm{Sym}(n),S_n^0}(c_i)$, the lemma follows. ☐

Lemma 15. *Let $g = c_1 c_2 \dots c_k \in \mathrm{Sym}(n)$ and $m = |\sup(g)|$ be as in the previous lemma. Then*

$$\begin{aligned} \ell_{\mathrm{Sym}(n),S_n^0}(g) \;&=\; \begin{cases} m+k & \text{if } g(0)=0, \\ m+k-2 & \text{if } g(0)\neq 0, \end{cases} \\ \kappa_{\mathrm{Sym}(n),S_n^0}(g) \;&=\; \quad m+k-2 \quad \text{as soon as } g \neq \mathrm{id}. \end{aligned}$$

Proof. Set $L = \ell_{\mathrm{Sym}(n),S_n^0}(g)$; there exist $r_1, \dots, r_L \in S_n^0$ such that $g = r_1 r_2 \cdots r_L$. For $i \in \{1, \dots, k\}$, there are distinct elements $x_1^i, \dots, x_{\mu_i}^i \in \{0, 1, \dots, n-1\}$ such that $c_i = (x_1^i, x_2^i, \dots, x_{\mu_i}^i)$; and $\mu_1 + \cdots + \mu_k = m$. Observe that, for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, \mu_i\}$, the transposition $(0, x_j^i)$ occurs in the list $r_1, \dots, r_L$, at least once.

Suppose first that $0 \notin \sup(g)$. We know from Lemma 14 that $L \leq m+k$. If one had $L < m+k$, there would exist $i \in \{1, \dots, k\}$ such that $(0, x)$ occurs only one time in the list $r_1, \dots, r_L$ for each $x \in \sup(c_i)$; but this is not possible since $0 \notin \sup(c_i)$. Hence $L = m+k$.

Suppose now that $0 \in \sup(g)$; we can assume that $x_1^1 = 0$. We know from Lemma 14 that $L \leq m+k-2$. If one had $L < m+k-2$, at least one of the two following situations would hold:

(a) there exists $i \in \{2, \dots, k\}$ such that $(0, x)$ occurs only one time in the list $r_1, \dots, r_L$ for each $x \in \sup(c_i)$,
(b) there exists $j \in \{2, 3, \dots, \mu_1\}$ such that the transposition $(0, x_j^1)$ does not occur in the list $r_1, \dots, r_L$;

but this is not possible. Hence $L = m+k-2$, and the formula for $\ell_{\mathrm{Sym}(n),S_n^0}(g)$ follows.

For all $g \neq \mathrm{id}$ in $\mathrm{Sym}(n)$, there exists a conjugate h of g such that $h(0) \neq 0$ to which the same computation applies. The formula for $\kappa_{\mathrm{Sym}(n),S_n^0}(g)$ follows. □

Similarly:

Lemma 16. *Let $g = c_1 c_2 \cdots c_k \in \mathrm{Sym}(\mathbf{N})$, where $c_1, \dots, c_k$ are disjoint cycles, each of length at least 2; set $m = |\sup(g)|$. Then $\ell_{\mathrm{Sym}(\mathbf{N}),S_{\mathbf{N}}^0}(g)$ and $\kappa_{\mathrm{Sym}(\mathbf{N}),S_{\mathbf{N}}^0}(g)$ are given by the formulas of the previous lemma.*

2.d. **Proof of Proposition 9.** We record a minor variation of Observation 2, as follows. Given $m \geq 2$ and $k \geq 1$, there is a bijection between

(a) the set of partitions of m with k parts, all at least 2,
(i.e. partitions of the form $\mu = (\mu_1, \dots, \mu_k) \vdash m$ with $\mu_1 \geq \cdots \geq \mu_k \geq 2$),

and

(b) the set of conjugacy classes of elements $g \neq 1$ in $\mathrm{Sym}(\mathbf{N})$ or $\mathrm{Sym}(n)$, with $|\sup(g)| = m$, which are products of k disjoint cycles, where moreover $m \leq n$ in the case of $\mathrm{Sym}(n)$
(i.e. of elements of the form $g = c_1 \cdots c_k$ with $\mathrm{length}(c_i) = \mu_i$).

For each μ as in (a), set

$\nu = (\nu_1, \dots, \nu_k) := (\mu_1 - 1, \cdots, \mu_k - 1) \vdash m - k$.
which is a partition in k positive parts.

The relevant length of the conjugacy class of g as in (b) is $m+k-2$, by Lemmas 15 and 16.

For (i) of Proposition 9, it follows that

$$\begin{aligned} C_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(q) &= \sum_{m=0}^{\infty} \gamma_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(m) q^m \\ &= 1 + \sum_{m=2}^{\infty} \sum_{k=1}^{\lfloor m/2 \rfloor} p_k(m-k) q^{m+k-2} \\ &= 1 + \sum_{k=1}^{\infty} q^{2k-2} \sum_{m=2k}^{\infty} p_k(m-k) q^{m-k} \\ &= 1 + \sum_{k=1}^{\infty} q^{2k-2} \sum_{n=k}^{\infty} p_k(n) q^n \\ &= 1 + \sum_{k=1}^{\infty} q^{3k-2} \prod_{j=1}^{k} \frac{1}{1-q^j} \end{aligned}$$

where the last equality holds by (EP_2) of Appendix B.b.

(ii) Similarly:

$$\begin{aligned} C_{\mathrm{Sym}(n),S^0_n}(q) &= 1 + \sum_{m=2}^{n} \sum_{k=1}^{\lfloor m/2 \rfloor} p_k(m-k) q^{m+k-2} \\ &= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{2k-2} \sum_{m=2k}^{n} p_k(m-k) q^{m-k} \\ &= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{2k-2} \sum_{j=k}^{n} p_k(j) q^j \ . \end{aligned}$$

(*Note:* $\sum_{j=k}^{n} p_k(j) q^j = \sum_{j=0}^{n} p_k(j) q^j$.) It is now clear that these polynomials converge coefficientwise to $1 + \sum_{m=2}^{\infty} \sum_{k=1}^{\lfloor m/2 \rfloor} p_k(m-k) q^{m+k-2}$, that is to $C_{\mathrm{Sym}(\mathbf{N}),S^0_{\mathbf{N}}}(q)$. □

We end this section with a sharpening of Claim (iv) of Proposition 1; this applies more generally to the situation of Proposition 8. Let $S \subset \mathrm{Sym}(\mathbf{N})$ be a partition-complete set of transpositions, and let L be a non-negative integer. Set

$$\mathcal{K}_L(S) = \{ g \in \mathrm{Sym}(\mathbf{N}) \mid \kappa_{\mathrm{Sym}(\mathbf{N}),S}(g) = L \}.$$

Observe that $\mathcal{K}_L(S)$ is a union of conjugacy classes in $\mathrm{Sym}(\mathbf{N})$. For $g \in \mathrm{Sym}(\mathbf{N})$, we denote by k_g the number of disjoint cycles of which g is the product.

Lemma 17. *Let S, L, and $\mathcal{K}_L(S)$ be as above.*

(i) Let $g \in \mathcal{K}_L(S)$. Then $|\mathrm{sup}(g)| = L + k_g \leq 2L$ for all $g \in \mathcal{K}_L(S)$. Equality $k_g = L$ holds if and only if g is a product of L disjoint transpositions.

(ii) Let $s \in \mathbf{N}$ be such that $0 \leq s \leq L/2$. Then $\mathcal{K}_L(S)$ contains exactly $p(s)$ conjugacy classes of elements g such that $|\mathrm{sup}(g)| = 2L - s$.

Proof. (i) Let $g \in \mathcal{K}_L(S)$ be written as a product $c_1 \cdots c_{k_g}$ of disjoint cycles of decreasing sizes. For $i \in \{1, \dots, k_g\}$, denote by $\lambda_i + 1$ the length of c_i; set $\lambda = (\lambda_1, \dots, \lambda_{k_g})$, so that $\lambda \vdash L$ by Lemma 7. Since $k_g \leq L$, we have $|\mathrm{sup}(g)| = L + k_g \leq 2L$. If $|\mathrm{sup}(g)| = 2L$, then $\lambda_i = 1$ for $i = 1, \dots, k_g$, and every c_i is a transposition.

(ii) Let s be such that $0 \leq s \leq L/2$. We proceed to establish a bijection between the set of partitions of s on the one hand, and the set of conjugacy classes of elements $g \in \mathrm{Sym}(\mathbf{N})$ such that $g \in \mathcal{K}_L(S)$ and $|\mathrm{sup}(g)| = 2L - s$ on the other hand; this will end the proof. As Claim (i) covers the case $s = 0$, we could assume that $s \geq 1$.

Choose a partition $\mu = (\mu_1, \dots, \mu_m) \vdash s$. Since $s \leq L/2$, we have $L - s \geq m$. Set

$$\lambda \,=\, (\lambda_1, \dots, \lambda_{L-s}) \,=\, (\mu_1 + 1, \dots, \mu_m + 1, 1, \dots 1),$$

a partition of L with $L - (s + m)$ parts 1. Let $g \in \mathrm{Sym}(\mathbf{N})$ be a product of disjoint cycles of lengths $\lambda_1 + 1, \dots, \lambda_{L-s} + 1$. Then

$$\kappa_{\mathrm{Sym}(\mathbf{N}),S}(g) \,=\, \sum_{j=1}^{L-s} \lambda_j \,=\, \Big(\sum_{j=1}^{m} \mu_j \Big) + L - s \,=\, L,$$

in particular $g \in \mathcal{K}_L(S)$, and

$$|\mathrm{sup}(g)| \,=\, \sum_{j=1}^{L-s} (\lambda_j + 1) \,=\, 2L - s.$$

Conversely, choose $g \in \mathcal{K}_L(S)$ with $|\mathrm{sup}(g)| = 2L - s$. Let $\lambda_1 + 1, \dots, \lambda_{L-s} + 1$ be the lengths, in decreasing order, of the disjoint cycles of which g is the product; note that $\lambda = (\lambda_1, \dots, \lambda_{L-s}) \vdash L$. Define a partition $\mu = (\mu_1, \dots, \mu_m)$ by $m = \max\{j \in \{1, \dots, L - s\} \mid \lambda_j \geq 2\}$, and $\mu_j = \lambda_j - 1$ for $j \in \{1, \dots, m\}$. Then $\mu \vdash L - (L - s) = s$. $\square$

Here is the announced sharpening, see Propositions 1 and 8.

Proposition 18. *Let S be a partition-complete set of transpositions in* $\mathrm{Sym}(\mathbf{N})$ *and, for each $m \geq 1$, let S_m be a partition-complete set of transpositions in*

$\mathrm{Sym}(m)$. *Write* $C_\infty(q)$ *for* $C_{\mathrm{Sym}(\mathbf{N}),S}(q)$ *and* $C_m(q)$ *for* $C_{\mathrm{Sym}(m),S_m}(q)$. *Then:*

$$\lim_{n\to\infty}\frac{1}{q^{n+1}}\big(C_\infty(q)-C_{2n+1}(q)\big) \;=\; \sum_{i=0}^{\infty} p(\leq 2i)q^i\ ,$$

$$\lim_{n\to\infty}\frac{1}{q^{n+1}}\big(C_\infty(q)-C_{2n}(q)\big) \;=\; \sum_{i=0}^{\infty} p(\leq (2i+1))q^i\ ,$$

where $p(\leq j) := p(0)+p(1)+\cdots+p(j)$ *for all* $j\in\mathbf{N}$.

Proof. Note first that, for $L,m,k\in\mathbf{N}$, a conjugacy class in $\mathcal{K}_L(S)$ of elements g such that $|\mathrm{sup}(g)| = L+k$ intersects $\mathrm{Sym}(m)$ if and only if $L+k\leq m$.

Let $n\geq 1$. Choose an integer k such that $1\leq k\leq \frac{n+4}{3}$. Let $\mathcal{C}$ be a conjugacy class in $\mathrm{Sym}(\mathbf{N})$ such that $\mathcal{C}\subset\mathcal{K}_{n+k}(S)$.

Suppose that $\mathcal{C}$ contributes to the coefficient of q^{n+k} in $C_\infty(q)$ and not to the coefficient of q^{n+k} in $C_{2n+1}(q)$. Equivalently, suppose that, for every $g\in\mathcal{C}$, we have $|\mathrm{sup}(g)|\geq 2n+2$; if $s\geq 0$ is defined by $|\mathrm{sup}(g)| = 2(n+k)-s$, this means that $s\leq 2k-2$. Since $k\leq\frac{n+4}{3}$, i.e. $\frac{3k-4}{2}\leq\frac{n}{2}$, we have $s\leq\frac{3k-4}{2}+\frac{k}{2}\leq\frac{n+k}{2}$, so that $\mathcal{C}$ is one of the $\sum_{s=0}^{2k-2}p(s)$ classes which appear in Lemma 17(ii). It follows that the coefficient of q^{n+k} in $C_\infty(q)-C_{2n+1}(q)$ is $p(\leq(2k-2))$, so that the coefficient of q^{k-1} in $\frac{1}{q^{n+1}}(C_\infty(q)-C_{2n+1}(q))$ is $p(\leq(2k-2))$ for k with $1\leq k\leq\frac{n+4}{3}$. Consequently, for given $i\in\mathbf{N}$, the coefficient of q^i in $\frac{1}{q^{n+1}}(C_\infty(q)-C_{2n+1}(q))$ is $p(\leq 2i)$ as soon as n is large enough.

Similarly, suppose that $\mathcal{C}$ contributes to the coefficient of q^{n+k} in $C_\infty(q)$ and not to the coefficient of q^{n+k} in $C_{2n}(q)$. A similar argument shows that $\mathcal{C}$ is one of the $\sum_{s=0}^{2k-1}p(s)$ classes which appear in Lemma 17(ii), and finally that, for $i\in\mathbf{N}$, the coefficients of q^i in $\frac{1}{q^{n+1}}(C_\infty(q)-C_{2n}(q))$ is $p(\leq(2i+1))$ for n large enough. □

3. **Some wreath products**

Consider a non-empty set X, a group H, and the **permutational wreath product** $H\wr_X\mathrm{Sym}(X) := H^{(X)}\rtimes\mathrm{Sym}(X)$. Here, $H^{(X)}$ denotes the group of functions from X to H having finite support, for the pointwise multiplication, and the semi-direct product "$\rtimes$" refers to the natural action of $\mathrm{Sym}(X)$ on $H^{(X)}$, i.e. to $f\in\mathrm{Sym}(X)$ acting on $\psi\in H^{(X)}$ by $\psi\longmapsto f(\psi) := \psi\circ f^{-1}$. The multiplication in this wreath product is given by $(\varphi,f)(\psi,g) = (\varphi f(\psi), fg)$, for $\varphi,\psi\in H^{(X)}$ and $f,g\in\mathrm{Sym}(X)$. There is a natural action of the group $H\wr_X\mathrm{Sym}(X)$ on the set $H\times X$, for which (φ,f) acts by $(h,x)\longmapsto(\varphi(f(x))h, f(x))$; this action is faithful.

