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# ON GROTHENDIECK'S TAME TOPOLOGY

NORBERT A'CAMPO, LIZHEN JI  
AND ATHANASE PAPADOPOULOS

ABSTRACT. Grothendieck's *Esquisse d'un programme* is often referred to for the ideas it contains on dessins d'enfants, the Teichmüller tower, and the actions of the absolute Galois group on these objects or their étale fundamental groups; see the surveys [3] and [11] in the present volume. But this program contains several other important ideas. In particular, motivated by surface topology and moduli spaces of Riemann surfaces, Grothendieck calls there for a recasting of topology, in order to make it fit to the objects of semialgebraic and semianalytic geometry, and in particular to the study of the Mumford-Deligne compactifications of moduli spaces. A new conception of manifold, of submanifold and of maps between them is outlined. We review these ideas in the present chapter, because of their relation to the theory of moduli and Teichmüller spaces. We also mention briefly the relations between Grothendieck's ideas and earlier theories developed by Whitney, Lojasiewicz and Hironaka and especially Thom, and with the more recent theory of o-minimal structures.

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The final version of this paper will appear as a chapter in Volume VI of *the Handbook of Teichmüller theory*. This volume is dedicated to the memory of Alexander Grothendieck.

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## 1. INTRODUCTION

Grothendieck's *Esquisse d'un programme* [9] is often referred to for the notions of dessin d'enfant and of Teichmüller tower, and for the actions of the absolute Galois group on these objects or their étale fundamental groups. But the *Esquisse* also contains several other important ideas. In particular, motivated by surface topology and the theory of moduli spaces, Grothendieck calls there for a recasting of topology so that it becomes more adapted to the objects of semianalytic and semialgebraic geometry. The name of the new field that he aims to is usually translated by *tame topology* (cf. the English translation of [9]). The French term that Grothendieck uses is “topologie modérée” (moderate topology). One of Grothendieck's aims is

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to obtain a branch of topology which would give a satisfactory theory of *dévisage* (“unscrewing”) of stratified structures. In the *Esquisse*, he proposes some of the main axioms and he formulates the foundational theorems of such a field. His motivation stems from the structure of the moduli spaces  $\mathcal{M}_{g,n}$ , a class of natural spaces which turn out to be subspaces of real analytic spaces, and maps between them. The moduli spaces are also the building blocks of the so-called modular tower, which is a basic object in the *Esquisse*; cf. [3]. The relation with Teichmüller spaces is the main reason why this chapter is included in the present Handbook.

Stratified structures (e.g. those underlying algebraic sets), rather than being given by atlases, are defined using attaching maps, and these attaching maps are required to have some regularity (for instance, Peano curves will not appear as attaching maps, the Jordan theorem should easily follow from the definitions, etc.). Grothendieck proposed then a new branch of topology based on a system of axioms that would be a natural setting for semianalytic and semialgebraic geometries and that would rule out pathological situations like the one we mentioned.

At the time Grothendieck wrote his *Esquisse*, there was already a theory for stratified spaces; such a theory was first developed by Hassler Whitney, see [32] [33]. It was generally admitted that this theory deals with the stratification of an algebraic variety in an effective and algorithmic way. René Thom also developed a theory of stratified spaces; cf. [31], where he uses in particular his notion of *controlled submersions*. Thom’s theory applies to finite complexes, manifolds with corners and semianalytic sets. However, it appears from the *Esquisse* that Grothendieck considered that the existing theories were not flexible enough to include the study of objects like moduli spaces. For instance, in Thom’s theory in [31], a stratified set is obtained by gluing a *finite* union of  $C^\infty$  manifolds. This is obviously not the case for the augmented Teichmüller space boundary, which covers the one of the Mumford-Deligne compactification of moduli of curves, whose natural decomposition into a stratified space is not locally finite. In any case, a major issue in all these theories concerns the description of the attaching maps since these theories involve several complications regarding these maps. This is at the origin of Grothendieck’s remark that classical topology, with the pathological cases that it necessarily involves, is not adapted to the examples he had in mind.

More recently, some new axiomatic topological theories were developed, which are adapted to the setting of semialgebraic geometry and also with the aim of ruling out the pathological phenomena that one encounters in the setting of classical topology. One of these theories, which is considered to be in the lineage of Grothendieck’s ideas, is that of o-minimal structures. This theory is based on a certain number of axioms which specify the kind of subsets of  $\mathbb{R}^n$  that are accepted, and the functions are defined as those whose graphs belong to the admissible sets. Certain authors (cf. [7] and [8]) consider that this is the theory which makes precise Grothendieck’s ideas of tame topology. There is a theory, in mathematical logic, which carries the same name (o-minimal theory), and which is part of the general theory of *quantifier elimination* in model theory. In the theory of quantifier elimination, one tries to replace (mathematical) formulae which contains quantifiers ( $\forall, \exists$ ) by formulae without quantifiers and admitting the same models. One example, in the theory of real fields (that is, the fields that have the same first-order properties as the field of real numbers), is the formula  $\exists x, x^2 + bx + c = 0$  which is equivalent to the formula  $b^2 - ac \geq 0$  which uses no quantifiers. The two theories (the geometric o-minimal theory and o-minimal theory as a branch of mathematical logic) are related. We shall say more about that in the last section of this chapter. Let us recall that the combination of geometry with mathematical logic is not a new subject; it can be traced back (at least) to the works of Hilbert and Frege.

