SOLUTION OPERATORS OF THREE TIME VARIABLES FOR FRACTIONAL LINEAR PROBLEMS

Hassan Emamirad, Arnaud Rougirel

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ABSTRACT. It is well-known that the time fractional equation $\mathbf{D}_{\tau,t}^{\alpha} u = Au$, where $\mathbf{D}_{\tau,t}^{\alpha} u$ is the fractional time derivative in the sense of Caputo of u does not generate a dynamical system in the standard sense.

In this paper, we study the algebraic properties of the solution operator $\boldsymbol{\mathcal{T}}(t, s, \tau)$ for that equation with $u(s) = v$. We apply this theory to linear time fractional PDEs with constant coefficients. These equations are solved by the Fourier multiplier technics. It appears that their solution exhibits some singularity, which leads us to introduce a new kind of solution for abstract time fractional problems.

1. INTRODUCTION

Time fractional differential equations have shown their efficiency to model many physical phenomena. They are used for instance in viscoelasticity ([Pod99]), control theory ([MAM97]), seismology ([KK90]) or biophysics ([GN95]).

One of the reasons of their success is that fractional derivatives allow relaxation process with algebraic decay rates unlike the usual first order derivative. Indeed, in dimension one, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u = \lambda u, \qquad \lambda < 0, \tag{1.1}$$

is the usual way to describe relaxation toward equilibrium by a linear differential equation. That gives rise of course to an exponential decay. However, in some situations, an algebraic decay is more realistic. It turns out that solutions to

$$\mathbf{D}_{0,t}^{\alpha} u = \lambda u, \tag{1.2}$$

where $\mathbf{D}_{0,t} u$ denotes the *fractional derivative* of u in the sense of Caputo (see Definition 3.3), decay like

$$\frac{1}{\Gamma(1-\alpha)|\lambda|} \frac{1}{t^{\alpha}}$$

at infinity (see Propositions 3.1 and 5.1 for details).

Another feature of time fractional derivative is that it involve the history of the system. So it is useful for modelling various phenomena where memory takes place. See [DWH13].

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From a mathematical point of view, fractional differential equations are intensively studied. The MathSciNet website inventories more than 600 papers whose title contains the keyword "*time fractional*". Tough time fractional initial value problems are rather well understood the dynamics generated by such equations is less investigated. If the structure of the solution to (1.1) and (1.2) are quiet closed, their dymanics differs in an essential way, as we will explain now.

It is well known that (1.1) is *autonomous.* More precisely, for s and v in $\mathbb{R}$, if u and w denote respectively the solutions to

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \lambda u, \quad u(0) = v \tag{1.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}w = \lambda w, \quad w(s) = v, \tag{1.4}$$

one has clearly

$$w(t) = u(t-s), \quad \forall t \in \mathbb{R}. \tag{1.5}$$

Thus the solution to (1.4) is obtained from the solution to (1.3) by a time shift and the trajectories are the same.

For fractional time derivatives, the situation is more involved. Indeed, if u and w denote respectively the solutions to

$$\mathbf{D}_t^\alpha u = \lambda u, \quad u(0) = v \tag{1.6}$$

$$\mathbf{D}_t^\alpha w = \lambda w, \quad w(s) = v, \tag{1.7}$$

we have (see Proposition 5.2)

$$w(t) = \frac{v}{u(s)}u(t) \neq u(t-s). \tag{1.8}$$

Then the trajectories are different; that fact, pointed out in [PL10], is the main difference between the dynamics of (1.1) and (1.2). So Equation (1.2) is not autonomous thus Problem (1.7) and its generalisations need to be studied.

Also (1.5) holds in a PDE context but the transposition of (1.8) in such a setting is not clear. These issues will be addressed in this paper.

A convenient tool to investigate dynamics is *solution operator*. For instance, the solution operator, S, for (1.3) is defined by

$$S(t)v := u(t) = \exp(\lambda t)v.$$

The solution operator, T, for (1.4) depends on two independent time variables, namely

$$T(t,s)v = w(t).$$

Clearly,

$$T(t,s) = S(t-s),$$

that is to say T is *translation invariant.*

What is the *solution operator* for (1.7)? That question is central in the present work. It turns out that the solution operator for (1.7) depends on three independent time variables. That observation is the main contribution of this paper. Let us roughly describe that solution operator, denoted by $\boldsymbol{\mathcal{T}}$. The first two variables are t and s as above. The third is hidden in Equation (1.2) and is, in fact, the lower limit of the definite integral involved in the fractional time derivative (see Definition 3.1

below). We will construct $\boldsymbol{\mathcal{T}}$ for some infinite dimensional time fractional differential linear equations and highlight some of these properties.

The paper is organised as follows. In Section 2, we introduce a theory for three variables maps that will be applied to solution operator of time fractional problems. We identify their main features and investigate some of their algebraic properties. In Section 3, we recall the required background on *Caputo's derivatives* and *Mittag-Leffler functions.* Section 4 is devoted to abstract linear time fractional problems. We define two type of solutions for these problems, namely *strong solutions* in the sense of [Prü93, Definition I-1.1] and a weaker and new type of solutions in Definition 4.1. Assuming some existence and uniqueness results, we construct a three time variables solution operators for these fractional problems and prove some of their properties by using theoretical results of Section 2.

Section 5 and 6 are concerned with applications. The former deals with one dimensional problems. The latter is concerned with time fractional differential equations with constant coefficients differential operators acting on the space variables. The main result is the proof of existence and uniqueness of solution in the sense of Definition 4.1. It turns out that the solutions to the non autonomous problems under consideration have in general a singularity and therefore are not *strong solutions*; see Remark 6.2. That is the reason why we introduce solutions in the sense of Definition 4.1. We give a representation of these solutions in term of Fourier multipliers, construct solution operators and state their properties by using the abstract results of Section 2 and 4.

In this context, many of the standard properties of non-fractional differential equations such as the regularizing effect or energy estimates are no more valid and in our forthcoming paper we will give the new version of these properties.