For $a\in H\smallsetminus\{1\}$ and $u\in X$, denote by $\varphi_u^a\in H\wr_X\mathrm{Sym}(X)$ the permutation that maps $(h,x)\in H\times X$ to (ah,u) if $x=u$, and to (h,x) otherwise; the support of φ_u^a is the set $\{(h,u)\}_{h\in H}$. Observe that $(\varphi_u^a)_{a\in H\smallsetminus\{1\},u\in X}$ generates the subgroup $H^{(X)}$, and that φ_u^a, φ_v^b are conjugate in $H \wr_X \mathrm{Sym}(X)$ if and only if a, b are conjugate in H.

For $u \in X$, we denote by H_u the set of elements φ_u^a for $a \in H \smallsetminus \{1\}$, and by T_H the subset $\bigcup_{u \in X} H_u$ of $H^{(X)}$; recall that T_X is the subset of all transpositions in $\mathrm{Sym}(X)$. Consider subsets $S_H \subset T_H$ and $S_X \subset T_X$, and define S to be the disjoint union $S_H \sqcup S_X$, inside $H \wr_X \mathrm{Sym}(X)$. It is again elementary to check that

$$\text{(GCwr)} \qquad \begin{array}{l} \text{if } \Gamma(S_X) \text{ is connected and if } S_H = \{\varphi_{u_1}^{a_1}, \dots, \varphi_{u_r}^{a_r}\} \\ \text{for some generating subset } \{a_1, \dots, a_r\} \subset H \\ \text{and some sequence } u_1, \dots, u_r \text{ of points of } X. \\ \text{then the group } H \wr_X \mathrm{Sym}(X) \text{ is generated by } S. \end{array}$$

When X is infinite, we consider subsets of $H \wr_X \mathrm{Sym}(X)$ of the form $S = S_H \sqcup S_X$ that satisfy the following condition:

$$\text{(PCwr)} \qquad \begin{array}{l} \text{the transposition graph } \Gamma(S_X) \text{ is connected and,} \\ \text{for all } L \geq 0 \text{ and partition } \lambda = (\lambda_1, \dots, \lambda_k) \vdash L, \\ \Gamma(S_X) \text{ contains a forest of } k \text{ trees } T_1, \dots, T_k, \\ \text{with } T_i \text{ having } \lambda_i \text{ vertices, including one of them, say } x^{(i)}, \\ \text{such that } \varphi_{x^{(i)}}^a \in S_H \text{ for all } a \in H \smallsetminus \{1\}. \end{array}$$

(The conditions "for all $a \in H \smallsetminus \{1\}$" could be replaced by "for all a in a set of representatives of the conjugacy classes in H distinct from $\{1\}$".)

Proposition 19. *Let H be a finite group; denote by M the number of conjugacy classes in H. Consider an infinite set X, the wreath product $W = H \wr_X \mathrm{Sym}(X)$, and a generating subset S that satisfies Condition* (PCwr)*. Then*

$$C_{W,S}(q) \; = \; \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^M} \, .$$

Set $\prod_{k=1}^{\infty} \frac{1}{(1-q^k)^M} = \sum_{n=0}^{\infty} p(n)_{(M)} q^n$. For low values of the integer M, the sequences $\big(p(n)_{(M)}\big)_{n=0,1,2,\dots}$ are well documented. For example, with A000041 and other similar numbers referring to those of [OEIS], we have:

$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \dots$ for $(p(n)_{(1)})_{n \geq 0}$, see A000041;

$1, 2, 5, 10, 20, 36, 65, 110, 185, 300, 481, \dots$ for $(p(n)_{(2)})_{n \geq 0}$, see A000712;

$1, 3, 9, 22, 51, 108, 221, 429, 810, 1479, 2640, \dots$ for $(p(n)_{(3)})_{n \geq 0}$, see A000716;

$1, 12, 90, 520, 2535, 10908, 42614, 153960, \dots$ for $(p(n)_{(12)})_{n \geq 0}$, see A005758;

for $(p(n)_{(M)})_{n \geq 0}$ when $4 \leq M \leq 23$, $M \neq 12$, see A023003 to A023021.

See also Section 6 and Appendix C for some congruence relations satisfied by the coefficients $p(n)_{(M)}$.

Proof of Proposition 19. In this proof, we write G for $\mathrm{Sym}(X)$ and W for $H \wr_X \mathrm{Sym}(X) = H^{(X)} \rtimes G$,

Preliminary Remark. There are several ways to associate a conjugacy class in a symmetric group to a partition. For example, when $X = \mathbf{N}$, in Observation 2 above and many other places of this article, the conjugay class associated to a partition such as $(3, 3, 1) \vdash 7$ is that of

$$(1, 2, 3, 4)(5, 6, 7, 8)(9, 10) \in \mathrm{Sym}(\mathbf{N}).$$

In other places, in particular at some point of the present proof, some fixed points of permutations are counted as parts of size 1, so that the conjugacy class associated to the same partition is that of[1]

$$(1, 2, 3)(4, 5, 6) \in \mathrm{Sym}(\mathbf{N}).$$

This is the reason for which we use below one symbol, λ, for a partition indexed by $1 \in H_*$ and a different symbol, μ, for a partition indexed by $\eta \neq 1$ in H_*.

First step: reminder on the conjugacy classes of W. The set of conjugacy classes of W is in bijection with the set of H_*-decorated partitions, as we now describe, much as in [Macd–95]. Here, H_* denotes the set of conjugacy classes of H; we write $1 \in H_*$ rather than $\{1\} \in H_*$ for the class $\{1\} \subset H$.

Let $w = (\varphi, f) \in H^{(X)} \rtimes_X \mathrm{Sym}(X)$. We proceed to associate a H_*-indexed family of partitions

$$(\dagger) \qquad \left(\lambda^{(1)}, \left(\mu^{(\eta)}\right)_{\eta \in H_* \smallsetminus 1}\right)$$

to w.

Let $X^{(w)}$ be the finite subset of X that is the union of the supports of φ and f. Denote by $c_1, \dots, c_k$ the disjoint cycles of which f is the product. Here, we include a cycle of length 1 for each point $x \in X$ such that $x \in \sup(\varphi)$ and $x \notin \sup(f)$, so that we have a disjoint union $X^{(w)} = \bigsqcup_{1 \le i \le k} \sup(c_i)$. For $i \in \{1, \dots, k\}$, there are points x_j^i in $X^{(w)}$, with $1 \le j \le \nu_i := \mathrm{length}(c_i)$, such that $c_i = (x_1^{(i)}, x_2^{(i)}, \dots, x_{\nu_i}^{(i)})$. Define $\eta_*^w(c_i) \in H_*$ to be the conjugacy class of the product $\varphi(x_{\nu_i}^{(i)})\varphi(x_{\nu_i - 1}^{(i)}) \cdots \varphi(x_1^{(i)}) \in H$. Observe that the product itself is not well-defined by c_i, since the $x_j^{(i)}$ are well-defined up to cyclic permutation only, but that its conjugacy class is well-defined. Observe also that, if $\nu_i = 1$, then $\eta_*^w(c_i) \neq 1$.

For $\eta \in H_*$ and $\ell \ge 1$, let $m_\ell^{w,\eta}$ denote the number of cycles c in $\{c_1, \dots, c_k\}$ that are of length ℓ and are such that $\eta_*^w(c) = \eta$. Let $\mu^{w,\eta}$ be the partition with $m_\ell^{w,\eta}$ parts equal to ℓ, for all $\ell \ge 1$; let $n^{w,\eta}$ be the sum of the parts of this partition, so that $\mu^{w,\eta} \vdash n^{w,\eta}$. We have $\sum_{\eta \in H_*} n^{w,\eta} = \sum_{\eta \in H_*, \ell \ge 1} \ell m_\ell^{w,\eta} = |X^{(w)}|$.

[1]At this point, it could be more consistent to include some fixed points in cycle decompositions of permutations, and thus to write $(1, 2, 3)(4, 5, 6)(7)(8)(9) \cdots \in \mathrm{Sym}(\mathbf{N})$.

We define the **pretype** of w as the family $\left(\mu^{w,\eta}\right)_{\eta \in H_*}$. By a routine argument, it can now be checked that

(i) for all $w = (\varphi, f) \in W$ and $g \in \mathrm{Sym}(X)$,
the pretypes of w and $(1, g)w(1, g^{-1})$ coincide;

(ii) for all $w = (\varphi, f) \in W$ and $\psi \in H^{(X)}$,
the pretypes of w and $(\psi, 1)w(\psi^{-1}, 1)$ coincide;

hence conjugate elements in W have the same pretype. Moreover:

(iii) two elements in W that have the same pretype are conjugate.

For details, we refer to [Macd–95, Appendix I.B, No. 3].

For $w = (\varphi, f) \in W$, observe that the partition $\mu^{w,1}$ does not have parts of size 1. With the same notation as above, denote by $\lambda^{w,1}$ the partiton with $m_\ell^{w,1}$ parts equal to $\ell - 1$. We define the **type** of w as the family $\left(\lambda^{w,1}, \left(\mu^{w,\eta}\right)_{\eta \in H_* \smallsetminus 1}\right)$. Then (i) to (iii) hold with "type" instead of "pretype". Moreover:

(iv) every H_*-indexed family of partitions, i.e., $\left(\lambda^{(1)}, \left(\mu^{(\eta)}\right)_{\eta \in H_* \smallsetminus 1}\right)$ as in (†), is the type of one conjugacy class in W.

Second step: proof of the formula for $C_{W,S}(q)$. Consider a H_*-index family of partitions $\left(\lambda^{(1)}, \left(\mu^{(\eta)}\right)_{\eta \in H_* \smallsetminus 1}\right)$ as in (†) and the corresponding conjugacy class in W. Denote by $n^{(1)}, n^{(\eta)}$ the sum of the parts and by $k^{(1)}, k^{(\eta)}$ the number of the parts of $\lambda^{(1)}, \mu^{(\eta)}$, respectively. Choose a representative $w = (\varphi, f)$ of this class, with f of the form $f = \prod_{i=1}^k c_i = \prod_{i=1}^k (x_1^{(i)}, x_2^{(i)}, \dots, x_{\mu_i}^{(i)})$ and

$$\begin{aligned} \varphi(x_j^{(i)}) \; &= \; 1 \in H \;\text{ for all }\; j \in \{1, \dots, \mu_i\} && \text{when } \eta_*^w(c_i) = 1 \\ \varphi(x_j^{(i)}) \; &= \; \begin{cases} \quad 1 & \text{for all } \; j \in \{1, \dots, \mu_i - 1\} \\ h \neq 1 & \text{for } \; j = \mu_i \end{cases} && \text{when } \eta_*^w(c_i) \neq 1. \end{aligned}$$

Recall that $\eta_*^w(c_i) \neq 1$ when $\mu_i = 1$, and observe that

$$\begin{aligned} k \; &= \; k^{(1)} + \sum_{\eta \in H_*, \eta \neq 1} k^{(\eta)} \\ |X^{(w)}| \; &= \; n^{(1)} + k^{(1)} + \sum_{\eta \in H_*, \eta \neq 1} n^{(\eta)}. \end{aligned}$$

The contribution of $(\varphi_{|\sup(c_i)}, c_i)$ to $\kappa_{W,S}(q)$ is $\mu_i - 1$ if $\eta_*^w(c_i) = 1$, and μ_i if $\eta_*^w(c_i) \neq 1$. Hence, the contribution of the type $\left(\lambda^{(1)}, \left(\mu^{(\eta)}\right)_{\eta \in H_* \smallsetminus 1}\right)$ to $C_{W,S}(q)$ is $q^{n^{(1)}} \prod_{\eta \in H_*, \eta \neq 1} q^{n^{(\eta)}}$. It follows that

$$\begin{aligned} C_{W,S}(q) \; &= \; \Big(\sum_{n_1=0}^{\infty} p(n_1) q^{n_1} \Big) \prod_{\eta \in H_*, \eta \neq 1} \Big(\sum_{n_\eta=0}^{\infty} p(n_\eta) q^{n_\eta} \Big) \\ &= \prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^{|H_*|}}. \end{aligned}$$

This ends the proof of Proposition 19. □

4. A sample of growth polynomials and conjugacy growth polynomials for finite symmetric groups

The purpose of the present section is to compute for $\mathrm{Sym}(n)$ growth polynomials $L_{\mathrm{Sym}(n),S}(q)$ and conjugacy growth polynomials $C_{\mathrm{Sym}(n),S}(q)$, with respect to a sample of generating sets S. Our computations rely partly on Lemmas 3 of Section 1 and 15 of Section 2.

Before this, we review part of what is known in the broader and classical setting of finite Coxeter groups. Though we will not recall precise statements, this is strongly related to the topology of connected compact Lie groups and their homogenous spaces.

Let (W, S) be a finite Coxeter system; set $l = |S|$. Denote the corresponding Coxeter exponents by $m_1, \dots, m_l$; they are positive integers. The growth polynomial is known to be

$$(L_{W,S}) \qquad L_{W,S}(q) \; = \; \prod_{k=1}^{l} (1 + q + \cdots + q^{m_k}).$$

This has received much attention; see for example [Solo–66] and [Bour–68, exercises of § IV.1 and VI.4]. As Solomon observes, the computation of $L_{W,S}$ for the particular case of the symmetric groups goes back to Rodrigues, in the first half of XIXth century (with a different formulation). Set $T = \bigcup_{w \in W} wSw^{-1}$. The word length $\ell_{W,T}$ is sometimes called the *reflection length* [Cart–72] and the corresponding growth polynomial is known to be

$$(L_{W,T}) \qquad L_{W,T}(q) \; = \; \prod_{k=1}^{l} (1 + m_k q).$$

For a group $W \subset \mathrm{GL}(V)$ generated by reflections, define $\rho : W \longrightarrow \mathbf{N}$ by

$$\rho(w) = \dim(V) - \dim(\{v \in V \mid w(v) = v\})$$

and set $R_W(q) = \sum_{w \in W} q^{\rho(w)}$. Then $R_W(q) = \prod_{k=1}^{l}(1 + m_k q)$; this is a special case of [ShTo–54, Number 5.3], verified there by inspection, and shown again more conceptually in [Solo–63]. For a finite *Weyl group*, it is easy to show that $\rho(w) = \ell_{W,S}(w)$, see e.g. [Cart–72, Lemma 2], so that $L_{W,T} = R_W$, and $(L_{W,T})$ holds; this carries over to every finite *Coxeter group*, see e.g. [Lehr–87]. Other avatars of these polynomials are discussed in [BaGo–94].

We do not know whether the companion polynomials $C_{W,S}, C_{W,T}$ have already been given any attention.