In the following, we shall review Grothendieck's considerations. They are mainly contained in § 5 and § 6 of the *Esquisse*. We shall mention relations with the theory of stratified sets developed by Thom. Thom's theories already contain an important part of Grothendieck's program on this subject (before Grothendieck formulated it), but without the number-theoretic background and motivations.

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## 2. GROTHENDIECK'S IDEAS ON TAME TOPOLOGY IN THE *Esquisse*

Regarding his program on tame topology, Grothendieck writes: "I was first and foremost interested by the modular algebraic multiplicities, over the absolute base-field  $\mathbb{Q}$ , and by a 'dévissage' at infinity of their geometric fundamental groups (i.e. of the profinite Teichmüller groups) which would be compatible with the natural operations of  $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ."<sup>1</sup> He then comments on the difficulties that one encounters as soon as he wants to make big changes in what is commonly considered as basic mathematics:

What is again lacking is not the technical virtuosity of mathematicians, which is sometimes impressive, but the audacity (or simply the innocence... ) to free oneself from a familiar context accepted by a flawless consensus...

Grothendieck recalls that the field of topology at the time he wrote his *Esquisse* was still dominated by the development, done during the 1930s and 1940s, by analysts, in a way that fits their needs, rather than by geometers. He writes that the problem with such a development is that one has to deal with several pathological situations that have nothing to do with geometry. He declares that the fact that "the foundations of topology are inadequate is manifest from the very beginning, in the form of 'false problems' (at least from the point of view of the topological intuition of shape)." These false problems include the existence of wild phenomena (space-filling curves, etc.) that add complications which are not essential. He states that a new field of topology is needed, one which should be adapted to a theory of "dévissage" (unscrewing) of stratified structures, a device which he was led to use several times in his previous works. Stratifications naturally appear in real or complex analytic geometry, where singular sets of maps appear as decreasing sequences of nested singular loci of decreasing dimensions. For Grothendieck, moduli spaces of geometric structures are naturally stratified sets, and the stratification also appears in the degeneration theory of these structures. The main examples are the moduli spaces  $\mathcal{M}_{g,n}$  of algebraic curves, equipped with their Mumford-Deligne boundaries. Grothendieck calls these spaces the Mumford-Deligne multiplicities, and he denotes them by  $\widehat{\mathcal{M}}_{g,\nu}$ .

It is interesting to stop for a while on the word "multiplicity," which is a term used by Riemann and which is an ancestor of our word "manifold."<sup>2</sup> In fact, after Riemann, Poincaré and others used this word in its french form ("multiplicité") to describe moduli space. Their view on that space was close to what we intend today by a manifold, given as a subset of a Euclidean space defined by a certain

<sup>1</sup>All the quotes that we make here are from § 5 and § 6 of the *Esquisse*. The English translation is due to P. Lochak and L. Schneps.

<sup>2</sup>The German word is *Mannigfaltigkeit*; it is the word still used today to denote a manifold, and it is close to the English word "manifold." But the French term, which Grothendieck uses in the *Esquisse*, was replaced by the word "variété," and is no more used to denote a mathematical object.

number of equations. Poincaré mentioned explicitly that the moduli spaces of Riemann surfaces have singularities (although he did not specify the nature of these singularities). We refer the interested reader to the historical survey [1] and to the more specialized one [23], for the notion of multiplicity and the origin of the notion of manifold. At several places in the *Esquisse*, Grothendieck explains the meaning he gives to the word “multiplicity,” with a reference to a stratification indexed by graphs that parametrize the possible combinatorial structures of stable curves (§5). In fact, his use of the word “multiplicity” is close to what we call today an orbifold:

Two-dimensional geometry provides many other examples of such modular stratified structures, which all (if not using rigidification) appear as “multiplicities” rather than as spaces or manifolds in the usual sense (as the points of these multiplicities may have non-trivial automorphism groups).

Other examples of stratified spaces that arise from the geometry of surfaces mentioned by Grothendieck are polygons (and he specifies, Euclidean, spherical or hyperbolic), systems of straight lines in a projective plane, systems of “pseudo-straight lines” in a projective topological plane, and more general immersed curves with normal crossings.