2. Propagators and translation invariant maps

In this section, we introduce a theory for maps of three variables that will be applied to solution operators of time fractional problems. We identify the main features of these maps and investigate some of their algebraic properties.

For sets E and F, we denote by $\mathcal{F}(E,F)$ the set of all maps defined on E with values in F. In the case where $E=F$, we will write $\mathcal{F}(E)$ instead of $\mathcal{F}(E,E)$.

Regarding two variables maps with values in $\mathcal{F}(E)$, their main properties are *translation invariance* and *propagation.*

Definition 2.1. Let $D \subset \mathbb{R}^2$ and $T : D \to \mathcal{F}(E)$. We say that T is *positively translation invariant* if for each $(t,s) \in D$ and $x \geq 0$, one has

$$(t+x, s+x) \in D \quad \text{and} \quad T(t+x,s+x) = T(t,s).$$

We say that T is a *propagator* if for each (t,s) and (s,τ) in D, one has

$$(t,\tau) \in D \quad \text{and} \quad T(t,s)T(s,\tau) = T(t,\tau).$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. For $\mathcal{D} \subset \mathbb{R}^3$, we denote by $p_f(\mathcal{D})$ the subset of $\mathbb{R}^2$ defined by

$$p_f(\mathcal{D}) := \big\{(t,s) \in \mathbb{R}^2 \mid \big(t,s,f(s)\big) \in \mathcal{D}\big\}. \tag{2.1}$$

Let also

$$\mathcal{S}_3 := \big\{\mathcal{D} \subset \mathbb{R}^3 \mid \forall t \geq 0,\ (t,0,0) \in \mathcal{D}\big\}. \tag{2.2}$$

For $\mathcal{D} \subseteq \mathbb{R}^3$ and $\boldsymbol{\mathcal{T}} : \mathcal{D} \to \mathcal{F}(E)$, we put

$$T_{\boldsymbol{\mathcal{T}},f} : p_f(\mathcal{D}) \to \mathcal{F}(E), \quad (t,s) \mapsto \boldsymbol{\mathcal{T}}\big(t,s,f(s)\big). \tag{2.3}$$

Notice that $\boldsymbol{\mathcal{T}}$ is well defined at $(t,s,f(s))$ due to (2.1).

Definition 2.2. For each $\mathcal{D} \subseteq \mathbb{R}^3$, we say that $\boldsymbol{\mathcal{T}} : \mathcal{D} \to \mathcal{F}(E)$ is *translation invariant* if for each $(t,s,\tau) \in \mathcal{D}$ and x in $\mathbb{R}$, one has

$$(t+x, s+x, \tau+x) \in \mathcal{D} \;\text{ and }\; \boldsymbol{\mathcal{T}}(t+x, s+x, \tau+x) = \boldsymbol{\mathcal{T}}(t,s,\tau).$$

Definition 2.3. For $\mathcal{D}$ in $\mathcal{S}_3$, let $\boldsymbol{\mathcal{T}} : \mathcal{D} \to \mathcal{F}(E)$. By definition of $\mathcal{S}_3$,

$$(t,0,0) \in \mathcal{D}, \quad \forall t \geq 0,$$

hence we may define the map $S_{\boldsymbol{\mathcal{T}}} : [0,\infty) \to \mathcal{F}(E)$ by

$$S_{\boldsymbol{\mathcal{T}}}(t) = \boldsymbol{\mathcal{T}}(t,0,0), \quad \forall t \geq 0.$$

Definition 2.4. We say that $S : [0,\infty) \to \mathcal{F}(E)$ is *additive* if

$$S(t)S(s) = S(t+s), \quad \forall s,\ t \geq 0.$$

With these definitions in mind, we may state sufficient conditions on $\boldsymbol{\mathcal{T}}$ for $S_{\boldsymbol{\mathcal{T}}}$ to be additive.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$, $\mathcal{D} \in \mathcal{S}_3$ and $\boldsymbol{\mathcal{T}} : \mathcal{D} \to \mathcal{F}(E)$. We assume that*

(i) *$\boldsymbol{\mathcal{T}}$ is translation invariant;*
(ii) *$T_{\boldsymbol{\mathcal{T}},f}$ is positively translation invariant;*
(iii) *$T_{\boldsymbol{\mathcal{T}},f}$ is a propagator;*
(iv) *there exists $\tau_0 \in \mathbb{R}$ such that*

$$f\big(f(\tau_0)\big) = f(\tau_0).$$

Then $S_{\boldsymbol{\mathcal{T}}}$ is additive.

Proof. For each $t \geq 0$, we recall that $(t,0,0)$ belongs to $\mathcal{D}$ due to (2.2). Then for $s \geq 0$, we have

$$\begin{aligned}
S_{\boldsymbol{\mathcal{T}}}(t) &= \boldsymbol{\mathcal{T}}(t,0,0) \\
&= \boldsymbol{\mathcal{T}}\big(t+f(\tau_0), f(\tau_0), f(\tau_0)\big) && \text{(since } \boldsymbol{\mathcal{T}} \text{ is translation invariant)} \\
&= \boldsymbol{\mathcal{T}}\big(t+f(\tau_0), f(\tau_0), f\big(f(\tau_0)\big)\big) && \text{(by assumption (iv))} \\
&= T_{\boldsymbol{\mathcal{T}},f}\big(t+f(\tau_0), f(\tau_0)\big) && \text{(by notation (2.3))} \\
&= T_{\boldsymbol{\mathcal{T}},f}\big(t+s+f(\tau_0), s+f(\tau_0)\big) && \text{(by assumption (ii)).}
\end{aligned}$$

In the same way, the following equality holds.

$$S_{\boldsymbol{\mathcal{T}}}(s) = T_{\boldsymbol{\mathcal{T}},f}\big(s+f(\tau_0), f(\tau_0)\big).$$