In the next proposition, we particularize $L_{W,S}(q)$ and $L_{W,T}(q)$ to $W = \mathrm{Sym}(n)$, and we provide expressions for the corresponding conjugacy growth polynomials. In the special case of finite symmetric groups, there is an ad hoc proof for $(L_{W,T})$ in Remark 21 and one for $(L_{W,S})$ in [Harp–91].

Proposition 20. *Consider an integer $n \geq 1$, the symmetric group* $\mathrm{Sym}(n)$ *and its generating sets*

$$\begin{aligned} S_n^{Cox} &= \{(1,2),(2,3),\cdots,(n-1,n)\}, \\ T_n &= \{(i,j) \mid 1 \leq i,j \leq n,\ i<j\}, \end{aligned}$$

as in Proposition 1. The corresponding growth polynomial and conjugacy growth polynomial are

$$\begin{aligned} L_{\mathrm{Sym}(n),S_n^{Cox}}(q) &= \prod_{k=1}^{n-1}(1+q+\cdots+q^k), \\ L_{\mathrm{Sym}(n),T_n}(q) &= \prod_{k=1}^{n-1}(1+kq), \\ C_{\mathrm{Sym}(n),S_n^{Cox}}(q) &= C_{\mathrm{Sym}(n),T_n}(q) \;=\; \sum_{k=0}^{n-1} p_{n-k}(n) q^k\ , \end{aligned}$$

where $p_{n-k}(n)$ is as in Appendix B.b.

Proof. The equalities involving the two products are particular cases of $(L_{W,S})$ and $(L_{W,T})$, since the Coxeter exponents of $(\mathrm{Sym}(n), S_n^{\mathrm{Cox}})$ are $1, 2, \ldots, n-1$. The equality for $C_{\mathrm{Sym}(n),S_n^{\mathrm{Cox}}}(q)$ is that of Proposition 1(ii), and $C_{\mathrm{Sym}(n),T_n}(q)$ is the same polynomial, see Remark 13. □

The polynomials $C_{\mathrm{Sym}(n),T_n}(q)$ for small n 's are given by

$$\begin{aligned} C_{\mathrm{Sym}(2),T_2}(q) &= 1+q, \\ C_{\mathrm{Sym}(3),T_3}(q) &= 1+q+q^2, \\ C_{\mathrm{Sym}(4),T_4}(q) &= 1+q+2q^2+q^3, \\ C_{\mathrm{Sym}(5),T_5}(q) &= 1+q+2q^2+2q^3+q^4, \\ C_{\mathrm{Sym}(6),T_6}(q) &= 1+q+2q^2+3q^3+3q^4+q^5\ . \end{aligned}$$

(Compare with the polynomials written after Proposition 22.)

Remark 21. (i) The second polynomial of Proposition 20 can also be written

$$L_{\mathrm{Sym}(n),T_n}(q) \;=\; 1 + \sum_{m=2}^{n} \binom{n}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} d_k(m) q^{m-k},$$

where $d_k(m)$ is as in Appendix B.d.

(ii) It is easy to check directly from (i) that we have also

$$L_{\mathrm{Sym}(n),T_n}(q) \;=\; \prod_{k=1}^{n-1}(1+kq),$$

as in Proposition 20.

Proof. (i) For $m \in \{0, 1, \dots, n\}$, there are $\binom{n}{m}$ subsets of size m in $\{1, 2, \dots, n\}$. For each such subset, say A, and each $k \in \{0, 1, \dots, n\}$, there are $d_k(m)$ permutations in $\mathrm{Sym}(n)$ with support A which are products of k disjoint cycles, and these elements have T_n-word length $m - k$, by Lemma 3. The growth polynomial of the situation is therefore

$$\sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{n} d_k(m) q^{m-k} .$$

To end this computation, we observe that the contribution of $m = 0$ is 1, that of $m = 1$ is 0, and $d_k(m) = 0$ for $2k > n$.

(ii) We proceed by induction on n. There is nothing to check for $n = 1$; we assume now that $n \geq 2$, and that the statement holds for $n - 1$.

Consider an element $g \in \mathrm{Sym}(n)$ which is not in $\mathrm{Sym}(n-1)$. There is a unique pair consisting of $i \in \{1, \dots, n-1\}$ and $h \in \mathrm{Sym}(n-1)$ such that $g = (i, n)h$. This implies that

$$C_{\mathrm{Sym}(n),T_n}(q) \;=\; C_{\mathrm{Sym}(n-1),T_{n-1}}(q) + (n-1)qC_{\mathrm{Sym}(n-1),T_{n-1}}(q).$$

Hence

$$C_{\mathrm{Sym}(n),T_n}(q) \;=\; C_{\mathrm{Sym}(n-1),T_{n-1}}(q) \,\big(1 + (n-1)q\big) \;=\; \prod_{i=1}^{n-1}(1 + kq)$$

by the induction hypothesis. □

The final proposition of this section shows polynomials L and C for finite symmetric groups and a third generating set S_n^0, essentially distinct from the generating sets S_n^{Cox} and T_n of Proposition 20 for $n \geq 4$. It is convenient to see $\mathrm{Sym}(n)$ as the symmetric group of $\{0, 1, \dots, n-1\}$; the generating set S_n^0 is that already considered in Lemmas 14 and 15.

Proposition 22. *Consider an integer $n \geq 1$, the symmetric group* $\mathrm{Sym}(n)$ *and the generating set $S_n^0 = \{(0, i) \mid 1 \leq i \leq n-1\}$. The corresponding growth polynomial and conjugacy growth polynomial are*

$$\begin{aligned} L_{\mathrm{Sym}(n),S_n^0}(q) \;&=\; 1 + \sum_{m=2}^{n-1} \binom{n-1}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} d_k(m) q^{m+k} \\ &\quad + \sum_{m=2}^{n} \binom{n-1}{m-1} \sum_{k=1}^{\lfloor m/2 \rfloor} d_k(m) q^{m+k-2} , \\ C_{\mathrm{Sym}(n),S_n^0}(q) \;&=\; 1 \;+\; \sum_{k=1}^{\lfloor n/2 \rfloor} q^{2k-2} \sum_{j=k}^{n} p_k(j) q^j \;\; \textit{(as in Proposition 9).} \end{aligned}$$

For example:

$$\begin{aligned}
L_{\mathrm{Sym}(4),S_4^0}(q) &= 1+3q+6q^2+9q^3+5q^4,\\
L_{\mathrm{Sym}(5),S_5^0}(q) &= 1+4q+12q^2+30q^3+44q^4+26q^5+3q^6,\\
L_{\mathrm{Sym}(6),S_6^0}(q) &= 1+5q+20q^2+70q^3+170q^4+250q^5+169q^4+35q^7,
\end{aligned}$$

and

$$\begin{aligned}
C_{\mathrm{Sym}(4),S_4^0}(q) &= 1+q+q^2+q^3+q^4,\\
C_{\mathrm{Sym}(5),S_5^0}(q) &= 1+q+q^2+q^3+2q^4+q^5,\\
C_{\mathrm{Sym}(6),S_6^0}(q) &= 1+q+q^2+q^3+2q^4+2q^5+2q^6+q^7\ .
\end{aligned}$$

(Compare with the polynomials written after Proposition 20.)

Proof. Let us deal with the polynomial L. Consider first elements $g \in \mathrm{Sym}(n)$ with $0 \notin \sup(g)$. For each $m \in \{0, 1, \dots, n-1\}$, there are $\binom{n-1}{m}$ subsets of size m in $\{1, 2, \dots, n-1\}$. For each such subset, say A, and each $k \in \{0, 1, 2, \dots, m\}$, there are $d_k(m)$ elements with support A which are products of k cycles, and these elements have S_n^0-word length $m+k$, by Lemma 15. The contribution to the growth polynomial of elements with $0 \notin \sup(g)$ is therefore

$$(0 \notin \sup) \qquad \sum_{m=0}^{n-1} \binom{n-1}{m} \sum_{k=0}^{m} d_k(m) q^{m+k}\ .$$

The contribution of $m=0$ is 1 and that of $m=1$ is zero; for $m \geq 2$, the contributions of terms with $k=0$ or $k > m/2$ is also zero.

Consider now elements $g \in \mathrm{Sym}(n)$ with $0 \in \sup(g)$. For each $m \in \{1, 2, \dots, n\}$, there are $\binom{n-1}{m-1}$ subsets of size m in $\{0, 1, \dots, n-1\}$ containing 0. For each such subset, say B, and each $k \in \{1, 2, \dots, m\}$, there are $d_k(m)$ elements with support B which are products of k cycles, and these elements have S_n^0-word length $m+k-2$. The contribution of these elements is therefore

$$(0 \in \sup) \qquad \sum_{m=1}^{n} \binom{n-1}{m-1} \sum_{k=1}^{m} d_k(m) q^{m+k-2}\ .$$

As above, the contributions of terms with $m=1$ or $k > m/2$ vanish.

The formula for $L_{\mathrm{Sym}(n),S_n^0}(q)$ follows. That for $C_{\mathrm{Sym}(n),S_n^0}(q)$ is a repetition of part of Proposition 9. □

5. **Alternating groups**

For a non-empty set X, we denote by $\mathrm{Alt}(X)$ the **finitary alternating group** of X, i.e. the subgroup of $\mathrm{Sym}(X)$ of permutations of even signature. Set

$$\begin{aligned}
T_X^A &= \big\{(x,y,z) \in \mathrm{Alt}(X) \mid x,y,z \in X \text{ are distinct}\big\},\\
U_X^A &= \big\{(x,y)(z,u) \in \mathrm{Alt}(X) \mid x,y,z,u \in X \text{ are distinct}\big\}.
\end{aligned}$$

Recall from the introduction that, when $X = \mathbf{N}$, we have defined

$$S^A_{\mathbf{N}} = \{(i, i+1, i+2) \in \mathrm{Alt}(\mathbf{N}) \mid i \in \mathbf{N}\},$$

and we consider also

$$R^A_{\mathbf{N}} = \big\{(1, i, i+1) \in \mathrm{Alt}(\mathbf{N}) \mid i \geq 2\big\}.$$

When $X = \{1, \dots n\}$ for some $n \geq 3$, we write

$$\begin{aligned} \mathrm{Alt}(n) &= \mathrm{Alt}(\{1, 2, \dots, n\}), \\ S^A_n &= \{(i, i+1, i+2) \in \mathrm{Alt}(n) \mid 1 \leq i \leq n-2\}, \\ R^A_n &= \{(1, i, i+1) \in \mathrm{Alt}(n) \mid 2 \leq i \leq n-1\}. \end{aligned}$$

When X is either $\mathbf{N}$ or $\{1, \dots, n\}$ for some $n \geq 1$, we write S^A_X to denote the relevant set, either $S^A_{\mathbf{N}}$ or S^A_n, and similarly for R^A_X.

The following lemma is well-known, even if we did not find a convenient reference.

Lemma 23. *With the notation above:*
for all $n \geq 3$, the sets S^A_n and R^A_n both generate $\mathrm{Alt}(n)$*;*
the sets $S^A_{\mathbf{N}}$ and $R^A_{\mathbf{N}}$ both generate $\mathrm{Alt}(\mathbf{N})$*;*
and the set T^A_X generates $\mathrm{Alt}(X)$*.*

Proof. Let H_n denote the subgroup of $\mathrm{Alt}(n)$ generated by S^A_n; we claim that $H_n = \mathrm{Alt}(n)$. The case of $n = 3$ is obvious; we proceed by induction on n, assuming that $n \geq 4$ and that the claim holds for $n - 1$.

The group H_n acts transitively on $\{1, \dots, n\}$, because it contains the 3-cycle $(n-2, n-1, n)$ as well as $H_{n-1} = \mathrm{Alt}(n-1)$. Hence the order of H_n is n times the index of the isotropy group $\{h \in H_n \mid h(n) = n\}$, that is $|H_n| = n\frac{1}{2}(n-1)! = \frac{1}{2}n!$. It follows that $H_n = \mathrm{Alt}(n)$.

As a consequence, R^A_n also generates $\mathrm{Alt}(n)$, since
$(1, i+1, i)(1, i+2, i+1)(1, i, i+1) = (i, i+1, i+2)$ for all $i \in \{2, \dots, n-1\}$.

The claims for $S^A_{\mathbf{N}}$, $R^A_{\mathbf{N}}$ and T^A_X follow. $\square$

Note that $T^A_X \cup U^A_X$ is the set of products of two distinct elements of the generating set S_X of $\mathrm{Sym}(X)$. It follows that

$$\kappa_{\mathrm{Alt}(X), T^A_X \cup U^A_X}(g) = \frac{1}{2}\kappa_{\mathrm{Sym}(X), S_X}(g) \quad \text{for all } g \in \mathrm{Alt}(X).$$

Since, for X infinite, two elements of $\mathrm{Alt}(X)$ are conjugate in $\mathrm{Alt}(X)$ if and only if they are conjugate in $\mathrm{Sym}(X)$, we obtain the following straightforward consequence of Proposition 8:

Proposition 24. *Let X be an infinite set, and T_X^A, U_X^A as above. Then*

$$\begin{aligned} C_{\mathrm{Alt}(X),T_X^A\cup U_X^A}(q) &= \sum_{m=0}^{\infty} p(2m)q^m \\ &= 1+2q+5q^2+11q^3+22q^4+42q^5+77q^6 \\ &+ 135q^7+231q^8+385q^9+627q^{10}+1002q^{11}+1575q^{12}+\cdots, \end{aligned}$$

where the numerical coefficients are those of the series [OEIS, A058696].

Let X be a set containing at least 5 elements. It is easy to check that U_X^A generates $\mathrm{Alt}(X)$, and it can be shown that

$$\kappa_{\mathrm{Alt}(X),U_X^A}(g) = \begin{cases} \kappa_{\mathrm{Alt}(X),T_X^A\cup U_X^A}(g) & \text{if } g=\mathrm{id} \text{ or } |\sup(g)|>3, \\ 2 & \text{if } g \text{ is a 3-cycle.} \end{cases}$$

It follows that

Proposition 25. *Let X be an infinite set, and U_X^A as above. Then*

$$C_{\mathrm{Alt}(X),U_X^A}(q) = q^2-q+\sum_{m=0}^{\infty} p(2m)q^m.$$

Remark. For the generating set

$V_{\mathbf{N}}^A := \{(i,i+1,i+2)\in \mathrm{Alt}(\mathbf{N}) \mid i\ge 0\}\cup\{(i,i+1)(i+2,i+3)\in\mathrm{Alt}(\mathbf{N}) \mid i\ge 0\}$, it can be shown that

$$C_{\mathrm{Alt}(\mathbf{N}),V_{\mathbf{N}}^A}(q) = C_{\mathrm{Alt}(\mathbf{N}),T_{\mathbf{N}}^A\cup U_{\mathbf{N}}^A}(q).$$

Our next target is to identify $C_{\mathrm{Alt}(X),T_X^A}(q)$.