Beyond these examples, Grothendieck declares he had “the premonition of the ubiquity of stratified structures in practically all domains of geometry.” One should note that a similar idea was expressed by Thom, who considered that all the sets that one encounters in geometry (in particular in generalized singular loci) are, at least in generic stable situations, stratified sets.

As the simplest example of a stratified structure, Grothendieck mentions pairs  $(X, Y)$  of topological manifolds, where  $X$  is a closed submanifold of  $Y$  such that  $X$  along  $Y$  is “equisingular.” In order to fit into this theory of stratification, such a notion needs to be defined with care, and for that purpose, one has to specify in a precise way the kind of *tubular neighborhood* of  $X$  in  $Y$  that is needed in the presence of such a structure which is more rigid than the topological one. The classical cases are the piecewise-linear, the Riemannian, and the metric. Such a tubular neighborhood has to be canonical, that is, well defined up to an automorphism of the structure. Grothendieck says that for that purpose, one has to work systematically in the *isotopic categories* associated with the categories of topological nature that arise a priori. Isotopic categories are categories where two maps are considered to be the same if they are isotopic. Let us mention again Thom, who, in the early 1960s, developed a notion of tubular neighborhood in the setting of stratified spaces, and a theory of locally trivial stratification [30]. In particular, his two theorems on isotopy provide a template for a deep reflection on stratified sets and morphisms. For an overview of the work of Thom on this subject, we refer the reader to the report [28] by Teissier.

As already said, Grothendieck calls a topology which avoids the pathological situations a *tame topology*, (in French, “topologie modérée”) and he declares that such a topology, which he wishes to develop, will not be unique, but that there is a “vast infinity” of possibilities. They range “from the strictest of all, the one which deals with the ‘piecewise-algebraic spaces’ (with  $\overline{\mathbb{Q}}_r = \overline{\mathbb{Q}} \cap \mathbb{R}$ ) to the piecewise-analytic.” One must mention here the theory of subanalytic spaces developed by Lojasiewicz and Hironaka, and in particular the latter’s work on “resolution of singularities.” In this theory, objects are defined not only by analytic equations, but also by analytic inequalities (they may have “corners” in the analytic sense an even more complicated singularities). The term “subanalytic” was introduced by Hironaka in his paper [13]. The theory dealing with these objects is rich. There

is a so-called *uniformization theorem* for subanalytic sets, which is a consequence of Hironaka's theory of resolution of singularities [12] [14]. This theorem says that any closed subanalytic subset of  $\mathbb{R}^n$  is the image by an analytic map of a proper analytic manifold of the same dimension.

There are also algebraic versions of this theory. An important observation in this setting is that projections of algebraic varieties (say, over the reals) onto affine subspaces are defined by inequations, and not only by equations. The standard example is that the projection of the circle in  $\mathbb{R}^2$  defined by the equation  $x^2 + y^2 - 1 = 0$  on the  $x$ -axis is the interval  $[0, 1]$ . This is not an algebraic set. With this in mind, a subset of  $\mathbb{R}^n$  is called *semialgebraic* if it can be obtained using combinations (finite operations of unions, intersections and complements) of polynomial equations and polynomial inequalities. (If one uses only polynomial equations – with no inequalities – then one gets the usual definition of an *algebraic set*.) A theorem of Lojasiewicz [20] says that a semialgebraic set can be triangulated, that is, transformed into a linearly embedded simplicial complex by a semialgebraic map of the ambient space. His main tool was an inequality known as the Lojasiewicz inequality. It says that if  $U$  is an open subset of a Euclidean space  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}$  an analytic function, then for any compact subset  $K \subset U$  there exists  $\alpha > 0, C > 0$  such that for any  $p$  in  $K$ ,  $d(p, Z)^\alpha \leq C|f(p)|$ , where  $Z$  is the analytic subset of  $U$  where  $f$  vanishes. See also [21] for the work of Lojasiewicz.

The Cartesian product of two semialgebraic sets is semialgebraic. Stratified sets naturally appear in this theory, since every semialgebraic set admits a stratification by semialgebraic sets of decreasing dimension. In particular, the boundary of a semialgebraic set is a semialgebraic set of lower dimension. These properties and others are summarized in the report [22].

The *Tarski-Seidenberg theorem* says that the projection onto an affine subspace of a semialgebraic set is semialgebraic. This theorem also implies that the closure of a semialgebraic set is semialgebraic. In fact, for any semialgebraic subset  $X$  of  $\mathbb{R}^n$ , the closure  $\overline{X}$  of  $X$ , its interior, and its frontier are semi-algebraic sets. (See [4] Proposition 2.3.7.) The Tarski-Seidenberg theorem is also discussed in the books [15] and [6], and it is summarized, with several other related things, in the review [22]. From the point of view of model theory, this theorem is an illustration of the theory of quantifier elimination over the reals.