Then

$$
\begin{aligned}
&S_{\boldsymbol{\mathcal{T}}}(t)S_{\boldsymbol{\mathcal{T}}}(s)\\
&\qquad = T_{\boldsymbol{\mathcal{T}},f}\big(t+s+f(\tau_0), s+f(\tau_0)\big)T_{\boldsymbol{\mathcal{T}},f}\big(s+f(\tau_0), f(\tau_0)\big)\\
&\qquad = T_{\boldsymbol{\mathcal{T}},f}\big(t+s+f(\tau_0), f(\tau_0)\big) && \text{(by assumption (iii))}\\
&\qquad = \boldsymbol{\mathcal{T}}\big(t+s+f(\tau_0), f(\tau_0), f(\tau_0)\big) && \text{(by (2.3) and (iv))}\\
&\qquad = \boldsymbol{\mathcal{T}}(t+s,0,0) && \text{(by (i))}\\
&\qquad = S_{\boldsymbol{\mathcal{T}}}(t+s).
\end{aligned}
$$

□

For D in $\mathbb{R}^2$, we denote by $\mathcal{F}_{\text{p.t.i.}}(D,\mathcal{F}(E))$ the set of all maps $T: D \to \mathcal{F}(E)$ that are positively translation invariant.

Proposition 2.2. *Let $D \subseteq \mathbb{R}^2$ be such that*

$$[0,\infty)\times\{0\} \in D; \tag{2.4}$$

$$(t,s)\in D \Longrightarrow t \geq s; \tag{2.5}$$

$$(t,s)\in D,\ x>0 \Longrightarrow (t+x, s+x)\in D. \tag{2.6}$$

Then the map

$$
\begin{aligned}
\mathcal{F}_{\text{p.t.i.}}\big(D,\mathcal{F}(E)\big) &\to \mathcal{F}\big([0,\infty), F(E)\big)\\
T &\mapsto S_T,
\end{aligned}
\tag{2.7}
$$

where

$$S_T(t) := T(t,0), \quad \forall t\geq 0,$$

is a bijection. Its inverse bijection is the map $S\mapsto T_S$ defined by

$$T_S(t,s) := S(t-s), \quad \forall (t,s)\in D.$$

Moreover, if

(i) *$S:[0,\infty)\to\mathcal{F}(E)$ is additive;*
(ii) *$(t,s)\in D,\ (s,\tau)\in D \Longrightarrow (t,\tau)\in D$*

then T_S is a propagator.

Remark 2.1. The set

$$\big\{(t,s)\in\mathbb{R}^2 \mid t\geq s\big\}$$

fulfills assumptions (2.4)-(2.6). For each $s_- < 0$ and $t_+ > 0$, the set

$$\big\{(t,s)\in\mathbb{R}^2 \mid t\geq s\geq 0\big\} \cup \big\{(t,s)\in\mathbb{R}^2 \mid t\geq s\geq t-t_+,\ s_-\leq s<0\big\}$$

satisfies also these assumptions.

Remark 2.2. In applications, $\boldsymbol{\mathcal{T}}(\cdot,s,\tau)v$ will be the solution of a fractional differential problem and v, the prescribed value of that solution at time s (i.e. $\boldsymbol{\mathcal{T}}(s,s,\tau)v = v$). The parameter τ is hidden in the fractional time derivative and is related of the nonlocal character of that operator: τ is simply the lower limit of the definite integral involved in the fractional time derivative (see Definition 3.1 below).

For usual non autonomous differential problems, there are only two time variables, so that their solution operator reads $T = T(t,s)$.

In the case where the above solution operator T is defined on

$$D_2 := \{(t,s) \in \mathbb{R}^2 \mid t \geq s\}$$

the following properties usually hold.

(i) $T(s,s)v = v$ for $s \in \mathbb{R}$ and $v \in E$.
(ii) $T : D_2 \to \mathcal{F}(E)$ is a propagator.

If E is a Banach space and additional continuity properties hold then T is called a *process* or an *evolution operator* (see [CLR13] for further information).

Moreover, if T is positively translation invariant then T is called an *autonomous process* and the corresponding differential problem is said to be *autonomous*. Then the corresponding solution operator S defined by

$$S(t) = T(t,0), \quad \forall t \geq 0,$$

is a *semigroup*. In particular, S is *additive*.

If $S : [0,\infty) \to \mathcal{F}(E)$ is a *solution operator* (that is to say $S(\cdot)v$ is solution of some given fractional differential problem), we may address the question of finding solution operators $T : D \subseteq \mathbb{R}^2 \to \mathcal{F}(E)$ extending S, that is, satisfying $T(\cdot,0) = S(\cdot)$.

When only positively translation invariant maps are considered, Proposition 2.2 gives sufficient conditions for existence and uniqueness of T.

Proof of Proposition 2.2. Let us start to show that the map defined by (2.7) is one-to-one. For, if T_1, T_2 belong to $\mathcal{F}_{\text{p.t.i.}}(D, \mathcal{F}(E))$ and satisfy $S_{T_1} = S_{T_2}$ then for every $(t,s) \in D$, we have $t - s \geq 0$ by (2.5). Thus

$$(t-s, 0) \in D, \tag{2.8}$$

due to (2.4). Then, since T_1 and T_2 are positively translation invariant, we have

$$\begin{aligned} T_1(t,s) &= T_1(t-s,0) = S_{T_1}(t-s) \\ &= S_{T_2}(t-s) = T_2(t-s,0) = T_2(t,s). \end{aligned}$$

Moreover, the map in (2.7) is onto. Indeed, let $S : [0,\infty) \to \mathcal{F}(E)$. For each $(t,s) \in D$, one has $t \geq s$ by (2.5). Then we may define

$$T : D \to \mathcal{F}(E), \quad (t,s) \mapsto T(t,s) := S(t-s).$$

Let us show that T is positively translation invariant: for $(t,s) \in D$ and $x > 0$,

$$(t+x, s+x) \in D$$

due to (2.6) and

$$T(t+x, s+x) = S(t-s) = T(t,s).$$

Then, clearly, $S_T = S$. That proves the first part of the proposition.