Lemma 26. *Let $g\in\mathrm{Alt}(X)$ and $g=t_1\cdots t_L$ a writing of g as a word of minimal length $L=\ell_{\mathrm{Alt}(X),T_X^A}(g)$ in the generators of T_X^A.*

Then $t_j\neq t_i^{\pm1}$, equivalently $\sup(t_i)\neq\sup(t_j)$, for all $i,j\in\{1,\dots,L\}$ with $i\neq j$.

Proof. Let $g=u_1\cdots u_M$ be a writing of g as a word in the generators of T_X^A.

Suppose first that there exist $j,k\in\{1,\dots,M\}$ with $j<k$ such that $u_k=u_j^{-1}$. If $k=j+1$, then deleting u_ju_k produces a new T_X^A-word of length $M-2$ representing g; if $k\ge j+2$, then g can be written as

$$u_1\cdots u_{j-1}\left(u_ju_{j+1}u_j^{-1}\right)\cdots\left(u_ju_{k-1}u_j^{-1}\right)u_{k+1}\cdots u_m,$$

i.e. g can again be written as a T_X^A-word of length $M-2$ representing g.

Suppose now that there exist $j,k\in\{1,\dots,M\}$ with $j<k$ such that $u_k=u_j$. If $k=j+1$, then replacing u_ju_k by u_j^{-1} produces a new T_X^A-word of length $M-1$ representing g; if $k\ge j+2$, then g can be written as

$$u_1\cdots u_{j-1}u_ju_{j+1}\cdots u_{k-1}u_j^{-1}u_j^{-1}u_{k+1}\cdots u_M$$

and the previous procedure provides a T_X^A-word representing g of length $M-1$.

The lemma follows. $\square$

For $g \in \mathrm{Alt}(X)$ a product of disjoint cycles, we denote by k'_g the number of cycles of odd lengths ≥ 3 and by $2k''_g$ the number of cycles of even lengths ≥ 2. Note that $k_g = k'_g + 2k''_g$ for k_g as in Lemma 17.

Lemma 27. *Let X be a set and S a generating set of* $\mathrm{Alt}(X)$. *Let $g \in \mathrm{Alt}(X)$ be a product of disjoint cycles, with k'_g, k''_g as above. Suppose either that $S = T_X^A$ or that X is one of* $\mathbf{N}$, *$\{1, \dots, n\}$ for some $n \geq 1$, and that $S_X^A \subset S \subset T_X^A$. We have*

$$\ell_{\mathrm{Alt}(X), T_X^A}(g) \; = \; \kappa_{\mathrm{Alt}(X),S}(g) \; = \; \frac{1}{2}\big(|\sup(g)| - k'_g\big).$$

In the proof below, we write ℓ for $\ell_{\mathrm{Alt}(X),T_X^A}$ and κ for $\kappa_{\mathrm{Alt}(X),S}$.

Proof of the upper bounds $\kappa(g), \ell(g) \leq \frac{1}{2}\big(|\sup(g)| - k'_g\big)$. We show the bound for $\ell(g)$, and leave it to the reader to check that a minor modification of the same argument shows the bound for $\kappa(g)$. Whenever convenient, we write k', k'' rather than k'_g, k''_g.

Consider a cycle of odd length, say

$$c_\alpha \; = \; (x_1, x_2, \dots, x_{2p+1})$$

for $x_1, \dots, x_{2p+1} \in X$. We have

$$c_\alpha \; = \; (x_1, x_2, x_3)(x_3, x_4, x_5)(x_5, x_6, x_7) \cdots (x_{2p-1}, x_{2p}, x_{2p+1})$$

and therefore $\ell(c_\alpha) \leq p = \frac{1}{2}(|\sup(c_\alpha)| - 1)$.

Consider a pair of disjoint cycles of even lengths, say

$$c_\beta c_\gamma \; = \; (x_1, x_2, \dots, x_{2r})(y_1, y_2, \dots, y_{2s})$$

for $x_1, \dots, x_{2r}, y_1 \dots, y_{2s} \in X$ (where we consider an appropriate conjugate of g and $2r+2s$ *consecutive* integers $y_1, y_2, \dots, y_{2s}, x_1, x_2, \dots, x_{2r}$ for the case of $\kappa(g)$). We have

$$\begin{aligned} c_\beta c_\gamma = & (y_1, y_2, y_3)(y_3, y_4, y_5) \cdots (y_{2s-3}, y_{2s-2}, y_{2s-1})(y_{2s-1}, y_{2s}, x_1) \\ & (x_1, x_2, x_3)(x_3, x_4, x_5) \cdots (x_{2r-3}, x_{2r-2}, x_{2r-1})(x_{2r-1}, x_{2r}, y_{2s}) \end{aligned}$$

and therefore $\ell(c_\beta c_\gamma) \leq r + s = \frac{1}{2}\left(|\sup(c_\beta)| + |\sup(c_\gamma)|\right)$.

For $g = c_1 c_2 \cdots c_{k'} c_{k'+1} c_{k'+2} \cdots c_{k'+2k''}$, where $c_1, \dots, c_{k'+2k''}$ are disjoint cycles, c_ν of odd length for $1 \leq \nu \leq k'$ and of even length for $k'+1 \leq \nu \leq k'+2k''$, it follows that

$$\begin{aligned} \ell(g) \; \leq \; \sum_{\nu=1}^{k'+2k''} \ell(c_\nu) \; & \leq \; \frac{1}{2}\Big(\sum_{\alpha=1}^{k'} (|\sup(c_\alpha)| - 1) + \sum_{\beta=k'+1}^{2k''} |\sup(c_\beta)| \Big) \\ & = \; \frac{1}{2}\big(|\sup(g)| - k'\big) \, , \end{aligned}$$

as was to be shown. $\square$

Proof of the lower bounds $\ell(g), \kappa(g) \geq \frac{1}{2}\big(|\sup(g)| - k'_g\big)$. For $g \neq \mathrm{id}$ in $\mathrm{Alt}(X)$ such that $|\sup(g)| \leq 3$, we have obviously $1 = \ell(g) = \kappa(g) \geq \frac{1}{2}\big(|\sup(g)| - k'_g\big) = \frac{1}{2}(3-1)$. We consider from now on an element g in $\mathrm{Alt}(X)$ with $|\sup(g)| > 3$, and therefore with $\ell(g) > 1$ and $\kappa(g) > 1$. As above, we continue and deal with $\ell(g)$ only.

Suppose by contradiction that there exists $g \in \mathrm{Alt}(X)$ with $|\sup(g)| > 3$ and

$$(\flat) \qquad \ell(g) \,<\, \frac{1}{2}\big(|\sup(g)| - k'_g\big) \;;$$

suppose moreover that $\ell(g)$ is minimal for the elements for which $(\flat)$ holds. We can write

$$(\flat\flat) \qquad g = t_1 \cdots t_L$$

for some $t_1, \ldots, t_L \in T_X^A$ with $1 \,<\, L \,=\, \ell(g) \,<\, \frac{1}{2}\big(|\sup(g)| - k'_g\big)$. By Lemma 26, we know that the supports $\sup(t_i)$ are pairwise distinct.

For each $i \in \{1, \ldots, L\}$, let $x_i, y_i, z_i \in X$ be such that $t_i = (x_i, y_i, z_i)$. Set $Y_i = \sup(t_i) = \{x_i, y_i, z_i\}$ and $Z_i = \bigcup_{1 \leq j \leq L,\, j \neq i} Y_j$.

Claim: We have

$$(\sharp) \qquad |Y_i \cap Z_i| \,\geq\, 2 \;\text{ for all }\; i \in \{1, \ldots, L\}.$$

Upon conjugating g by $t_{i+1} \cdots t_L$, we can assume that $i = L$ for the proof of the claim.

Let us first check that $|Y_L \cap Z_L| \geq 1$. Indeed, otherwise, set

$$(\ddagger) \qquad h \,=\, \prod_{i=1}^{L-1} t_i \,.$$

Observe that $\ell(h) \leq L-1$. We have $|\sup(h)| = |\sup(g)| - 3$, and also $k'_h = k'_g - 1$, since the cycle of odd length t_i has been deleted in the product defining h. It follows that $\ell(h) < \frac{1}{2}\big(|\sup(h)| - k'_h\big)$. This contradicts the minimality hypothesis on g made above; hence $|Y_L \cap Z_L| \geq 1$.

Let us now show that $|Y_L \cap Z_L| \geq 2$. Indeed, otherwise, $|Y_L \cap Z_L| = 1$. Let again h be defined by $(\ddagger)$; observe again that $\ell(h) \leq L - 1$, and that $|\sup(h)| = |\sup(g)| - 2$; it can be shown that $k'_h = k'_g$ (details below). It follows that $\ell(h) < \frac{1}{2}\big(|\sup(h)| - k'_h\big)$. This contradicts again the minimality hypothesis above; hence $|Y_L \cap Z_L| \geq 2$.

Here are the announced details. Let $x, y, z \in X$ be such that $Y_L \cap Z_L = \{x\}$ and $t_L = (x, y, z)$; Then x is contained in the support of a cycle d of h of length $\ell \geq 2$, and also by Lemma 31 in the support of a cycle $c = dt_L$ of $g = ht_L$ of length $\ell + 2$. Hence $k'_h = k'_g$.

This ends the proof of the Claim.

Lemma 26 and the claim just proven imply that, for each $i \in \{1, \ldots, L\}$, there are $x_i, y_i, z_i \in X$ such that

$t_i = (x_i, y_i, z_i)$,
$y_i, z_i \in Z_i$.

Consider the product of $2L$ transpositions, equal to g, obtained from the product ($\flat\flat$) by changing each t_i to $(x_i, z_i)(x_i, y_i)$, say

$$g \; = \; s_1 s_2 \cdots s_{2L-1} s_{2L}.$$

Set $S = \{s_1, \dots, s_{2L}\}$; define $\widetilde{\Gamma}(S)$ to be the multigraph with vertex set $V := \bigcup_{j=1}^{2L} \operatorname{sup}(s_j)$, and one edge connecting $x, y \in V$ for every $j \in \{1, \dots, 2L\}$ with $s_j = (x, y)$; here, "multigraph" means that $\widetilde{\Gamma}(S)$ may have multiple edges. On the one hand, the number of vertices of this graph is bounded below by $|\operatorname{sup}(g)|$; on the other hand, what we have shown so far implies that the degree of each vertex of $\widetilde{\Gamma}(S)$ is at least 2; it follows that the number of edges of this graph, which is at least twice its number of vertices, is bounded below by $|\operatorname{sup}(g)|$; in other words, $L \geq \frac{1}{2}|\operatorname{sup}(g)|$. This is strongly in contradiction with ($\flat$); hence the inequality of ($\flat$) is not true, and this ends the proof of the lemma. $\square$

Remark concerning the claim of the previous proof. Consider an element $g \in \operatorname{Alt}(X)$ which is a word $g = t_1 \cdots t_L$ in the letters of T_X^A of minimal length $L = \ell(g)$, now with $2 \leq L \leq \frac{1}{2}(\operatorname{sup}(g)| - k'_g)$. The cardinality $|Y_1 \cap Z_1|$ can be any of $0, 1, 2, 3$, as the following examples show:

$$\begin{aligned}
g_0 &= (1,2,3)(4,5,6) \text{ for which} \\
&\quad L = 2, \;\; |\operatorname{sup}(g_0)| - k'_{g_0} = 6 - 2, \;\text{ and } Y_1 \cap Z_1 = \emptyset, \\
g_1 &= (1,4,5)(1,2,3) = (1,2,3,4,5) \text{ for which} \\
&\quad L = 2, \;\; |\operatorname{sup}(g_1)| - k'_{g_1} = 5 - 1, \;\text{ and } Y_2 \cap Z_2 = \{1\}, \\
g_2 &= (5,6,7)(2,3,4)(1,4,7) = (1,2,3,4,5,6,7) \text{ for which} \\
&\quad L = 3, \;\; |\operatorname{sup}(g_2)| - k'_{g_2} = 7 - 1, \;\text{ and } Y_3 \cap Z_3 = \{4,7\}, \\
g_3 &= (1,8,9)(5,6,7)(2,3,4)(1,4,7) = (1,2,3,4,5,6,7,8,9) \text{ for which} \\
&\quad L = 4, \;\; |\operatorname{sup}(g_3)| - k'_{g_3} = 9 - 1, \;\text{ and } Y_4 \cap Z_4 = \{1,4,7\}.
\end{aligned}$$

Proposition 28 is a minor generalization of Proposition 11. Recall from Appendix B.c that $p_e(n)$ denotes the number of partitions of $n \in \mathbf{N}$ involving an even number of positive parts.

Proposition 28. *Let X be an infinite set and S a generating set of* $\mathrm{Alt}(X)$. *Suppose either that $S = T_X^A$ or that $X = \mathbf{N}$ and that $S_{\mathbf{N}}^A \subset S \subset T_{\mathbf{N}}^A$. Then*

$$\begin{aligned} C_{\mathrm{Alt}(X),S}(q) &= \sum_{u=0}^{\infty} p(u)q^u \sum_{v=0}^{\infty} p_e(v)q^v \\ &= \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{(1-q^j)^2} + \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1-q^{2j}} \\ &= 1 + q + 3q^2 + 5q^3 + 11q^4 + 18q^5 + 34q^6 \\ &\quad + 55q^7 + 95q^8 + 150q^9 + 244q^{10} + \cdots . \end{aligned}$$

Proof. We write κ for $\kappa_{\mathrm{Alt}(X),S}$.

Let $g \in \mathrm{Alt}(X)$ be written as a product of disjoint cycles, say k' of them of odd lengths and $2k''$ of them of even lengths. Denote by g_o the product of the cycles of odd lengths and by g_e the product of the cycles of even lengths, so that $g = g_o g_e$. Let $\lambda^{(g)} = (\lambda_1^{(g)}, \dots, \lambda_{k'}^{(g)}) \vdash u$ and $\nu^{(g)} = (\nu_1^{(g)}, \dots, \nu_{2k''}^{(g)}) \vdash v$ be the partitions such that g_o is the product of cycles of lengths $2\lambda_1^{(g)} + 1, \dots, 2\lambda_{k'}^{(g)} + 1$, and g_e the product of cycles of lengths $2\nu_1^{(g)}, \dots, 2\nu_{2k''}^{(g)}$; note that $|\mathrm{sup}(g_o)| = 2u + k'$ and $|\mathrm{sup}(g_e)| = 2v$. By Lemma 27, we have

$$\kappa(g_o) = u, \quad \kappa(g_e) = v, \quad \text{and} \quad \kappa(g) = \kappa(g_o) + \kappa(g_e) = u + v.$$

The set of conjugacy classes in $\mathrm{Alt}(X)$ is naturally parametrized by pairs (λ, ν) of partitions such that ν has an even number of positive parts. (It is important here that the set X is infinite, otherwise some pairs correspond to *two* conjugacy classes in the alternating group). The contribution to $C_{\mathrm{Alt}(X),S}(q)$ of classes of elements such that $g = g_o$ is therefore $\sum_{u=0}^{\infty} p(u)q^u = \prod_{i=1}^{\infty}\frac{1}{1-q^i}$; the contribution of classes of elements such that $g = g_e$ is $\sum_{v=0}^{\infty} p_e(v)q^v = \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1-q^j} + \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1+q^j}$ (this uses Proposition 34); finally $C_{\mathrm{Alt}(X),S}$ is the product of these two contributions. $\square$

Remark 29. (i) Recall from Observation 12 that we denote by

$$\left(p^A(n)\right)_{n\geq 0} = (1, 1, 3, 5, 11, 18, 34, 55, 95, 150, 244, \dots)$$

the sequence of coefficients of the series of Proposition 28. At the day of writing, this sequence does not appear in [OEIS].