After talking about semialgebraic sets, one needs to talk about maps between them. There are some natural properties that such maps must satisfy, and this leads to the notion of isomorphism between two semialgebraic sets.

Algebraic and semialgebraic sets are tame in the sense that the pathological examples of Cantor sets, space filling curves, Sierpiński sponges, etc. do not occur as level sets of algebraic or semialgebraic functions. In some sense, this justifies Grothendieck's assertions in the *Esquisse*, that such pathologies are irrelevant in this kind of geometry.

In his program, Grothendieck states some of the foundational theorems he expects to hold in tame topology. One of them is a *comparison theorem*, which says the following:

*We essentially find the same isotopic categories (or even  $\infty$ -isotopic) whatever the tame theory we work with.*

He makes more precise statements about this idea in §5 of the *Esquisse*. The maps considered in this category may be embeddings, fibrations, smooth, étale fibrations, etc. Among the axioms that he introduces is the *triangulability axiom* “in the tame sense, of a tame part of  $\mathbb{R}^n$ .” He considers a “piecewise  $\mathbb{R}$ -algebraic” theory of complex algebraic varieties, and his setting also includes varieties defined over number fields. Again, one has to mention here that Thom considered that

any semianalytic set, in a neighborhood of any of its points, is equivalent, by an ambient isotopy, to a semialgebraic set. Several variants of this fact were proved, in particular by T. Mostowski.

After the comparison theorem, the next fundamental theorem that Grothendieck mentions concerns the existence and uniqueness of a tubular neighborhood  $T$  for a closed tame subspace  $Y$  in a tame space  $X$ , together with a way of constructing it, using for instance a tame map  $X \rightarrow \mathbb{R}^+$  having  $Y$  as a zero set, and the description of the boundary (not in the usual sense of a manifold with boundary)  $\partial T$  of  $T$ . This is where the “equisingularity” hypothesis on  $X$  is needed. One expects that the tubular neighborhood  $T$  will be endowed, “in an essentially unique way,” with the structure of a locally trivial fibration over  $Y$ , with  $\partial T$  as a subfibration. In fact, this is one of the isotopy theorems of Thom. In this respect, one of the basic ideas of Thom concerns tubular neighborhoods of strata; it says that each stratum has a “tubular neighborhood” in the union of adjacent strata, and that the isotopies between these various neighborhoods are insured by appropriate transversality conditions.

Concerning his project of tame topology, Grothendieck writes: “This is one of the least clear points in my temporary intuition of the situation, whereas the homotopy class of the predicted structure map  $T \rightarrow Y$  has an obvious meaning, independent of any equisingularity hypothesis, as the homotopic inverse of the inclusion map  $Y \rightarrow T$ , which must be a homotopism.” He declares ([9] §5):

It will perhaps be said, not without reason, that all this may be only dreams, which will vanish in smoke as soon as one sets to work on specific examples, or even before, taking into account some known or obvious facts which have escaped me. Indeed, only working out specific examples will make it possible to sift the right from the wrong and to reach the true substance. The only thing in all this which I have no doubt about, is the very necessity of such a foundational work, in other words, the artificiality of the present foundations of topology, and the difficulties which they cause at each step. It may be however that the formulation I give of a theory of dévissage of stratified structures in terms of an equivalence theorem of suitable isotopic (or even  $\infty$ -isotopic) categories is actually too optimistic. But I should add that I have no real doubts about the fact that the theory of these dévissages which I developed two years ago, although it remains in part heuristic, does indeed express some very tangible reality.

For what concerns stratified spaces, the theory of tubular neighborhoods that Grothendieck sketches is included in a more general theory of “local retraction data which make it possible to construct a canonical system of spaces, parametrized by the ordered set of flags  $\text{Fl}(I)$  of the ordered set  $I$  indexing the strata; these spaces [...] are connected by embedding and proper fibration maps, which make it possible to reconstitute in an equally canonical way the original stratified structure, including these additional structures.” The main examples are again the Mumford-Deligne multiplicities  $\widehat{\mathcal{M}}_{g,\nu}$  with their canonical stratification at infinity. Grothendieck writes ([9] §5):

Another, probably less serious difficulty, is that this so-called moduli “space” is in fact a multiplicity – which can be technically expressed by the necessity of replacing the index set  $I$  for the strata with an (essentially finite) category of indices, here the “MD [Mumford-Deligne] graphs” which “parametrize” the possible “combinatorial structures” of a stable curve of type  $(g, \nu)$ . This said, I can assert that the general theory of dévissage, which has been developed especially to meet the needs of this example, has indeed proved to be a precious guide, leading to a progressive understanding, with flawless coherence, of some essential



aspects of the Teichmüller tower (that is, essentially the “structure at infinity” of the ordinary Teichmüller groups). It is this approach which finally led me, within some months, to the principle of a purely combinatorial construction of the tower of Teichmüller groupoids.