Finally, if S is additive then for (t,s) and (s,τ) in D, we have

$$T_S(t,s)T_S(s,\tau) = S(t-s)S(s-\tau) = S(t-\tau) = T_S(t,\tau).$$

Hence T_S is a propagator. This completes the proof of the proposition. □

Let us denote by *id* the identity function of $\mathbb{R}$ that is to say

$$id(t) = t, \quad \forall t \in \mathbb{R}.$$

Recalling the notation (2.1), the following results holds true.

Corollary 2.3. *Let $\mathcal{D} \in \mathcal{S}_3$ be such that*

$$p_{id}(\mathcal{D}) = \big\{(t,s) \in \mathbb{R}^2 \mid t \geq s\big\} \tag{2.9}$$

and $\boldsymbol{\mathcal{T}} : \mathcal{D} \to \mathcal{F}(E)$ be translation invariant. Then the unique positively translation invariant map $T : p_{id}(\mathcal{D}) \to \mathcal{F}(E)$ satisfying

$$T(t,0) = \boldsymbol{\mathcal{T}}(t,0,0), \quad \forall t \geq 0$$

is

$$T = T_{\boldsymbol{\mathcal{T}},id}$$

that is to say

$$T(t,s) = \boldsymbol{\mathcal{T}}(t,s,s), \quad \forall t \geq s.$$

Proof. By assumption (2.9), the set $p_{id}(\mathcal{D})$ satisfies clearly (2.4)-(2.6). By Proposition 2.2, there exists an unique

$$T \in \mathcal{F}_{\text{p.t.i.}}\big(p_{id}(\mathcal{D}), \mathcal{F}(E)\big)$$

such that

$$T(t,s) = \boldsymbol{\mathcal{T}}(t-s,0,0), \quad \forall (t,s) \in p_{id}(\mathcal{D}).$$

Since $\boldsymbol{\mathcal{T}}$ is translation invariant, we have

$$T(t,s) = \boldsymbol{\mathcal{T}}(t,s,s) = T_{\boldsymbol{\mathcal{T}},id}(t,s).$$

□

3. Fractional derivatives

In this section, we recall the required background on *Caputo's derivatives* and *Mittag-Leffler fonctions.*

For X a complex Banach space, let us introduce the convolution of functions defined on semi-infinite intervalls.

Definition 3.1. Let $\tau \in \mathbb{R}$, g be a function of $L^1_{loc}(0,\infty)$ and f be an element of $L^1_{loc}((\tau,\infty),X)$. Then the convolution of g and f is the function of $L^1_{loc}((\tau,\infty),X)$ defined by

$$g *_\tau f(t) = \int_\tau^t g(t-y)f(y)\mathrm{d}y, \qquad \text{a.e. } \; t \in [\tau,\infty).$$

Remark 3.1. This definition is a natural extension of the usual convolution of functions. Indeed, for f, g as in Definition 3.1, let us denote by $\tilde{f}$ and $\tilde{g}$ the functions

$$\tilde{g}(t) = \begin{cases} g(t) & \text{if } \; t \geq 0 \\ 0 & \text{if } \; t < 0 \end{cases}, \qquad \tilde{f}(t) = \begin{cases} f(t) & \text{if } \; t \geq \tau \\ 0 & \text{if } \; t < \tau \end{cases}.$$

Then for a.e. $t \in [\tau,\infty)$,

$$y \mapsto \tilde{g}(t-y)\tilde{f}(y) \in L^1(\mathbb{R},X)$$

and

$$\begin{aligned} g *_\tau f(t) &= \int_{\mathbb{R}} \tilde{g}(t-y)\tilde{f}(y)\mathrm{d}y \\ &= \tilde{g} * \tilde{f}(t). \end{aligned}$$

The following kernel is of fundamental importance in the theory of fractional derivatives.

Definition 3.2. For $\beta \in (0,\infty)$, let us denote by g_β the function of $L^1_{loc}(0,\infty)$ defined for a.e. $t > 0$ by

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}.$$

Let us notice that, for $\alpha \in (0,1)$, we have

$$g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}, \qquad \text{a.e.} \quad t > 0.$$

Moreover, for each α, $\beta \in (0,\infty)$, the following identity holds.

$$g_\alpha * g_\beta = g_{\alpha+\beta}. \tag{3.1}$$

Then we are able to introduce the well known *fractional derivative* of a function in the sense of Caputo.

Definition 3.3. Let $\alpha \in (0,1)$, $\tau \in \mathbb{R}$ and $f \in C([\tau,\infty), X)$. Let also I be any sub-interval of $[\tau,\infty)$. We say that f *admits a (fractional) derivative of order* α *in* $C(I,X)$ if

$$g_{1-\alpha} *_\tau \big(f - f(\tau)\big) \in C^1(I,X).$$

In this case, its *(fractional) derivative of order* α is the function of $C(I,X)$ defined by

$$\mathbf{D}^\alpha_{\tau,t} f := \frac{\mathrm{d}}{\mathrm{d}t}\big\{g_{1-\alpha} *_\tau (f - f(\tau))\big\}.$$

In applications, we will have $I = [\tau,\infty)$ or $I = (\tau,\infty)$.

In fractional differential problems, *Mittag-Leffler functions* play the role of exponential functions in differential equations.

Definition 3.4. For α, $\beta \in (0,\infty)$, we define the *generalised Mittag-Leffler function*, $E_{\alpha,\beta}$ by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad \forall z \in \mathbb{C}.$$

If $\beta = 1$ then we put $E_\alpha := E_{\alpha,1}$ and E_α is called the *Mittag-Leffler function of order* α.

The following proposition gathers the results on the Mittag-Leffler function that we will use in the sequel.