(ii) The sums and products in the previous proposition converge again for q complex with $|q| < 1$. Numerically, the roots of smallest absolute value of $C_{\mathrm{Alt}(\mathbf{N}),T_{\mathbf{N}}^A}(q)$ are simple and located at $\sim 0.67 \pm 0.43i$.

(iii) As in the case of $C_{\mathrm{Sym}(X),S}(q)$, see Proposition 8(a), it can be observed that the series $C_{\mathrm{Alt}(X),T_X^A}(q)$ does not depend on the cardinality of X, as long as X is infinite.

6. Congruences à la Ramanujan for the coefficients of the series of Proposition 28

Ramanujan, and later Watson, Atkin, Andrews, and others, have discovered remarkable congruence properties for the partition function, including

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}, \\ p(25n+24) &\equiv 0 \pmod{5^2}, \\ p(125n+99) &\equiv 0 \pmod{5^3}, \\ p(49n+47) &\equiv 0 \pmod{7^2}, \\ p(121n+116) &\equiv 0 \pmod{11^2}. \end{aligned}$$

See for example [Hard–40], or [Bern–06] and references there.

Consider a finite group H with M conjugacy class, an infinite set X, the permutational wreath product $W = H \wr_X \mathrm{Sym}(X)$, a generating set S that satisfies Condition (PCwr), and the corresponding conjugacy growth series

$$C_{W,S}(q) = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^M} = \sum_{n=0}^{\infty} p(n)_{(M)} q^n$$

as in Proposition 19. There is an important literature on congruence properties of the sequences $\big(p(n)_{(M)}\big)_{n=0,1,2,\dots}$ of so-called *multipartition numbers*. In particular:

(Gandhi)
$$\begin{aligned} p(5n+3)_{(2)} &\equiv 0 \pmod{5}, \\ p(11n+4)_{(8)} &\equiv 0 \pmod{11}, \end{aligned}$$

(Andrews)
$$p(5n+B)_{(2)} \equiv 0 \pmod{5} \quad \text{for } B \in \{2,3,4\},$$

(CDHS)
$$p(25n+23)_{(2)} \equiv 0 \pmod{25}.$$

See [Gand–63], a particular case of Theorem 1 in [Andr–08], and Fromula (1.17) in [CDHS–14], respectively.

Like the partition numbers $p(n)$ and the multipartition numbers $p(n)_{(M)}$, the coefficients of the conjugacy growth series

$$C_{\mathrm{Alt}(X),S}(q) = \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{(1-q^j)^2} + \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1-q^{2j}} = \sum_{n=0}^{\infty} p^A(n) q^n$$

of Proposition 28 verify intriguing congruence relations, as was recorded in Observation 12 of the Introduction. With the notation of Appendix C, the coefficients of this series can be written as

$$p^A(n) = \frac{1}{2}\left(p(n)_{(2)} + p(n)_{(0,1)}\right).$$

Proposition 30. *With the notation above, we have*

$$\begin{aligned} p^A(5n+3) &\equiv 0 \pmod 5, \\ p^A(10n+7) &\equiv 0 \pmod 5, \\ p^A(10n+9) &\equiv 0 \pmod 5, \\ p^A(25n+23) &\equiv 0 \pmod{25}. \end{aligned}$$

Proof. One the one hand, as recorded above in (Gandhi), it is known that $p(5n+3)_{(2)} \equiv 0 \pmod 5$ for all $n \geq 0$. On the other hand, it follows from the definitions that

$$p(k)_{(0,1)} = \begin{cases} p(m) & \text{if } k = 2m \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Since $p(5n+4) \equiv 0 \pmod 5$ for all $n \geq 0$, we have also $p(5n+3)_{(0,1)} \equiv 0 \pmod 5$ for all $n \geq 0$. Hence $p^A(5n+3) = \frac{1}{2}\left(p(4n+3)_{(2)} + p(4n+3)_{(0,1)}\right) \equiv 0 \pmod 5$ for all $n \geq 0$.

Similarly, since $p(n)_{(0,1)} = 0$ for all odd n, the congruences for $p^A(10n+7)$ and $p^A(10n+9)$ follows from (Andrews), and for $p^A(25n+23)$ from (CDHS). □

On the conjectured relations of Observation 12. For $p^A(\cdot)$, Proposition 30 contains the established part of Observation 12. The remaining congruences of this observation follow from the congruences

$$\begin{aligned} p(49n+17)_{(2)} &\equiv 0 \pmod 7, & p(49n+33)_{(1)} &\equiv 0 \pmod 7, \\ p(49n+31)_{(2)} &\equiv 0 \pmod 7, & p(49n+40)_{(1)} &\equiv 0 \pmod 7, \\ p(49n+38)_{(2)} &\equiv 0 \pmod 7, & p(49n+19)_{(1)} &\equiv 0 \pmod 7, \\ p(49n+45)_{(2)} &\equiv 0 \pmod 7, & p(49n+47)_{(1)} &\equiv 0 \pmod 7, \\ p(121n+111)_{(2)} &\equiv 0 \pmod{11}, & p(121n+116)_{(1)} &\equiv 0 \pmod{11}. \end{aligned}$$

For what we know, the congruences of the left-hand side are conjectural, with numerical evidence recorded in our Appendix C. The congruences on the right-hand side are all established, and are indeed particular cases of the classical congruences $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$.

Appendix A. Three lemmas on symmetric and alternating groups

For reference elsewhere, we state here three elementary facts. Recall from the introduction that, for $a, b \in \mathrm{Sym}(X)$, we agree that ab denotes b *followed* by a. The first lemma is straightforward:

Lemma 31. *Let X be a set with at least 3 elements, and $a, b \in \mathrm{Sym}(X)$ two cycles such that their supports have exactly one element in common.*

Then ab is a cycle and $sup(ab) = sup(a) \cup sup(b)$. More precisely, if $a = (x_1, \ldots, x_r)$ and $b = (x_r, \ldots, x_{r+s-1})$, then $ab = (x_1, \ldots, x_{r+s-1})$.

The next lemma is well-known. See e.g. [GoRo–01, Lemmas 3.10.1 and 3.10.2], where the proof of (2) is left as an exercise.

Lemma 32. *Let X be a non-empty set, S a set of transpositions of X, and $\Gamma(S)$ the transposition graph, as in Definition 4.*

(1) S generates $\mathrm{Sym}(X)$ if and only if $\Gamma(S)$ is connected.

(2) Suppose that X is finite, say of cardinality n, and that $\Gamma(S)$ is a tree. Let $s_1, s_2, \dots, s_{n-1}$ be an enumeration of the elements of S.

Then the product $s_1 s_2 \cdots s_{n-1}$ is a cycle of length n.

Proof. (1) Denote by G the subgroup of $\mathrm{Sym}(X)$ generated by S.

Suppose that $\Gamma(S)$ is not connected. Choose a connected component of $\Gamma(S)$, denote by X_1 its vertex set, and set $X_2 = X \smallsetminus X_1$. Then G is a subgroup of the proper subgroup $\mathrm{Sym}(X_1) \times \mathrm{Sym}(X_2)$ of $\mathrm{Sym}(X)$, hence S does not generate $\mathrm{Sym}(X)$.

Assume that $\Gamma(S)$ is connected. We have to show that $G = \mathrm{Sym}(X)$. Since this is trivial when $|X| \le 2$, we assume that $|X| \ge 3$. Let x, y, z be three distinct elements in X; observe that $(y,z)(x,y)(y,z) = (x,z)$. For two distinct elements u, v in X, it follows that $(u,v) \in G$ by induction on the length of a path connecting u and v in $\Gamma(S)$. Hence G contains all transpositions of elements of X, and therefore $G = \mathrm{Sym}(X)$.

(2) We proceed by induction on n. Note that the lemma is obvious for $n = 2$; suppose that $n > 2$, and that the lemma holds up to $n-1$.

Choose a leaf x of $\Gamma(S)$. There is a unique $i(x) \in \{1, \dots, n-1\}$ such that $x \in \sup(s_{i(x)})$. Upon replacing the product $s_1 \cdots s_{n-1}$ by a conjugate element, we can assume that $s_{i(x)} = s_{n-1}$. By the induction hypothesis, the product $s_1 \cdots s_{n-2}$ is now a cycle c' of length $n-1$. By Lemma 31, $s_1 \cdots s_{n-2} s_{n-1} = c' s_{n-1}$ is a cycle of length n. $\square$

The third lemma is a cheap confirmation of the fact that most pairs of elements of $\mathrm{Sym}(n)$ generate either $\mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$ [Baba–89].

Lemma 33. *Let X be a non-empty set with at least 3 elements, $a, b \in \mathrm{Sym}(X)$ two cycles, respectively of lengths $\ell, m \ge 2$, such that their supports have exactly one element in common (as in Lemma 31). Let G be the subgroup of $\mathrm{Sym}(X)$ generated by $\{a, b\}$.*

Then G is isomorphic to the alternating group $\mathrm{Alt}(\ell + m - 1)$ if ℓ, m are both odd, and to $\mathrm{Sym}(\ell + m - 1)$ otherwise.

Proof. Denote by x the element in $\sup(a) \cap \sup(b)$; set $y = a^{-1}(x)$ and $z = b^{-1}(x)$. The commutator $a^{-1}b^{-1}ab$ is the 3-cycle $c := (x, y, z)$. By Lemma 23 for $R^A_{\ell+1}$, the conjugates of c by the powers of a generate $\mathrm{Alt}(\sup(a) \cup \{z\})$; similarly the conjugates of c by the powers of b generate $\mathrm{Alt}(\{x\} \cup \sup(b))$.

Observe that the intersection $\mathrm{Alt}(\sup(a) \cup \{z\}) \cap \mathrm{Alt}(\{x\} \cup \sup(b))$ contains c, and the union $\mathrm{Alt}(\sup(a) \cup \{z\}) \cup \mathrm{Alt}(\{x\} \cup \sup(b))$ contains a set of 3-cycles similar to $S^A_{\ell+m-1}$. By Lemma 23 again, this time for $S^A_{\ell+m-1}$, the group G contains $\mathrm{Alt}(\sup(a) \cup \sup(b))$, isomorphic to $\mathrm{Alt}(\ell + m - 1)$.

If ℓ and m are both odd, every element in G has an even signature, hence $G = \mathrm{Alt}(\sup(a) \cup \sup(b)) \simeq \mathrm{Alt}(\ell + m - 1)$. Otherwise, G is a subgroup of $\mathrm{Sym}(\sup(a) \cup \sup(b))$ in which $\mathrm{Alt}(\sup(a) \cup \sup(b))$ is a proper subgroup, hence $G = \mathrm{Sym}(\sup(a) \cup \sup(b)) \simeq \mathrm{Sym}(\ell + m - 1)$. □

This lemma implies for example that the set

$$\{(0, 1, 2), (2, 3, 4), (4, 5, 6), \dots, (2i, 2i + 1, 2i + 2), \dots\}$$

generates $\mathrm{Alt}(\mathbf{N})$. It is a proper subset of the generating set $S^A_{\mathbf{N}}$ introduced in the beginning of Section 5.

Appendix B. **Reminder on partitions and derangements**

B.a. **The partition function.** For $n \in \mathbf{N}$, let $p(n)$ denote the number of partitions of n. The first values are given by the table

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

(more values in [OEIS, A000041]).

In our context $p(n)$ is the number of conjugacy classes in the finite symmetric group $\mathrm{Sym}(n)$, alternatively the number of conjugacy classes in $\mathrm{Sym}(\mathbf{N})$ of elements of supports of size at most n. For this reason, the partition function appears already in Propositions 1 and 9.

It is known since Euler that the generating series for $p(n)$ has a product expansion

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1 - q^k} \, . \tag{EP$_1$}$$

See [Eule–48, Caput XVI], as well as, for example, [HaWr–79, Section 19.3]. The equality can be viewed either between formal expressions, or between absolutely converging sum and product for $q \in \mathbf{C}$ with $|q| < 1$.

There is an asymptotic formula for $n \to \infty$

$$\begin{aligned} p(n) \, &= \, \frac{1}{4\sqrt{3}\left(n - \frac{1}{24}\right)} \, \exp\left(\pi\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right) \\ &+ \, O\left(\frac{1}{\left(n - \frac{1}{24}\right)^{3/2}} \, \exp\left(\pi\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right)\right) \end{aligned}$$

due to Hardy and Ramanujan [HaRa–18, Formula (1.41)]. For this and more on $p(n)$ when $n \to \infty$, see e.g. [Chan–70, Chapter VII] and [Hard–40, Chapters VI and VIII]. This shows in particiular that the sequence $(p(n))_{n \geq 0}$ has **intermediate growth**, i.e. that its growth is superpolynomial and subexponential.

B.b. **Partitions with k parts.** For $n, k \in \mathbf{N}$, we denote by $p_k(n)$ the number of partitions of n in exactly k positive parts, equivalently the number of partitions of n with largest part k, equivalently the number of partitions of $n-k$ in k non-negative parts. Whenever needed, we set $p_k(n) = 0$ for all $n \in \mathbf{N}$ and $k < 0$. Numbers $p_k(\cdot)$ appear in connection with finite symmetric groups, in Propositions 1, 9, 20, and 22.

We have classically

$$\begin{aligned}
p_0(0) &= 1 \quad \text{and} \quad p_0(n) = 0 \quad \text{for all } n \geq 1, \\
p_1(0) &= 0 \quad \text{and} \quad p_1(n) = 1 \quad \text{for all } n \geq 1, \\
p_2(n) &= \lfloor n/2 \rfloor \quad \text{for all } n \geq 0, \\
p_3(n) &= \lfloor \frac{1}{12}(n^2+6) \rfloor \quad \text{for all } n \geq 0 \ \text{[OEIS, A069905]}, \\
\cdots & \quad \cdots \\
p_{n-2}(n) &= 2 \quad \text{for all } n \geq 4, \\
p_{n-1}(n) &= p_n(n) = 1 \quad \text{for all } n \geq 2, \\
p_k(n) &= 0 \quad \text{for all } k > n \geq 0, \\
p_k(n) &= p_k(n-k) + p_{k-1}(n-1) \quad \text{for all } n \geq k \geq 1, \\
\sum_{k=0}^{n} p_k(n) &= \sum_{k=1}^{n} p_k(n) = p(n) \quad \text{for all } n \geq 1 \,,
\end{aligned}$$

and the generating function

$$\text{(EP}_2\text{)} \qquad \sum_{n \geq 0} p_k(n) q^n = q^k \prod_{i=1}^{k} \frac{1}{1-q^i} \quad \text{for all } k \geq 0 \,.$$

(Observe that $\sum_{n \geq 0} p_k(n) q^n = \sum_{n \geq k} p_k(n) q^n$.) Up to the notation, Equality (EP$_2$) is contained in Number 312 of [Eule–48, Caput XVI].