In §6, Grothendieck writes that one of the most interesting foundational theorems in that theory would be a dévissage theorem for maps  $f : X \rightarrow Y$ , where  $Y$  is equipped with a filtration  $Y^i$  by closed tame subspaces and where above the strata  $Y^i \setminus Y^{i-1}$ ,  $f$  induces a trivial fibration (from this tame point of view). This theorem may also be generalized to the case where the space  $X$  is also equipped with a filtration. He declares that theories of locally and globally tame spaces, and of set-theoretic differences of tame spaces, and of globally tame maps, generalizing the notion of locally trivial fibration, must be developed. Again, we note that part of this program was realized by Thom, cf. [31].

### 3. GROTHENDIECK'S LATER COMMENTS

A few years after he wrote the *Esquisse*, Grothendieck, commenting on his ideas on tame topology, considered that they did not attract the attention of topologists or geometers. In Section 2.12, Note 42 of his *Récoltes et semailles* (1986) [10], he compares the introduction of this new “tame topology” to the introduction of schemes in the field of algebraic geometry. He writes:<sup>3</sup>

A multitude of new invariants, whose nature is more subtle than the invariants which are currently known and used, but which I feel as fundamental, are planned in my program on “moderate topology” (of which a very rough sketch is contained in my *Sketch of a program*, which will appear in Volume 4 of the Reflections). This program is based on a notion of “moderate topology,” or of “moderate space,” which constitute, may be like the notion of topos, a (second) “metamorphosis of the notion of space.” It seems to me that it is much more evident and less profound than the latter. Nevertheless, I suspect that its immediate impact on topology itself will be much stronger, and that it will transform from top to bottom the craft of the topologist-geometer, by a profound transformation of the conceptual context in which he works. (This also occurred in algebraic geometry, with the introduction of the point of view of schemes.) On the other hand, I sent my *Esquisse* to several old friends and eminent topologists, but it appears it did not have the gift of making anyone of them interested...<sup>4</sup>

In §15 Note 91<sub>3</sub> of *Récoltes et semailles* [10], he writes:

It is especially since my talks at the Cartan seminar on the foundations of complex analytic spaces and on the precise geometric interpretation of “modular varieties with level” à la Teichmüller, around the end of the 1950s, that I understood the importance of a double generalization of the usual notions of “manifold” with which we had worked until now

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<sup>3</sup>The translations from *Récoltes et semailles* are ours.

<sup>4</sup>[Une foule de nouveaux invariants, de nature plus subtile que les invariants actuellement connus et utilisés, mais que je sens fondamentaux, sont prévus dans mon programme de “topologie modérée” (dont une esquisse très sommaire se trouve dans l'*Esquisse d'un Programme*, à paraître dans le volume 4 des Réflexions). Ce programme est basé sur la notion de “théorie modérée” ou “d'espace modéré,” qui constitue, un peu comme celle de topos, une (deuxième) “métamorphose de la notion d'espace.” Elle est bien plus évidente (me semble-t-il) et moins profonde que cette dernière. Je prévois que ses retombées immédiates sur la topologie “proprement dite” vont être pourtant nettement plus percutantes, et qu'elle va transformer de fond en comble le “métier” de topologue géomètre, par une transformation profonde du contexte conceptuel dans lequel il travaille. (Comme cela a été le cas aussi en géométrie algébrique avec l'introduction du point de vue des schémas.) J'ai d'ailleurs envoyé mon “Esquisse” à plusieurs de mes anciens amis et illustres topologues, mais il ne semble pas qu'elle ait eu le don d'en intéresser aucun...]



(algebraic, real or complex analytic, differentiable – and later on, their ‘moderate topology’ variants). The one consisted in enlarging the definition in such a way that arbitrary “singularities” are admitted, as well as nilpotent elements in the structure sheaf of “scalar functions” – modelled on my foundational work with the notion of scheme. The other extension is towards a “relativisation” over appropriate locally ringed toposes (the “absolute” notions being obtained by taking as a basis a point topos). This conceptual work, which is mature since more than twenty-five years and which started in the thesis of Monique Hakim, is still waiting to be resumed. A particularly interesting case is the notion of relative rigid-analytic space which allows the consideration of ordinary complex analytic spaces and rigid-analytic spaces over local rings with variable residual characteristics, like the “fibers” of a unique relative rigid-analytic space; in the same way as the notion of relative scheme (which was eventually generally accepted) allows to relate to each other algebraic varieties defined over fields of different characteristics.<sup>5</sup>