Proposition 3.1. *For all* α, $\beta \in (0,1)$, $\lambda \in \mathbb{C}$, $\mu \in (\frac{\pi\alpha}{2},\pi)$, *we have*

$$\mathbf{D}^\alpha_{0,t} E_\alpha(t^\alpha\lambda) = \lambda E_\alpha(t^\alpha\lambda) \tag{3.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_\alpha(t^\alpha\lambda) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha\lambda), \qquad \forall t > 0 \tag{3.3}$$

$$E_\alpha(z) = -\frac{1}{\Gamma(1-\alpha)}\frac{1}{z} + O\Big(\frac{1}{z^2}\Big), \qquad |z| \to \infty,\ \mu \le |\arg(z)|. \tag{3.4}$$

If $\lambda \in (-\infty,0)$ *then* $t \mapsto E_\alpha(t^\alpha\lambda)$ *is decreasing and positive on* $[0,\infty)$.

Above, it is understood that the function $\arg(\cdot)$ ranges in $(-\pi, \pi]$.

Proof. The proofs of (3.2) and (3.3) are obvious. We refer respectively to [Pod99, theorem 1.4] and [Pol48] for the proofs of (3.4) and the monotonicity result. □

The consequence of the following proposition is that, in general, fractional differential Problems do not generate semigroups. This proposition is proved in [PL10]; the proof relies on the functional identity characterising exponential functions and on Laplace transforms. We give here a still more elementary proof based on the regularity properties of $E_\alpha(\cdot)$.

Proposition 3.2. *For every $\alpha \in (0,1)$, $\lambda \in \mathbb{C}$ and $T > 0$, if*

$$E_\alpha(t^\alpha \lambda)E_\alpha(s^\alpha \lambda) = E_\alpha\big((t+s)^\alpha \lambda\big), \quad \forall t,\ s \in [0,T] \tag{3.5}$$

then $\lambda = 0$.

Proof. Let us define the function $f : [0,\infty) \to \mathbb{C}$ by

$$f(t) = E_\alpha(t^\alpha \lambda), \quad \forall t \geq 0.$$

Since f is continuous at $t = 0$ and $f(0) = 1$, there exists some s_0 in $(0,T]$ such that $f(s_0) \neq 0$. Due to (3.5),

$$f(t)f(s_0) = f(t+s_0), \quad \forall t \in [0,T]. \tag{3.6}$$

Differentiating (3.6) w.r.t. $t > 0$, we get

$$f'(t)f(s_0) = f'(t+s_0). \tag{3.7}$$

Since f' is continuous on $(0,T]$ and $s_0 > 0$, we have

$$f'(t+s_0) \xrightarrow[t\to 0^+]{} f'(s_0).$$

With (3.7) and $f(s_0) \neq 0$, we infer

$$f'(t) \xrightarrow[t\to 0^+]{} \frac{f'(s_0)}{f(s_0)}.$$

Thus f' has a limit in $t = 0$. With (3.3), there results that $\lambda = 0$. □

4. Abstract linear fractional Problems

Let $A : D(A) \subseteq X \to X$ be a closed densely defined linear operator on a complex Banach space X. The domain $D(A)$ is equipped with the standard graph norm, so that it is a Banach space. For $\alpha \in (0,1)$ and $\tau \leq s$, let us consider the following linear fractional non autonomous problem.

$$\begin{cases} \mathbf{D}^\alpha_{\tau,t} u = Au \\ u(s) = v \in D(A). \end{cases} \tag{4.1}$$

Let us notice that the non autonomous fractional problem (4.1) depends on three time variables τ, s and t. Usual differential non autonomous problems depend only on two time variables.

Definition 4.1. Let s, $\tau \in \mathbb{R}$ with $\tau \leq s$, let also $v \in D(A)$ and $u \in C([\tau,\infty), X)$. We say that u is a *solution to* (4.1) on $[\tau,\infty)$ if

(i) u belongs to $C((\tau,\infty),D(A))$ and $u(s)=v$;
(ii) u admits a derivative of order α in $C((\tau,\infty),X)$;
(iii) $\mathbf{D}^{\alpha}_{\tau,t}u$ belongs to $L^1_{\mathrm{loc}}([\tau,\infty),X)$;
(iv) $\mathbf{D}^{\alpha}_{\tau,t}u=Au$ in $C((\tau,\infty),X)$.

Condition (iii) above is usefull only for t close to τ and allows to control the behaviour of the solution in a neighbourhood of τ.

When a solution to (4.1) is continuous at $t=\tau$, we say that u is a *strong solution*. More precisely, we set the following definition.

Definition 4.2. A function u in $C([\tau,\infty),X)$ is called a *strong solution* to (4.1) on $[\tau,\infty)$ if u is a solution to (4.1) in the sense of Definition 4.1 and u belongs to $C([\tau,\infty),D(A))$.

Remark 4.1. In the case $s=\tau$, the notion of strong solution is well adapted for Problem (4.1) and is commonly used in the literature (see for instance [Prü93, Definition I-1.1] or [Baz98]). However, if $s>\tau$ then (4.1) may have no strong solution but we may obtain solution in the sense of Definition 4.1: see Remark 6.2.

Proposition 4.1. *Let $v\in D(A)$ and s, $\tau\in\mathbb{R}$ with $\tau\le s$. If u is a strong solution to* (4.1) *then u has a derivative of order α in $C([\tau,\infty),X)$ and*

$$\mathbf{D}^{\alpha}_{\tau,t}u=Au,\qquad \textit{in}\ \ C([\tau,\infty),X).$$

Proof. Let u be a solution to (4.1). Let us first show that the function G defined by

$$G=g_{1-\alpha}*_{\tau}\big(u-u(\tau)\big)$$

belongs to $C^1([\tau,\infty),X)$. Since u is a solution in the sense of Definition 4.1, the function G belongs to $C^1((\tau,\infty),X)$. So it is enough to consider regularity near τ.