Moreover, if $P(n,t) := \sum_{k=0}^{n} p_k(n) t^k$, then

$$\text{(EP}_3\text{)} \qquad \sum_{n=0}^{\infty} P(n,t) q^n = \prod_{j=1}^{\infty} \frac{1}{1-tq^j} \,.$$

This appears in Number 304 of [Eule–48, Caput XVI], and is used in the proof of our Proposition 34.

For $n, \ell \in \mathbf{N}$ with $n \leq 2\ell$, every partition of $n - \ell$ has at most ℓ parts. Thus every partition of $n - \ell$ can be obtained from a unique partition of n in ℓ parts by substracting 1 from each part. Consequently

$$\text{(EP}_4\text{)} \qquad p_\ell(n) = p(n-\ell) \quad \text{for integers } n, \ell \text{ such that } 0 \leq \ell \leq n \leq 2\ell \,,$$

or, setting $k = n - \ell$,

$$\text{(EP}_4'\text{)} \qquad p_{n-k}(n) = p(k) \quad \text{for integers } n, k \text{ such that } k \geq 0 \text{ and } 2k \leq n \,.$$

The double sequence $(p_k(n))_{n \geq 0,\ 0 \leq k \leq n}$ gives rise to a generalized Pascal triangle of which the first rows are:

$$
\text{(PTp)} \qquad
\begin{array}{llllllll}
p_0(0) \\
p_0(1) & p_1(1) \\
p_0(2) & p_1(2) & p_2(2) \\
p_0(3) & p_1(3) & p_2(3) & p_3(3) \\
p_0(4) & p_1(4) & p_2(4) & p_3(4) & p_4(4) \\
p_0(5) & p_1(5) & p_2(5) & p_3(5) & p_4(5) & p_5(5) \\
p_0(6) & p_1(6) & p_2(6) & p_3(6) & p_4(6) & p_5(6) & p_6(6) \\
p_0(7) & p_1(7) & p_2(7) & p_3(7) & p_4(7) & p_5(7) & \dots
\end{array}
=
\begin{array}{ccccccc}
1 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 2 & 1 & 1 \\
0 & 1 & 3 & 3 & 2 & 1 & 1 \\
0 & 1 & 3 & 4 & 3 & 2 & \dots
\end{array}
$$

B.c. **Partitions with even or odd numbers of parts.** We denote by $p_e(n)$, respectively $p_o(n)$, the number of partitions of a non-negative integer n involving an even, respectively odd, number of non-zero parts. Working with conjugate partitions, we see that $p_e(n)$, respectively $p_o(n)$, is equivalently given by the number of partitions of n having an even largest part, respectively an odd largest part. We have the trivial identity $p(n) = p_e(n) + p_o(n)$. These numbers $p_e(n)$ appear in Proposition 28. Their values for $n \leq 15$ are given by

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p_e(n)$	1	0	1	1	3	3	6	7	12	14	22	27	40	49	69	86
$p_o(n)$	0	1	1	2	2	4	5	8	10	16	20	29	37	52	66	90
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

see A027187 and A027193 of [OEIS].

Proposition 34. *(1) The generating series of the sequence $p_e(n)$ is*

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_e(n) q^n &= \sum_{k=0}^{\infty} q^{2k} \prod_{j=1}^{2k} \frac{1}{1-q^j} \\
&= \frac{1}{2} \left(\prod_{j=1}^{\infty} \frac{1}{1-q^j} + \prod_{j=1}^{\infty} \frac{1}{1+q^j} \right) \\
&= \prod_{j=1}^{\infty} \frac{1}{1-q^j} \sum_{m=0}^{\infty} (-q)^{m^2}.
\end{aligned}
$$

(2) The generating series of the sequence $p_o(n)$ is

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_o(n) q^n &= \sum_{k=0}^{\infty} q^{2k+1} \prod_{j=1}^{2k+1} \frac{1}{1-q^j} \\
&= \frac{1}{2} \left(\prod_{j=1}^{\infty} \frac{1}{1-q^j} - \prod_{j=1}^{\infty} \frac{1}{1+q^j} \right) \\
&= -\prod_{j=1}^{\infty} \frac{1}{1-q^j} \sum_{m=1}^{\infty} (-q)^{m^2}.
\end{aligned}
$$

Proof. (1) Using (EP_2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} p_e(n)q^n \; &= \; \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} p_{2k}(n)q^n \; = \; \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p_{2k}(n)q^n \\ &= \sum_{k=0}^{\infty} q^{2k}\prod_{j=1}^{2k}\frac{1}{1-q^j}.\end{aligned}$$

Also, if $P(n,t) := \sum_{k=0}^{n} p_k(n)t^k$ as in (EP_3), then

$$\begin{aligned}\sum_{n=0}^{\infty} p_e(n)q^n \; &= \; \frac{1}{2}\left(\sum_{n=0}^{\infty} P(n,1)q^n + \sum_{n=0}^{\infty} P(n,-1)q^n\right) \\ &= \frac{1}{2}\left(\prod_{j=1}^{\infty}\frac{1}{1-q^j} + \prod_{j=1}^{\infty}\frac{1}{1+q^j}\right).\end{aligned}$$

For the third equality in (1), one way is to refer to [Fine–88]: see there Equation (7.324), Page 6, and also Example 7, Page 39.

The proof of (2) is similar.

Here is an alternative to citing [Fine–88]. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(p_e(n) - p_o(n)\right)q^n \; &= \; 2\sum_{n=0}^{\infty} p_e(n)q^n \; - \; \sum_{n=0}^{\infty} p(n)q^n \; = \; \prod_{j=1}^{\infty}\frac{1}{1+q^j} \\ = \prod_{j=1}^{\infty}\frac{1-q^j}{1-q^{2j}} \; &= \; \prod_{j=1}^{\infty}(1-q^{2j-1}) \; = \; \prod_{j=1}^{\infty}\frac{1}{1-q^j}\;\prod_{k=1}^{\infty}(1-q^{2k-1})^2(1-q^{2k}) \, .\end{aligned}$$

The *Jacobi triple product identity* reads

$$\prod_{k=1}^{\infty}\Big(1-x^{2k}\Big)\Big(1+x^{2k-1}y^2\Big)\Big(1+\frac{x^{2k-1}}{y^2}\Big) \; = \; \sum_{n=-\infty}^{\infty} x^{n^2}y^{2n}$$

(see e.g. [HaWr–79, Theorem 352]). For $x = q$ and $y = \sqrt{-1}$ it reduces to

$$\prod_{k=1}^{\infty}(1-q^{2k-1})^2(1-q^{2k}) \; = \; \sum_{n=-\infty}^{\infty}(-q)^{n^2} \, ,$$

hence

$$\sum_{n=0}^{\infty}\left(p_e(n) - p_o(n)\right)q^n \; = \; \prod_{j=1}^{\infty}\frac{1}{1-q^j}\;\sum_{n=-\infty}^{\infty}(-q)^{n^2} \, .$$

Finally:

$$\begin{aligned}\sum_{n=0}^{\infty} p_e(n)q^n &= \frac{1}{2}\sum_{n=0}^{\infty}\left(p_e(n)-p_o(n)\right)q^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(p_e(n)+p_o(n)\right)q^n \\ &= \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1-q^j}\sum_{n=-\infty}^{\infty}(-q)^{n^2} + \frac{1}{2}\prod_{j=1}^{\infty}\frac{1}{1-q^j} \\ &= \prod_{j=1}^{\infty}\frac{1}{1-q^j}\sum_{n=0}^{\infty}(-q)^{n^2},\end{aligned}$$

as was to be shown. □

Observation 35. We have

$$\left(\sum_{n=0}^{\infty} p_e(n)q^n\right)^2 - \left(\sum_{n=0}^{\infty} p_o(n)q^n\right)^2 = \sum_{n=0}^{\infty} p(n)q^{2n} = \prod_{j=1}^{\infty}\frac{1}{1-q^{2j}}.$$

Proof. The left-hand side can be written as

$$\begin{aligned}&\frac{1}{4}\left(\prod_{j=1}^{\infty}\frac{1}{1-q^j}+\prod_{j=1}^{\infty}\frac{1}{1+q^j}\right)^2 - \frac{1}{4}\left(\prod_{j=1}^{\infty}\frac{1}{1-q^j}-\prod_{j=1}^{\infty}\frac{1}{1+q^j}\right)^2 \\ &\qquad = \prod_{j=1}^{\infty}\frac{1}{1-q^{2j}},\end{aligned}$$

and the claim follows. □

B.d. **Derangements that are products of k cycles.** A **derangement** is a fixed point free permutation. For $n, k \in \mathbf{N}$, denote by $d_k(n)$ the number of derangements of $\{1, 2, \dots, n\}$ that are products of k disjoint cycles. These numbers appear in Remark 21 and Proposition 22.

Lemma 36. *With the notation above, we have*

(i) $d_0(0) = 1$ *;*
(ii) $d_k(1) = 0$ *for all* $k \in \mathbf{N}$ *;*
(iii) $d_k(n) = 0$ *for all* $n, k \in \mathbf{N}$ *with* $k = 0$ *or* $2k > n$ *;*

For all $n \geq 2$ *and* $k \geq 1$*, we have*

(iv) $d_k(n) = (n-1)\big(d_k(n-1) + d_{k-1}(n-2)\big)$ *;*
(v) $d_k(n) = \sum_{a=2}^{n}\binom{n-1}{a-1}(a-1)!\, d_{k-1}(n-a)$.

Proof. Claims (i) to (iii) are obvious.

For (iv), consider a derangement g of $\{1, \dots, n\}$ product of k cycles.

Either n is in the support of a cycle $(x_1, \dots, x_{\ell-1}, n)$ of length at least 3. Replacing it by the cycle $(x_1, \dots, x_{\ell-1})$ produces a derangement of $\{1, \dots, n-1\}$ product of k cycles, and each of the latter is obtained $n-1$ times in this way. This explains the contribution $(n-1)d_k(n-1)$ of the right-hand side.

Or n is in the support of a transposition, say (i, n) with $i \in \{1, \dots, n-1\}$, so that g is the product of (i, n) with a derangement h of $\{1, \dots, n-1\} \smallsetminus \{i\}$ product of $k-1$ cycles. For each of the $n-1$ possible values of i, there are $d_{k-1}(n-2)$ such permutations h, and this explains the contribution $(n-1)d_{k-1}(n-2)$.

For (v), a permutation contributing to $d_k(n)$ is the product of a cycle c of length $a \geq 2$, with $n \in \sup(c)$, and there are $\binom{n-1}{a-1}(a-1)!$ such cycles, with a derangement of $\{1, \dots, n\} \smallsetminus \sup(c)$ which is a product of $k-1$ cycles. $\square$

Remark 37. (i) The double sequence $(d_k(n))_{n \geq 0,\, 0 \leq k \leq n}$ gives rise to a generalized Pascal triangle of which the first rows are:

$$\text{(PTd)} \qquad \begin{array}{cccccc} d_0(0) & & & & & \\ d_0(1) & d_1(1) & & & & \\ d_0(2) & d_1(2) & d_2(2) & & & \\ d_0(3) & d_1(3) & d_2(3) & d_3(3) & & \\ d_0(4) & d_1(4) & d_2(4) & d_3(4) & d_4(4) & \\ d_0(5) & d_1(5) & d_2(5) & d_3(5) & d_4(5) & d_5(5) \\ d_0(6) & d_1(6) & d_2(6) & d_3(6) & d_4(6) & \dots \end{array} \; = \; \begin{array}{cccccc} 1 & & & & & \\ 0 & 0 & & & & \\ 0 & 1 & 0 & & & \\ 0 & 2 & 0 & 0 & & \\ 0 & 6 & 3 & 0 & 0 & \\ 0 & 24 & 20 & 0 & 0 & 0 \\ 0 & 120 & 130 & 15 & 0 & \dots \end{array}$$

(ii) Besides the relations of Lemma 36, we have also

$$(\Sigma d) \qquad \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{m} d_k(m) \; = \; n! \quad \text{for all } \; n \in \mathbf{N},$$

which is useful to check numerical values. Indeed, each of the $n!$ permutations g of $\{1, \dots, n\}$ induces a derangement of $\sup(g)$. For $m \in \{0, 1, \dots, n\}$, there are $\binom{n}{m}$ subsets of $\{1, \dots, n\}$ of size m. Since there are $\sum_{k=0}^{m} d_k(m)$ derangements of each of these subsets, we obtain the left-hand side. Relation (Σd) reduces to $d_0(0) = 1$ for $n = 0$, and to $d_0(0) + d_0(1) + d_1(1) = 1 + 0 + 0 = 1$ for $n = 1$. Otherwise, it can be written

$$(\Sigma d') \qquad 1 + \sum_{m=2}^{n} \binom{n}{m} \sum_{k=1}^{\lfloor m/2 \rfloor} d_k(m) \; = \; n! \quad \text{for all } \; n \geq 2.$$

The sum $d(m) := \sum_{k=0}^{m} d_k(m) = \sum_{k=0}^{\lfloor m/2 \rfloor} d_k(m)$ is the number of derangements of m objects, and there is a classical formula:

$$d(m) \; = \; \sum_{k=0}^{m} d_k(m) \; = \; m! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^m \frac{1}{m!} \right)$$

for all $m \geq 0$; it follows that we have the relations

$$d(m) \; = \; m d(m-1) + (-1)^m \quad \text{for all } \; m \geq 1 \, ,$$

$$d(m) \; = \; (m-1)\big(d(m-1) + d(m-2)\big) \quad \text{for all } \; m \geq 2 \, ;$$

see e.g. [Stan–97, Example 2.2.1]. The sequence

$$(d(m))_{m \geq 0} \; = \; (1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, \ldots)$$

is A000166 in [OEIS].