In the same manuscript ([10] § 18), Grothendieck returns to his subject; he writes:

There is a fourth direction, carried on during my past as a mathematician, which is directed towards a renewal from top to bottom of an existing field. This the “moderate topology” approach in topology, on which I somehow elaborated in the *Esquisse d'un Programme* (Sections 5 and 6). Here, as many times since my very faraway high-school years, it seems that I am still the only one to realize the richness and the emergency of a work on foundations that has to be done, and whose need here seems to me however more evident than ever. I have the clear impression that the development of the point of view of moderate topology, in the spirit alluded to in the *Esquisse d'un programme*, would represent for topology a renewal whose scope is comparable to the one the theory of schemes brought in algebraic geometry, and without requiring an energy investment of a comparable size. Furthermore, I think that such a moderate topology will end up being a valuable tool in the development of arithmetic geometry, in particular in order to be able to prove “comparison theorems” between the “profinite” homotopic structure associated to a stratified scheme of finite type over the field of complex numbers (or, more generally, to a scheme-stratified multiplicity of finite type over this field), and the corresponding “discrete” homotopic structure, defined using a transcendental way, and up to appropriate (in particular, equisingularity) hypotheses. This question only makes sense in terms of a precise “dévissage theory” for stratified structures which seems to me,

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<sup>5</sup>[C'est surtout depuis mes exposés au Séminaire Cartan sur les fondements de la théorie des espaces analytiques complexes, et sur l'interprétation géométrique précise des “variétés modulaires à niveau” à la Teichmüller, vers la fin des années cinquante, que j'ai compris l'importance d'une double généralisation des notions courantes de “variété” avec lesquelles on a travaillé jusqu'à présent (algébrique, analytique réelle ou complexe, différentiable – ou par la suite, leurs variantes en “topologie modérée”). L'une consiste à élargir la définition de sorte à admettre des “singularités” arbitraires, et des éléments nilpotent dans le faisceau structural des “fonctions scalaires” – sur le modèle de mon travail de fondements avec la notion de schéma. L'autre extension est vers une “relativisation” au-dessus de topos localement annelés convenables (les notions “absolues” étant obtenues en prenant comme base un topos ponctuel). Ce travail conceptuel, mûr depuis plus de vingt-cinq ans et amorcé dans la thèse de Monique Hakim, attend toujours d'être repris. Un cas particulièrement intéressant est celui d'une notion d'espace rigide-analytique relatif, qui permet de considérer des espaces analytiques complexes ordinaires et des espaces rigide-analytiques sur des corps locaux à caractéristiques résiduelles variables, comme les “fibres” d'un même espace rigide-analytique relatif ; tout comme la notion de schéma relatif (qui a fini par entrer dans les mœurs) permet de relier entre elles des variétés algébriques définies sur des corps de caractéristiques différentes.]

in the case of “transcendental” topology, to require the introduction of the “moderate” context.”<sup>6</sup>

In the section called *La vision – ou douze thèmes pour une harmonie* (“The vision – or twelve themes for a harmony”) of *Récoltes et semailles* ([10] § 2.8), the subject of tame topology is considered by Grothendieck as one of the twelve themes which he describes as his “great ideas” (*grandes idées*).

#### 4. O-MINIMAL SETS

In this section, we briefly mention a few facts on o-minimal structures.

The notion of o-minimal structure is a kind of a generalization of a semialgebraic and semianalytic structure, and it is in part motivated by it.<sup>7</sup> We already mentioned that from their very definition, semialgebraic sets are stable under the usual Boolean operations of intersection, union and taking the complement. More precisely, an o-minimal structure on  $\mathbb{R}$  is a collection of subsets  $S_n$  of  $\mathbb{R}^n$  for each  $n \geq 1$  satisfying the following:

- Each  $S_n$  is stable under the operations of finite union, intersection, and taking the complement;
- the elements of the collection  $S_1$  are finite unions of intervals and points;
- the projection maps from  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^n$  sends subsets in  $S_{n+1}$  to subsets in  $S_n$ .

We already recalled that the Tarski-Seidenberg theorem says that the projection to a lower-dimensional affine space of a semi-algebraic set is semi-algebraic, so that semi-algebraic sets constitute an o-minimal structure. The study of o-minimal structures is also a subfield of mathematical logic, and the theory can be wholly developed as a theory about quantifiers. It is closely related to model theory, even though the motivation behind it comes from the theory of semi-algebraic sets. In fact, the theory of semi-algebraic and subanalytic sets are prominent instances where the properties of quantifier elimination in model theory may be applied.