We have, on one hand, for each $t\ge\tau$,

$$\|G(t)\|_{D(A)}\le\frac{(t-\tau)^{1-\alpha}}{\Gamma(2-\alpha)}\sup_{y\in[\tau,t]}\|u(y)-u(\tau)\|_{D(A)}.$$

Since, by hypothesis, u lies in $C([\tau,\infty),D(A))$, we obtain that

$$G\in C([\tau,\infty),D(A)). \tag{4.2}$$

On the other hand, since u is solution to (4.1),

$$\mathbf{D}^{\alpha}_{\tau,t}u=Au,\qquad \text{in}\ \ C((\tau,\infty),X). \tag{4.3}$$

Since u lies in $C([\tau,\infty),D(A))$, we deduce that $\mathbf{D}^{\alpha}_{\tau,t}u$ has a limit in X as $t\to\tau$. Thus $\frac{\mathrm{d}}{\mathrm{d}t}G$ has a limit in X as $t\to\tau$. Combining this with (4.2), we deduce that $G\in C^1([\tau,\infty),X)$, that is to say, u has a derivative of order α in $C([\tau,\infty),X)$.

Then there results that (4.3) holds in $C([\tau,\infty),X)$, this completes the proof of the proposition. $\square$

It turns out that a time shift simplifies the situation by reducing the study to the case $\tau=0$. More precisely, for $\tau\le s$, let us consider the following problem.

$$\begin{cases}\mathbf{D}^{\alpha}_{0,t}w=Aw,\\ w(s-\tau)=v\in D(A).\end{cases} \tag{4.4}$$

According to Definition 4.1, w is a solution to (4.4) on $[0,\infty)$ if and only if

$$\begin{cases} w \in C([0,\infty),X) \cap C((0,\infty),D(A)) \\ g_{1-\alpha} * (w - w(0)) \in C^1((0,\infty),X) \\ \mathbf{D}^\alpha_{0,t} w \in L^1_{\mathrm{loc}}([0,\infty),X) \\ \mathbf{D}^\alpha_{0,t} w = Aw \quad \text{in } C((0,\infty),X) \\ w(s-\tau) = v. \end{cases} \tag{4.5}$$

Then we have

Proposition 4.2. *Let $\tau \le s$, $v \in D(A)$, $u \in C([\tau,\infty),X)$ and $w \in C([0,\infty),X)$. We suppose that*

$$u(t+\tau) = w(t), \qquad \forall t \ge 0.$$

Then u is a solution to (4.1) if and only if w is a solution to (4.4). Moreover, u is a strong solution to (4.1) if and only if w is a strong solution to (4.4).

Proof. The assertion regarding strong solutions is a straightforward consequence of the assertion for solutions. Let us prove the assertion for solutions. For every $t \ge 0$, we have, by a change of variable in the first integral below

$$\begin{aligned} g_{1-\alpha} *_0 w(t) &= \int_0^t g_{1-\alpha}(t-y)u(y+\tau)\mathrm{d}y \\ &= \int_\tau^{t+\tau} g_{1-\alpha}(t+\tau-y)u(y)\mathrm{d}y \\ &= (g_{1-\alpha} *_\tau u)(t+\tau). \end{aligned}$$

Thus

$$g_{1-\alpha} *_0 \big(w - w(0)\big) = \Big(g_{1-\alpha} *_\tau \big(u - u(\tau)\big)\Big)(\cdot + \tau), \qquad \text{on } [0,\infty)$$

and

$$\mathbf{D}^\alpha_{0,t} w = \big(\mathbf{D}^\alpha_{\tau,t} u\big)(\cdot + \tau), \qquad \text{in } C((0,\infty),X).$$

Then the assertion follows. □

Let us explain the consequences of Proposition 4.2 for solution operators. For, let us assume that for each $s > 0$ and $v \in D(A)$, the problem

$$\mathbf{D}^\alpha_{0,t} u = Au, \quad u(s) = v, \tag{4.6}$$

has a unique solution on $[0,\infty)$ and that the problem

$$\mathbf{D}^\alpha_{0,t} u = Au, \quad u(0) = v, \tag{4.7}$$

has a unique strong solution on $[0,\infty)$.

Due to Proposition 4.2, we infer that for each s, τ in $\mathbb{R}$ with $s > \tau$, the problem (4.1) has a unique solution on $[\tau,\infty)$ and that, for each s in $\mathbb{R}$, the problem

$$\mathbf{D}^\alpha_{s,t} u = Au, \quad u(s) = v, \tag{4.8}$$

has a unique strong solution on $[s,\infty)$.

As a consequence, for $s > \tau$, we put

$$\boldsymbol{\mathcal{T}}(t,s,\tau)v := u(t), \quad \forall t > \tau,$$

where u is the solution to (4.1). Let us notice that, in view of Definition 4.1,

$$\boldsymbol{\mathcal{T}}(t,s,\tau) \in \mathcal{F}\big(D(A)\big), \quad \forall t > \tau.$$

On the other hand, for $s = \tau$, we put

$$\boldsymbol{\mathcal{T}}(t,s,s)v := u(t), \quad \forall t \geq s,$$

where u is the strong solution to (4.8) on $[s,\infty)$.