(iii) Numbers $d_k(n)$ have some flavour of Stirling numbers. For $n, k \in \mathbf{N}$ with $0 \le k \le n$, recall that the **unsigned Stirling number of the fist kind** $\left[{n \atop k}\right]$ counts the number of ways to arrange n objects into k cycles (here, cycles of length 1 are included, unlike elsewhere in this article, and this is why entries in (PTd) are smaller or equal than entries in (PTStir). When $n \ge 1$, we have $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$. See for example [GrKP–89, Page 245] and [OEIS, A132393]. The generalized Pascal triangle for $\left(\left[{n \atop k}\right]\right)_{n\ge 0,\ 0\le k\le n}$ is

$$\text{(PTStir)} \qquad \begin{array}{ccccccc} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 6 & 11 & 6 & 1 \\ 0 & 24 & 50 & 35 & 10 & 1 \\ 0 & 120 & 274 & 225 & 85 & 15 & 1 \end{array}$$

Note that we have

$$\left[{n \atop k}\right] = \sum_{j=0}^{k} \binom{n}{j} d_{k-j}(n-j) \, .$$

Indeed, in the right-hand side, the term with a given value of j counts the number of contributions to $\left[{n \atop k}\right]$ with j fixed points.

Appendix C. Generalized Ramanujan congruences

This appendix is partly experimental. It grew out of our desire to understand the reasons for the congruences for the numbers $p^A(n)$ described in Observation 12 and Section 6.

C.a. Definitions.

Definition 38. Given a sequence $\mathbf{e} = (e_1, e_2, e_3, \dots) \in \mathbf{Z}^{(1,2,3,\dots)}$ of integers with $e_d = 0$ for d large enough, the corresponding **generalized partition numbers** $p(n)_{\mathbf{e}}$ are the coefficients of the power series

$$\text{(1)} \qquad \begin{aligned} \sum_{n=0}^{\infty} p(n)_{\mathbf{e}} q^n \; &= \; \prod_{n=1}^{\infty}\prod_{d=1}^{\infty} \frac{1}{(1-q^{dn})^{e_d}} \\ &= \; \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{e_1}(1-q^{2n})^{e_2}(1-q^{3n})^{e_3}\cdots} \, . \end{aligned}$$

Remark 39. As a shorthand, we also write a sequence $\mathbf{e}$ as above as $(e_1, e_2, \dots, e_k)$ when $e_k \neq 0$ and $e_d = 0$ for all $d \ge k+1$. For example:

$$\text{(2)} \qquad \sum_{n=0}^{\infty} p(n)_{(0,3)} q^n \; = \; \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^3} \, .$$

For a sequence of the form $(e_1, \dots, e_j, 0, \dots, 0, e_k)$ with $e_k \neq 0$, and $e_d = 0$ when $j < d < k$ or $d > k$, we also write $(e_1, \dots, e_j, (e_k)_k)$. For example:

$$\sum_{n=0}^{\infty} p(n)_{(0,1,2_8)} q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})(1-q^{8n})^2} \, . \tag{3}$$

For a positive integer M, the numbers $p(n)_{(M)}$ arising as coefficients of the series defined by $\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^M}$ are called **multi-partition numbers** in the literature, since $p(n)_{(M)}$ counts the number of ways of writing n as a sum of parts, each coloured in one of M colours. More generally, for $\mathbf{e} \in \mathbf{Z}^{(1,2,3,\dots)}$ as above, $p(n)_{\mathbf{e}}$ can be interpreted as multi-partition numbers which constraints on the parts; for example, the coefficient $p(n)_{(0,1,2_8)}$ of (3) counts the number of partitions of the form

$$n = \lambda_1 + \cdots + \lambda_i + \mu_1 + \cdots + \mu_j + \nu_1 + \cdots + \nu_k$$

where

$$\begin{aligned} &\lambda_1 \geq \cdots \geq \lambda_i \geq 1 \ \text{ and } \lambda_1, \dots, \lambda_i \text{ are even,} \\ &\mu_1 \geq \cdots \geq \mu_j \geq 1 \ \text{ and } \mu_1, \dots, \mu_j \text{ are multiples of 8,} \\ &\nu_1 \geq \cdots \geq \nu_k \geq 1 \ \text{ and } \nu_1, \dots, \nu_k \text{ are multiples of 8.} \end{aligned}$$

Definition 40. A **generalized Ramanujan congruence** is

- a sequence $\mathbf{e} = (e_1, e_2, e_3, \dots) \in \mathbf{Z}^{(1,2,3,\dots)}$ as above,
- an arithmetic progression $(An + B)_{n \geq 0}$ with $A \geq 2$ and $1 \leq B \leq A - 1$
- a prime power ℓ^f, with ℓ prime and $f \geq 1$,

such that

$$p(An + B)_{\mathbf{e}} \equiv 0 \pmod{\ell^f} \quad \text{for all } n \geq 0. \tag{4}$$

Observation 41. (1) Let $p(An+B)_{\mathbf{e}} \equiv 0 \pmod{\ell^f}$ be a generalized Ramanujan congruence as above, and let $m \geq 2$. Define a sequence $\mathbf{e}'$ by $e'_d = e_{d/m}$ if m divides d and $e'_d = 0$ otherwise. Then we have

$$\begin{aligned} p(mAn + mB)_{\mathbf{e}'} &\equiv 0 \pmod{\ell^f} \qquad \text{for all } n \geq 0, \\ p(mn + B')_{\mathbf{e}'} &= 0 \qquad \text{for all } n \geq 0 \text{ and } B' \in \{1, 2, \dots, m-1\}. \end{aligned} \tag{5}$$

Observe that the integers of the **support** $\{d \geq 1 \mid e'_d \neq 0\}$ of $\mathbf{e}'$ have a common divisor $m \geq 2$.

A generalized Ramanujan congruence is **primitive** if the integers in its support are coprime. All examples of generalized Ramanujan congruences appearing below are primitive.

(2) In lists of examples involving congruences modulo ℓ (and not ℓ^f with $f \geq 2$), we write shortly $p(\ell n + B)_{\mathbf{e}}$ for $p(\ell n + B)_{\mathbf{e}} \equiv 0 \pmod{\ell}$. The Ramanujan congruences of this sort in Section 6 can therefore be written

$$p(5n+4)_{(1)}, \ \ p(7n+5)_{(1)}, \ \ p(11n+6)_{(1)}, \ \ p(5n+B)_{(2)}, \ \ p(11n+4)_{(2)}.$$

(With $B \in \{2,3,4\}$.) Moreover, we also write

$$p(\ell n + B)_{\mathbf{e},\, \mathbf{e}',\, \ldots,\, \mathbf{e}''}$$

as a shorthand for $p(\ell n + B)_{\mathbf{e}}$, $p(\ell n + B)_{\mathbf{e}'}$, ..., $p(\ell n + B)_{\mathbf{e}''}$.

This shorthand notation will be used systematically in the lists of Subsections C.b to C.f.

(3) When we consider below generalized Ramanujan congruence involving a prime ℓ (and not a prime power ℓ^f with $f \geq 2$), it suffices to consider sequences $\mathbf{e} = (e_1, e_2, e_3, \ldots)$ with $0 \leq e_d \leq \ell - 1$ for all $d \geq 0$. This is a corollary of the following standard proposition, for which we did not find a convenient reference.

Proposition 42. *Let ℓ be a prime, $S(q) = \sum_{n=0}^{\infty} s_n q^n, T(q) = \sum_{n=0}^{\infty} t_n q^n$ two power series in $\mathbf{Z}[[q]]$, and $(pn + B)_{n \geq 0}$ an arithmetic progression of common difference ℓ and first term $B \geq 1$ not divisible by ℓ. Set*

$$U(q) \; = \; S(q)\big(T(q)\big)^{\ell} \; = \; \sum_{n=0}^{\infty} u_n q^n.$$

Assume that $s_{\ell n + B} \equiv 0 \pmod{\ell}$ for all $n \geq 0$.

Then $u_{\ell n + B} \equiv 0 \pmod{\ell}$ for all $n \geq 0$.

Proof. For the binomial coefficients, we have the well-known congruences

$$\binom{\ell}{j} \equiv 0 \pmod{\ell} \quad \text{for all } j \geq 0 \text{ with } j \not\equiv 0 \pmod{\ell}.$$

Hence the power series $\big(T(q)\big)^{\ell} = \sum_{n=0}^{\infty} t'_n q^n$ and $T(q^{\ell}) = \sum_{n=0}^{\infty} t_n q^{\ell n}$ have coefficients that are congruent modulo ℓ; in particular, $t'_n \equiv 0 \pmod{\ell}$ for all $n \geq 0$ with $n \not\equiv 0 \pmod{\ell}$.

In particular, if $s_{\ell n + B} \equiv 0 \pmod{\ell}$ for all $n \geq 0$, then $u_{\ell n + B} \equiv 0 \pmod{\ell}$ for all $n \geq 0$. □

Corollary 43. *Consider a sequence $\mathbf{e} = (e_1, e_2, e_3, \ldots) \in \mathbf{Z}^{(1,2,3,\ldots)}$, an arithmetic progression $(An + B)_{n \geq 0}$ with $A \geq 2$ and $1 \leq B \leq A - 1$, a prime ℓ, and another sequence $\mathbf{e}' = (e'_1, e'_2, e'_3, \ldots) \in \mathbf{Z}^{(1,2,3,\ldots)}$. Assume that $e'_d \equiv e_d \pmod{\ell}$ for all $d \geq 0$.*

If $p(An + B)_{\mathbf{e}} \equiv 0 \pmod{\ell}$ for all $n \geq 0$ (as in Definition 40), then $p(An + B)_{\mathbf{e}'} \equiv 0 \pmod{\ell}$ for all $n \geq 0$.

We now proceed to indicate a list of examples of generalized Ramanujan congruences. Except for a few exceptions, they are CONJECTURAL. In each case, they have been checked numerically, for $p(n)_{\mathbf{e}}$ with $n \leq 5000$.

We use the shorthand notation explained in Remark 41(2).

C.b. **Some examples of the form $p(3n + B)_{\mathbf{e}} \equiv 0 \pmod{3}$.**

$$p(3n+2)_{(1,1),\ (2,1,0,2),\ (2,1,0,1,2,1_{10},1_{20}),\ (1,1,0,2,1,1_{10},2_{20})}\,.$$

C.c. **Some examples of the form $p(5n+B)_{\mathbf{e}} \equiv 0 \pmod 5$.** For $\ell = 5$ and when $e_d = 0$ for all $d \geq 3$, we find the Ramanujan congruences

$$p(5n+2)_{(2),\,(3,1),\,(1,3)}, \quad p(5n+3)_{(2),\,(4),\,(3,1)}, \quad p(5n+4)_{(1),\,(2),\,(4),\,(2,2),\,(1,3)}\,.$$

When $e_d = 0$ for all d not dividing 4, we find moreover the Ramanujan congruences

$$p(5n+2)_{(2,0,0,2),\,(3,1,0,2),\,(3,1,0,3),\,(2,0,0,4),\,(4,1,0,4)}\,,$$
$$p(5n+3)_{(1,2,0,1),\,(2,0,0,2),\,(4,0,0,2),\,(3,1,0,3),\,(1,2,0,3)}\,,$$
$$p(5n+4)_{(1,2,0,1),\,(3,2,0,1),\,(2,1,0,3),\,(3,1,0,3),\,(3,3,0,3),\,(4,1,0,4),\,(4,3,0,4)}\,.$$

When $e_d = 0$ for all d not dividing 6, we find moreover

$$p(5n+1)_{(0,2,2),\,(0,4,2),\,(0,2,3,0,0,1)}\,,$$
$$p(5n+2)_{(1,3,2),\,(1,3,4,0,0,1),\,(4,1,1,0,0,3),\,(4,1,3,0,0,3),\,(3,1,1,0,0,4),\,(3,1,3,0,0,4)}\,,$$
$$p(5n+3)_{(1,1,1,0,0,1),\,(1,4,3,0,0,1),\,(1,3,4,0,0,1),\,(3,3,4,0,0,1)}\,,$$
$$p(5n+3)_{(3,1,0,0,0,2),\,(2,3,4,0,0,2),\,(4,2,2,0,0,3),\,(3,2,2,0,0,4)}\,,$$
$$p(5n+4)_{(0,2,2),\,(0,2,4),\,(1,4,3,0,0,1),\,(3,4,3,0,0,1),\,(2,4,3,0,0,2)}\,,$$
$$p(5n+4)_{(4,1,1,0,0,3),\,(4,3,1,0,0,3),\,(1,4,3,0,0,3),\,(3,1,1,0,0,4),\,(3,3,1,0,0,4)}\,.$$

When $e_d = 0$ for all d not dividing 8, we find moreover

$$p(5n+2)_{(2,2_8),\,(1,3,2_8),\,(3,1,0,3,2_8),\,(4,1,0,4,2_8)}\,,$$
$$p(5n+3)_{(3,1,0,1,1_8),\,(2,0,0,3,1_8)}\,,$$
$$p(5n+4)_{(4,4,3_8),\,(1,1,0,1,3_8),\,(2,3,0,1,3_8),\,(3,4,0,4,3_8),\,(2,4,0,1,4_8),\,(3,0,4,4_8)}\,.$$

C.d. **Some examples of the form $p(7n+B)_{\mathbf{e}} \equiv 0 \pmod 7$.**

$$p(7n+2)_{(4)},\ p(7n+3)_{(6)},\ p(7n+4)_{(4),\,(6)},\ p(7n+5)_{(1),\,(4)},\ p(7n+6)_{(4),\,(6)},$$
$$p(7n+2)_{(2,2),\,(1,5),\,(3,5)},\ p(7n+3)_{(5,1),\,(2,2)},\ p(7n+4)_{(1,2),\,(2,2),\,(4,4),\,(1,5)},$$
$$p(7n+5)_{(5,1),\,(1,5),\,(5,5)},\ p(7n+6)_{(2,1),\,(5,1),\,(2,2),\,(5,3)},$$

$$p(7n+2)_{(6,1,0,3),\,(3,5,0,3),\,(4,0,0,4),\,(1,5,0,4),\,(5,1,0,5),\,(6,1,0,6)},$$
$$p(7n+3)_{(1,4,0,1),\,(2,2,0,2),\,(5,1,0,4),\,(5,1,0,5),\,(2,2,0,6)},$$
$$p(7n+4)_{(1,4,0,1),\,(3,6,0,1),\,(3,2,0,3),\,(3,5,0,3),\,(4,1,0,5),\,(5,1,0,5),\,(6,1,0,6),\,(6,5,0,6)},$$
$$p(7n+5)_{(2,2,0,2),\,(2,6,0,2),\,(4,3,0,3),\,(3,5,0,3),\,(3,1,0,6),\,(6,1,0,6),\,(6,3,0,6)},$$
$$p(7n+6)_{(1,4,0,1),\,(4,5,0,1),\,(2,2,0,2),\,(6,2,0,2),\,(2,4,0,2),\,(3,5,0,3),\,(1,6,0,3),\,(3,3,0,4),\,(5,0,0,5),\,(5,1,0,5)}.$$