Loi reports in [18] that the name o-minimal structure was given by van den Dries, Knight, Pillay and Steinhorn, who developed the general theory ([8], [16], [24]). The author of [18] also says that Shiota had a similar program ([25], [26]). He

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<sup>6</sup>[Il y a enfin une quatrième direction de réflexion, poursuivie dans mon passé de mathématicien, allant en direction d’un renouvellement “de fond en comble” d’une discipline existante. Il s’agit de l’approche “topologie modérée” en topologie, sur laquelle je m’étends quelque peu dans l’“Esquisse d’un Programme” (par. 5 et 6). Ici, comme tant de fois depuis les années lointaines du lycée, il semblerait que je sois seul encore à sentir la richesse et l’urgence d’un travail de fondements à faire, dont le besoin ici me paraît plus évident pourtant que jamais. J’ai le sentiment très net que le développement du point de vue de la topologie modérée, dans l’esprit évoqué dans l’Esquisse d’un programme, représenterait pour la topologie un renouvellement de portée comparable à celui que le point de vue des schémas a apporté en géométrie algébrique, et ceci, sans pour autant exiger des investissements d’énergie de dimensions comparables. De plus, je pense qu’une telle topologie modérée finira par s’avérer un outil précieux dans le développement de la géométrie arithmétique, pour arriver notamment à formuler et à prouver des “théorèmes de comparaisons” entre la structure homotopique “profinie” associée à un schéma stratifié de type fini sur le corps des complexes (ou plus généralement, à une multiplicité schématique stratifiée de type fini sur ce corps), et la structure homotopique “discrète” correspondante, définie par voie transcendante, et modulo des hypothèses (d’équisingularité notamment) convenables. Cette question n’a de sens qu’en termes d’une “théorie de dévissage” précise pour les structures stratifiées, qui dans le cadre de la topologie “transcendante” me semble nécessiter l’introduction du contexte “modéré.”]

<sup>7</sup>According to Teissier, the theory is also motivated by the work of Lojasiewicz on semianalytic sets. The very definition of o-minimality generalizes the crucial observation made by Lojasiewicz that the proofs of (most of) the tameness properties do not really require the global finiteness of generic projections of an algebraic subset to an affine space of the same dimension (Noether’s theorem) that appears in the Tarski-Seidenberg theory, but only the local finiteness expressed by the Weierstrass preparation theorem.

then writes: “The theory of o-minimal structures is a wide-ranging generalization of semialgebraic and subanalytic geometry. Moreover, one can view the subject as a realization of Grothendieck’s idea of *topologie modérée*, or tame topology, in his *Esquisse d’Un Programme* (1984).” The paper [27] by Shiota is an example of the combination of topology, geometry and logic which is realized in o-minimal theory.

## 5. AS A WAY OF CONCLUSION

We saw that some of the basic ideas of Grothendieck on tame topology were worked out by Thom and others, even before Grothendieck formulated his program. We also saw that there are relations of these ideas with other subjects that grew up after Grothendieck’s work, like the theory of o-minimal structures. But Grothendieck’s project of recasting the whole foundations of topology or of creating a new field of topology based on these ideas has still not been realized. We consider this project as another aspect of his broad vision on mathematics.

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## REFERENCES

- [1] N. A’Campo, L. Ji and A. Papadopoulos, On the early history of moduli and Teichmüller spaces, In *Lipman Bers, a life in mathematics*, a volume dedicated to Lipman Bers, ed. L. Keen, I. Kra and Rubi Rodriguez, American Math. Society, 2015, 175-262.
- [2] N. A’Campo, L. Ji and A. Papadopoulos, On Grothendieck’s construction of Teichmüller space. In *Handbook of Teichmüller theory* (A. Papadopoulos, ed.), Volume VI, EMS Publishing House, Zürich, 2016, 35-69.
- [3] N. A’Campo, L. Ji and A. Papadopoulos, Actions of the absolute Galois group. In *Handbook of Teichmüller theory* (A. Papadopoulos, ed.), Volume VI, EMS Publishing House, Zürich, 2016, 397-435.
- [4] R. Benedetti and J.-J. Risler, *Real algebraic and semi-algebraic sets*, Hermann, 1990.
- [5] E. Bierstone and P. Milman, *Subanalytic Geometry*, In: *Model Theory, Algebra, and Geometry*, (D.Haskell, A. Pillay, and Ch. Steinhorn, ed.) MSRI Publications, Volume 39, 2000
- [6] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Springer Verlag, 1998.
- [7] L. van den Dries, *Tame topology and o-minimal structures*. London Mathematical Society lecture note series, no. 248. Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1998.
- [8] L. van den Dries, A generalization of the Tarski-Seidenberg theorem and some nondefinability results, *Bull. Amer. Math. Soc. (N.S.)* 15 (1986), 189-193.
- [9] A. Grothendieck, *Esquisse d’un programme* (Sketch of a program), unpublished manuscript (1984), English translation by P. Lochak and L. Schneps in *Geometric Galois actions, vol. 1, “Around Grothendieck’s Esquisse d’un Programme”* (L. Schneps and P. Lochak, ed.) London Math. Soc. Lecture Note Ser. vol. 242, pp. 5-48, Cambridge Univ. Press, Cambridge, 1997.
- [10] A. Grothendieck, *Récoles et semailles : Réflexions et témoignage sur un passé de mathématicien* (Harvesting and Sowing : Reflections and testimony on a mathematician’s past), manuscript, 1983-1986, to appear,
- [11] P. Guillot, *A primer on dessins*, In *Handbook of Teichmüller theory* (A. Papadopoulos, ed.), Volume VI, EMS Publishing House, Zürich, 2016.
- [12] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. of Math. (2)* 79 (1964), 109-203, 205-326.
- [13] H. Hironaka, *Subanalytic sets*, In: *Number theory, algebraic geometry and commutative algebra: in honor of Yasuo Akizuki*, (Y. Kusunoki et al. ed.), Kinokuniya, Tokyo, 1973, p. 453-493.
- [14] H. Hironaka, *Introduction to the theory of infinitely near singular points*, Mem. Mat. Instituto Jorge Juan 28, Consejo Superior de Investigaciones Científicas, Madrid, 1974.