Setting

$$\mathcal{D}_0 := \big\{(t,s,\tau) \in \mathbb{R}^3 \mid t > \tau,\ s > \tau\big\} \cup \big\{(t,s,s) \in \mathbb{R}^3 \mid t \geq s\big\}, \tag{4.9}$$

the above map $\boldsymbol{\mathcal{T}}$ is defined on $\mathcal{D}_0$ with values in $\mathcal{F}(D(A))$. Next, for $(t,s,\tau) \in \mathcal{D}_0$ and $x \in \mathbb{R}$, it is clear that

$$(t+x, s+x, \tau+x) \in \mathcal{D}_0,$$

thus, by Proposition 4.2,

$$\boldsymbol{\mathcal{T}}(t+x, s+x, \tau+x) = \boldsymbol{\mathcal{T}}(t,s,\tau). \tag{4.10}$$

Hence $\boldsymbol{\mathcal{T}}$ is *translation invariant.*

Let us highlight another property of $\mathcal{T}$. For each $\tau_0 \in \mathbb{R}$ fixed, let us denote with a slight abuse of notation, by τ_0 the function

$$\mathbb{R} \to \mathbb{R}, \quad t \mapsto \tau_0.$$

Then, in view of (2.1) and (2.3),

$$p_{\tau_0}(\mathcal{D}_0) = \big\{(t,s) \in \mathbb{R}^2 \mid t > \tau_0,\ s > \tau_0\big\} \cup \big\{(t,\tau_0) \in \mathbb{R}^2 \mid t \geq \tau_0\big\} \tag{4.11}$$

$$T_{\boldsymbol{\mathcal{T}},\tau_0}(t,s) = \boldsymbol{\mathcal{T}}(t,s,\tau_0), \quad \forall (t,s) \in p_{\tau_0}(\mathcal{D}_0). \tag{4.12}$$

We will show that $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is a propagator. Indeed, let (t,s) and (s,s') be in $p_{\tau_0}(\mathcal{D}_0)$. Then (t,s') belongs to $p_{\tau_0}(\mathcal{D}_0)$ by (4.11). Moreover, since $s' \geq \tau_0$, the problem

$$\mathbf{D}^\alpha_{\tau_0,t} w = Aw, \quad w(s') = v \in D(A) \tag{4.13}$$

has a unique solution w on $[\tau_0,\infty)$ and

$$w(t) = \boldsymbol{\mathcal{T}}(t,s',\tau_0)v.$$

Since $\boldsymbol{\mathcal{T}}$ maps $\mathcal{D}_0$ into $\mathcal{F}(D(A))$,

$$w(s) = \boldsymbol{\mathcal{T}}(s,s',\tau_0)v \in D(A).$$

Hence the solution u to

$$\mathbf{D}^\alpha_{\tau_0,t} u = Au, \quad u(s) = w(s). \tag{4.14}$$

satisfies

$$u(t) = \boldsymbol{\mathcal{T}}(t,s,\tau_0)\boldsymbol{\mathcal{T}}(s,s',\tau_0)v.$$

Moreover, w is obviously a solution to (4.14). Thus by our uniqueness assumption,

$$\boldsymbol{\mathcal{T}}(t,s',\tau_0) = \boldsymbol{\mathcal{T}}(t,s,\tau_0)\boldsymbol{\mathcal{T}}(s,s',\tau_0).$$

That is to say

$$T_{\boldsymbol{\mathcal{T}},\tau_0}(t,s') = T_{\boldsymbol{\mathcal{T}},\tau_0}(t,s)T_{\boldsymbol{\mathcal{T}},\tau_0}(s,s').$$

There results that $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is a propagator.

Let us introduce the third property of $\boldsymbol{\mathcal{T}}$. Usually the following fractional Cauchy Problem is considered.

$$\mathbf{D}^{\alpha}_{0,t}u = Au, \quad u(0) = v \in D(A). \tag{4.15}$$

Under the above assumptions and notation, the unique solution u to (4.15) reads

$$u(t) = \boldsymbol{\mathcal{T}}(t, 0, 0), \quad \forall t \geq 0.$$

Since

$$p_{id}(\mathcal{D}_0) = \{(t, s) \in \mathbb{R}^2 \mid t \geq s\}$$

and $\boldsymbol{\mathcal{T}}$ is translation invariant according to (4.10), we deduce thanks to Corollary 2.3 that the only one positively translation invariant map T defined on $p_{id}(\mathcal{D}_0)$ and satisfying

$$T(t, 0) = \boldsymbol{\mathcal{T}}(t, 0, 0), \quad \forall t \geq 0$$

is the map $T_{\boldsymbol{\mathcal{T}},id}$. Roughly speaking,

$$\mathbf{D}^{\alpha}_{s,t}u = Au, \quad u(s) = v$$

is the unique non autonomous translation invariant problem extending (4.15).

The following theorem gathers the main points of the latter analysis.

Theorem 4.3. *Let $A : D(A) \subseteq X \to X$ be a closed densely defined linear operator on a Banach space X. Let us assume that for each $v \in D(A)$,*

- (i) *(4.7) has a unique strong solution on $[0, \infty)$;*
- (ii) *for each s in $(0, \infty)$, (4.6) has a unique solution on $[0, \infty)$ in the sense of Definition 4.1.*

Then the following assertions hold true.

- (a) *For $s > \tau$, (4.1) has a unique solution $u \in C((\tau, \infty), D(A))$ and for each s in $\mathbb{R}$, (4.8) has a unique strong solution $u_s \in C([s, \infty), D(A))$.*
- (b) *The solution operator $\boldsymbol{\mathcal{T}} : \mathcal{D}_0 \to \mathcal{F}(D(A))$ defined for each $(t, s, \tau) \in \mathcal{D}_0$ and $v \in D(A)$, by*

$$\boldsymbol{\mathcal{T}}(t, s, \tau)v = \begin{cases} u(t) & \text{if } s > \tau \\ u_s(t) & \text{if } s = \tau \end{cases} \tag{4.16}$$

  is translation invariant. (See (4.9) for the definition of $\mathcal{D}_0$.)
- (c) *For each τ_0 in $\mathbb{R}$, the two variables solution operator $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is a propagator.*
- (d) *The solution operator $T_{\boldsymbol{\mathcal{T}},id}$ is the only positively translation invariant map defined on $p_{\tau_0}(\mathcal{D}_0)$ satisfying*

$$T_{\boldsymbol{\mathcal{T}},id}(t, 0) = \boldsymbol{\mathcal{T}}(t, 0, 0), \quad \forall t \geq 0.$$

Proposition 4.4 below is a technical result that will be usefull to prove uniqueness for solution in the sense of Definition 4.1. This proposition is well know and easily proved for strong solutions (see for instance [Baz98]).