C.e. **Some examples of the form $p(11n+B)_{\mathbf{e}} \equiv 0 \pmod{11}$.**

$$p(11n+2)_{(8)},\ p(11n+3)_{(10)},\ p(11n+4)_{(8)},$$
$$p(11n+5)_{(8)},\ p(11n+6)_{(1),\,(10)},$$
$$p(11n+7)_{(3),\,(8)},\ p(11n+8)_{(5),\,(8),\,(10)},$$
$$p(11n+9)_{(7),\,(8),\,(10)},\ p(11n+10)_{(10)}.$$

$$p(11n+2)_{(9,1),\,(2,6),\,(1,9)},\ p(11n+3)_{(4,1),\,(6,2),\,(2,6)},\ p(11n+4)_{(2,3),\,(2,6)},$$
$$p(11n+5)_{(6,2),\,(7,7),\,(1,9)},\ p(11n+6)_{(9,1),\,(6,2),\,(2,5),\,(2,6),\,(9,7)},$$
$$p(11n+7)_{(9,1),\,(2,6),\,(1,9),\,(7,9)},\ p(11n+8)_{(9,1),\,(6,2),\,(8,4),\,(1,9),\,(9,9)},$$
$$p(11n+9)_{(3,2),\,(6,2),\,(2,6),\,(6,6),\,(1,9)},\ p(11n+10)_{(9,1),\,(5,2),\,(6,2),\,(1,4),\,(4,8)},$$

$$p(11n+2)_{(3,2,0,2),\,(2,6,0,2),\,(6,6,0,2),\,(3,2,0,3)},$$
$$p(11n+2)_{(5,2,0,7),\,(9,1,0,9),\,(8,0,0,10),\,(10,1,0,10)},$$
$$p(11n+3)_{(1,8,0,1),\,(5,9,0,4),\,(5,9,0,5),\,(7,2,0,7),\,(9,1,0,9),\,(6,2,0,10)},$$
$$p(11n+4)_{(8,9,0,1),\,(3,2,0,3),\,(3,0,0,4),\,(9,2,0,7),\,(9,1,0,9),\,(2,7,0,9),\,(9,9,0,9)}.$$

$$p(11n+5)_{(6,0,0,1),\,(3,2,0,3),\,(10,5,0,3),\,(1,2,0,4)},$$
$$p(11n+5)_{(4,6,0,4),\,(5,9,0,5),\,(5,7,0,6),\,(10,1,0,10)},$$
$$p(11n+6)_{(4,2,0,1),\,(1,8,0,1),\,(2,1,0,2),\,(2,6,0,2),\,(8,7,0,3)},$$
$$p(11n+6)_{(5,9,0,5),\,(3,9,0,6),\,(7,3,0,8),\,(9,0,0,9),\,(9,1,0,9),\,(10,3,0,10)}.$$

$$p(11n+7)_{(4,1,0,2),\,(2,6,0,2),\,(3,2,0,3),\,(6,9,0,3),\,(4,8,0,4),\,(10,3,0,5),\,(1,0,0,6)},$$
$$p(11n+7)_{(8,2,0,6),\,(5,5,0,8),\,(8,9,0,8),\,(9,1,0,9),\,(7,2,0,9),\,(3,4,0,9),\,(10,1,0,10)},$$
$$p(11n+8)_{(1,8,0,1),\,(2,3,0,2),\,(2,6,0,2),\,(4,0,0,3),\,(3,2,0,3),\,(3,6,0,3)},$$
$$p(11n+8)_{(9,1,0,4),\,(8,5,0,5),\,(5,9,0,5),\,(10,2,0,6),\,(6,4,0,6)},$$
$$p(11n+8)_{(2,6,0,6),\,(1,10,0,7),\,(3,7,0,8),\,(7,1,0,10),\,(10,1,0,10)}.$$

$$p(11n+9)_{(1,8,0,1),\,(9,8,0,1),\,(2,2,0,3),\,(3,2,0,3),\,(9,4,0,3)},$$
$$p(11n+9)_{(4,10,0,4),\,(5,2,0,5),\,(6,7,0,5),\,(2,9,0,5),\,(5,9,0,5)},$$
$$p(11n+9)_{(10,1,0,7),\,(8,0,0,8),\,(1,9,0,8),\,(9,1,0,9),\,(10,1,0,10),\,(5,3,0,10)},$$
$$p(11n+10)_{(1,8,0,1),\,(7,10,0,1),\,(2,6,0,2),\,(9,7,0,2),\,(5,9,0,2),\,(2,1,0,4)},$$
$$p(11n+10)_{(7,2,0,5),\,(4,9,0,5),\,(5,9,0,5),\,(6,6,0,6),\,(8,3,0,7),\,(7,9,0,7)},$$
$$p(11n+10)_{(10,0,0,8),\,(6,2,0,8),\,(4,1,0,9),\,(1,8,0,9),\,(3,5,0,10),\,(10,7,0,10)}.$$

C.f. **Some examples of the form $p(13n+B)_{\mathbf{e}} \equiv 0 \pmod{13}$.** An incomplete list of (conjectural) primitive examples modulo 13 involving only unit-roots of order at most 4 is given by:

$$p(13n+2)_{(11,1),\ (2,8),\ (2,8,0,2),\ (8,8,0,6),\ (11,1,0,11),\ (5,6,0,11)},$$
$$p(13n+3)_{(12),\ (8,2),\ (1,10,0,1),\ (5,0,0,5),\ (10,6,0,6),\ (3,10,0,9)},$$
$$p(13n+4)_{(10),\ (12),\ (8,2),\ (2,8),\ (1,11),\ (2,6,0,1),\ (1,10,0,1),\ (3,4,0,3),\ \dots},$$
$$p(13n+5)_{(10),\ (11,1),\ (1,11),\ (6,1,0,2),\ (2,8,0,2),\ (3,4,0,3),\ \dots},$$
$$p(13n+6)_{(12),\ (11,1),\ (8,2),\ (2,8),\ (1,10,0,1),\ (2,8,0,2),\ (8,12,0,2),\ \dots},$$
$$p(13n+7)_{(10),\ (11,1),\ (8,2),\ (6,3),\ (1,11),\ (2,8,0,2),\ (10,10,0,2),\ (3,4,0,3),\ \dots},$$
$$p(13n+8)_{(10),\ (12),\ (8,1),\ (11,1),\ (8,2),\ (1,10,0,1),\ (2,8,0,2),\ (12,8,0,2),\ (8,10,0,2),\ \dots},$$
$$p(13n+9)_{(10),\ (2,8),\ (1,11),\ (10,12),\ (12,9,0,1),\ (1,6,0,2),\ (10,8,0,2),\ \dots},$$
$$p(13n+10)_{(12),\ (8,2),\ (2,8),\ (12,10),\ (8,12),\ (1,7,0,1),\ (1,10,0,1),\ (5,1,0,3),\ \dots},$$
$$p(13n+11)_{(10),\ (12),\ (11,1),\ (8,2),\ (1,8),\ (10,10),\ (1,11),\ (3,5,0,1),\ (2,8,0,2),\ \dots},$$
$$p(13n+12)_{(10),\ (3,6),\ (2,8),\ (12,8),\ (1,11),\ (5,3,0,1),\ (1,5,0,1),\ (7,0,0,2),\ \dots}.$$

C.g. **Computational aspects.** We outline here briefly the discovery of the (conjectural) generalized Ramanujan congruences previously described.

The computations where done in two steps. In a first step, we used series expansions of $\sum_{n=0}^{\infty} p(n)q^n$ (with coefficients reduced modulo a small fixed prime l) and its powers up to degree $N \sim 200$ in order to guess them. In a second step, we redid the computations up to degree $N = 5000$ for the discovered examples (we did not encounter false positives, they should be rare since the probability for a false positive should naively be close to $l^{-N/l}$ for examples of the kind considered here.)

Conjectural examples where guessed by considering all possible exponents $e_i \in \{0, \dots, p-1\}$ for i ranging over the set $\mathcal{D}(a)$ of all divisors of a small integer a (we considered mainly $a \in \{2, 3, 4, 6, 8\}$). We wrote a small Maple-program generating all $l^{|\mathcal{D}(a)|}$ possible series $\prod_{i \in \mathcal{D}(a)} A_i^{e_i}$ up to order N over $\mathbb{F}_l$ where $A_j = \sum_{n=0}^{\infty} P(n)q^{jn}$ (with coefficients reduced modulo l and working only up to degree N), and checking for generalized Ramanujan congruences up to order N.

Examples of generalized Ramanujan congruences seem surprisingly abundant, it is not hard to find them, they come in large numbers and seem to be very common.

References

[Andr–08] G.E. Andrews, *A survey of multipartitions: congruences and identities*, Survey in number theory, 1–19, Developments in Math. **17**, Springer, 2008.

[AnCi] Y. Antolín and L. Ciobanu, *Formal conjugacy growth in acylindrically hyperbolic groups*, Int. Math. Res. Not. IMRN, first published online April 14, 2016.

[Baba–89] L. Babai, *The probability of generating the symmetric group*, J. Combin. Theory Ser. A **52** (1989), no 1, 148–153.

[Babe–88] I. Babenko, *Closed geodesics, asymptotic volume and the characteristics of growth of groups*, Math. USSR-Izv. **33** (1989), 1–37 [Izb. Akad. Naud SSSR Ser. Math. **52** (1988), no 4, 675–711].

[BaGo–94] H. Barcelo and A. Goupil, *Combinatorial aspects of the Poincaré polynomial associated with a reflection group*, in "Jerusalem Combinatorics '93", Contemp. Math. **178** (Amer, Math. Soc. 1994), 21–44.

[Baud–63] C. Baudelaire, *Le peintre de la vie moderne*, Le Figaro, 26 novembre 1863, and `http://baudelaire.litteratura.com/ressources/pdf/oeu_29.pdf`

[Bern–06] B.C. Berndt, *Number theory in the spirit of Ramanujan*, Student Math. Library **34**, Amer. Math. Soc., 2006.

[Bour–68] N. Bourbaki, *Groupes et algèbres de Lie, chapitres 4, 5 et 6*, Hermann, 1968.

[BCLM–13] E. Breuillard, Y. de Cornulier, A. Lubotzky and C. Meiri, *On conjugacy growth of linear groups*, Math. Proc. Cambridge Philos. Soc. **154** (2013), 261–277.

[Cart–72] R. Carter, *Conjugacy classes in the Weyl group*, Compositio Math. **25** (1972), 1–50.

[Chan–70] K. Chandrasekharan, *Arithmetical functions*, Springer, 1970.

[CDHS–14] W.Y.C. Chen, D.K. Du, Q. Hou, and L.H. Sun, *Congruences of multipartition functions modulo powers of primes*, Ramanujan J. **35** (2015), no 1, 1–19.

[CoKn–04] M. Coornaert and G. Knieper, *An upper bound for the growth of conjugacy classes in torsion-free word hyperbolic groups*, Internat. J. Algebra Comput. **14** (2004), 395–401.

[Eule–48] L. Euler, *Introductio in analysin infinitorum*, M.-M. Bousquet & Socios, Lausannæ, MDCCXLVIII. [Available on ETH Bibliothek web page: `http://www.e-rara.ch/doi/10.3931/e-rara-8740`. English translation by Ian Bruce: `http://www.17centurymaths.com/contents/introductiontoanalysisvol1.htm`.]

[Fine–88] N.J. Fine, *Basic hypergeometric series and applications*, Math. Surveys and Monographs **27**, Amer. Math. Soc., 1988.

[Fink–14] E. Fink, *Conjugacy growth and conjugacy width of certain branch groups*, Internat. J. Algebra Comput. **24** (2014), 1213–1231.

[Gand–63] J.M. Gandhi, *Congruences for $p_r(n)$ and Ramanujan's τ function*, Amer. Math. Monthly **70** (1963), 265–274.

[GoRo–01] C. Godsil and G. Royle, *Algebraic graph theory*, Springer, 2001.

[GrKP–89] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete mathematics*, Addison-Wesley, 1989.

[GuSa–10] V. Guba and M. Sapir, *On the conjugacy growth functions of groups*, Illinois J. Math. **54** (2010), 301–313.

[HaRa–18] G.H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[Hard–40] G.H. Hardy, *Ramanujan: twelve lectures on subjects suggested by his life and work*, Chelsea, 1959 [originally published by Cambridge Univ. Press, 1940].

[HaWr–79] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Fifth Edition, Oxford Univ. Press, 1979.

[Harp–91] P. de la Harpe, *An invitation to Coxeter groups*, in "Group theory from a geometrical viewpoint (Trieste, 1990)", World Sci. Publ. (1991), 193–253.

[Hube–56] H. Huber, *Über eine neue Klasse automorpher Funktionen und eine Gitterpunktproblem in der hyperbolischen Ebene. I*, Comment. Math. Helv. **30** (1956), 20–62.

[HuOs–13] M. Hull and D. Osin, *Conjugacy growth of finitely generated groups*, Adv. Math. **235** (2013), 361–389.

[Knie–83] G. Knieper, *Das Wachstum der Äquivalenzklassen geschlossener Geodätischer in kompakten Mannigfaltigkeiten*, Arch. Math. (Basel) **40** (1983), no. 6, 559–568.

[Lehr–87] G.I. Lehrer, *On the Poincaré series associated with Coxeter group actions on complements of hyperplanes*, J. London Math. Soc. (2) **36** (1987), 275–294.

[Macd–95] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Second ed., Oxford Univ. Press, 1995.

[Mann–12] A. Mann, *How groups grow*, Cambridge Univ. Press, 2012.

[Marg–69] G. Margulis, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, Functional Anal. Appl. **3** (1969), no 4, 335–336.

[OEIS] The on-line encyclopedia of integer sequences, `oeis.org`.

[PaPa–15] J. Parkkonen and F. Paulin, *On the hyperbolic orbital counting problem in conjugacy classes*, Math. Z. **279** (2015), 1175–1196.

[Rivi–10] I. Rivin, *Growth in free groups (and other stories) – twelve years later*, Illinois J Math. **54** (2010), 327–370.

[Serg–93] V. Sergiescu, *Graphes planaires et présentations des groupes de tresses*, Math. Z. **214** (1993), 477–490.

[ShTo–54] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups*, Canadian J. Math., **6** (1954), 274–304.

[Solo–63] L. Solomon, *Invariants of finite reflection groups*, Nagoya Math. J. **22** (1963), 57–64.

[Solo–66] L. Solomon, *The orders of the finite Chevalley groups*, J. Algebra **3** (1966), 376–393.

[Stan–97] R.P. Stanley, *Enumerative combinatorics*, Volume 1, Cambridge Univ. Press, 1997.

Institut Fourier, Université de Grenoble, 100 rue des maths, BP74, 38402 Saint-Martin d'Hères, France

E-mail address: `Roland.Bacher@ujf-grenoble.fr`

Section de mathématiques, Université de Genève, C.P. 64, 1211 Genève 4, Switzerland

E-mail address: `Pierre.delaHarpe@unige.ch`