- [15] A. G. Khovanskii, *Fewnomials*. Translations of Mathematical Monographs 88. Translated from the Russian by Smilka Zdravkovska. Providence, RI: American Mathematical Society, 1991.
- [16] J. Knight, A. Pillay and C. Steinhorn, Definable sets in ordered structures II, *Trans. Amer. Math. Soc.* 295 (1986), 593-605.
- [17] T. L. Loi and P. Phan, *Acta Mathematica Vietnamica*, 29 (2014) No. 4, 637-647
- [18] T. L. Loi, o-minimal structures. In: *The Japanese-Australian workshop on real and complex singularities* (Fukui, Toshizumi et al., ed.), The University of Sydney, Sydney, Australia, September 15-18, 2009. Proceedings of the Centre for Mathematics and its Applications, Australian National University 43, 2010, 19-30.
- [19] T. L. Loi, Tame topology and Tarski-type systems, *Vietnam J. Math.* 31 (2003), No. 2, 127-136.
- [20] S. Lojasiewicz, Triangulation of semi-analytic sets, *Ann. Scuola Norm. Sup. Pisa* (3), 18 (1964), 449-474.
- [21] S. Lojasiewicz, Sur les ensembles semi-analytiques. In *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Volume 2, Gauthier-Villars, Paris, 1971, 237-241.
- [22] D. Marker, Book review of *Tame topology and o-minimal structures* by Lou van den Dries, Cambridge Univ. Press, New York, 1998. *Bull. Amer. Math. Soc.* (N.S.) 37 (2000) No. 3, 351-357.
- [23] K. Ohshika, The origin of the notion of manifold, In *From Riemann to differential geometry and relativity*, (L. Ji, A. Papadopoulos and S. Yamada, ed.), Springer Verlag, to appear in 2017.
- [24] A. Pillay and C. Steinhorn, Definable sets in ordered structures I, *Trans. Amer. Math. Soc.*, 295 (1986), 565-592.
- [25] M. Shiota, Geometry of subanalytic and semianalytic sets: Abstract, In: *Real analytic and Algebraic Geometry*, Proceedings of the international conference, Trento, Italy, September 21-25, 1992. Walter de Gruyter, Berlin, 1995, 251-275.
- [26] M. Shiota, *Geometry of Subanalytic and Semialgebraic Sets*, Progress in Math., vol. 150, Birkhäuser, Boston, 1997.
- [27] M. Shiota, O-minimal Hauptvermutung for polyhedra I. *Invent. math.* 196 (2014), Issue 1, 163-232.
- [28] B. Teissier, Travaux de Thom sur les singularités, *Publ. Math. Inst. Hautes Études Sci.* 68 (1988), 19-25.
- [29] B. Teissier, Tame and stratified objects. In: *Geometric Galois Actions, 1. Around Grothendieck's esquisse d'un programme*. London. Math. Soc. Lecture Note Series 242, 1997, 231-242.
- [30] R. Thom, La stabilité topologique des applications polynomiales, *L'Œns. Math.* (2) 8 (1962), 24-33.
- [31] R. Thom, Ensembles et morphismes stratifiés. *Bull. Am. Math. Soc.* 75 (1969), 240-284.
- [32] H. Whitney, Complexes of manifolds, *Proc. Nat. Acad. Sci. U.S.A.* 33, (1947), 10-11.
- [33] H. Whitney, Elementary structure of real algebraic varieties, *Ann. Math.* 66 (1957), 545-556.

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