Proposition 4.4. *Let $\alpha \in (0, 1)$, $s \geq 0$ and u be a solution to (4.6) in the sense of Definition 4.1. Then*

$$u = u(0) + g_\alpha * Au, \qquad \text{in } C\big([0, \infty), X\big). \tag{4.17}$$

The proof of that proposition uses the following lemma.

Lemma 4.5. *If G belongs to $C^1([0,\infty),X)$ then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\{g_\alpha * G\big\} = g_\alpha(\cdot)G(0) + g_\alpha * \frac{\mathrm{d}}{\mathrm{d}t}G, \qquad \textit{in} \ \ L^1_{\mathrm{loc}}\big([0,\infty),X\big).$$

The proof of this lemma is obvious hence we will skip it. Let us just notice that g_α is not continuous at $t=0$ but g_α belongs to $L^1_{\mathrm{loc}}([0,\infty))$.

Proof of Proposition 4.4. It is enough to prove (4.17) in $C([0,T],X)$ for any positive T. By Definition 4.1, the function

$$G := g_{1-\alpha} * \big(u - u(0)\big)$$

lies in $W^{1,1}([0,T],X)$. Thus we may consider a regularised sequence $(G_n)_{n\geq 0}$ in $C^1([0,\infty),X)$ such that

$$G_n \xrightarrow[n\to\infty]{} G, \qquad \text{in} \ \ W^{1,1}([0,T],X).$$

According to Lemma 4.5, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\{g_\alpha * G_n\big\} = g_\alpha(\cdot)G_n(0) + g_\alpha * \frac{\mathrm{d}}{\mathrm{d}t}G_n, \qquad \text{in} \ \ L^1\big([0,T]),X\big). \tag{4.18}$$

Moreover, since

$$\|g_\alpha * G\|_{L^1([0,T]),X)} \leq \|g_\alpha\|_{L^1([0,T]))}\|G\|_{L^1([0,T]),X)}$$

and $W^{1,1}([0,T],X)$ is continuously embedded in $C([0,T],X)$, we infer

$$g_\alpha(\cdot)G_n(0) + g_\alpha * \frac{\mathrm{d}}{\mathrm{d}t}G_n \to g_\alpha(\cdot)G(0) + g_\alpha * \frac{\mathrm{d}}{\mathrm{d}t}G, \qquad \text{in} \ \ L^1\big([0,T]),X\big). \tag{4.19}$$

Since $G(0)=0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}G = \mathbf{D}^\alpha_{0,t}u = Au,$$

we get with (4.18) and (4.19)

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\{g_\alpha * G_n\big\} \to g_\alpha * Au, \qquad \text{in} \ \ L^1\big([0,T]),X\big). \tag{4.20}$$

Let us show, by a distribution-type argument, that the above limit is also equal to $u-u(0)$. Indeed, for each φ in $C^1([0,T],X)$ with compact support in $(0,T)$, one has

$$\begin{aligned}\int_0^T \varphi(t)\frac{\mathrm{d}}{\mathrm{d}t}\big\{g_\alpha * G_n\big\}\,\mathrm{d}t &= -\int_0^T \varphi'(t) g_\alpha * G_n\,\mathrm{d}t \\ &\to -\int_0^T \varphi'(t) g_\alpha * G\,\mathrm{d}t \qquad \text{in} \ \ X.\end{aligned}$$

Besides,

$$\begin{aligned} g_\alpha * G &= g_\alpha * g_{1-\alpha} * \big(u-u(0)\big) \\ &= g_1 * \big(u - u(0)\big) && \text{(by (3.1))} \\ &= \int_0^{\cdot} u(y) - u(0)\mathrm{d}y. \end{aligned}$$

Thus
$$\int_0^T \varphi(t)\frac{\mathrm{d}}{\mathrm{d}t}\big\{g_\alpha * G_n\big\}\,\mathrm{d}t \to \int_0^T \varphi(t)\big(u-u(0)\big)\,\mathrm{d}t.$$
Combining this limit with (4.20), we deduce
$$u-u(0)=g_\alpha * Au, \qquad \text{in } \ L^1\big([0,T]),X\big).$$
Then (4.17) follows by regularity of u. □

5. One dimensional fractional problems

For λ, $v\in\mathbb{C}$ and $\alpha\in(0,1)$, let us consider the problem
$$\mathbf{D}^\alpha_{0,t}u=\lambda u, \quad u(0)=v. \tag{5.1}$$
In this setting, Definitions 4.1 and 4.2 are equivalent. So we will state our results for strong solutions.

Proposition 5.1. *For λ, v and α as above,* (5.1) *admits an unique strong solution $u(\cdot)$ on* $[0,\infty)$*. Moreover,*
$$u(t)=E_\alpha(t^\alpha\lambda)v, \quad \forall t\geq 0. \tag{5.2}$$

Proof. By (3.2), (5.2) gives rise to a solution to (5.1). The uniqueness may be obtain as in [SKM93, Proof of Theorems 42.1-42.6]; see also the uniqueness part of the proof of Theorem 6.1. □

Let us set
$$S(t)=E_\alpha(t^\alpha\lambda), \quad \forall t\geq 0, \tag{5.3}$$
so that, thanks to Proposition 5.1, S is the solution operator corresponding to (5.1).

In view of (4.1), we introduce the following non autonomous problem.
$$\mathbf{D}^\alpha_{\tau,t}u=\lambda u, \quad u(s)=v. \tag{5.4}$$

Proposition 5.2. *Let λ, $v\in\mathbb{C}$, $\alpha\in(0,1)$ and $\tau\leq s$.*

(i) *If $E_\alpha((s-\tau)^\alpha\lambda)\neq 0$ then* (5.4) *admits an unique strong solution $u(\cdot)$ on* $[\tau,\infty)$*. Moreover,*
$$u(t)=\frac{E_\alpha\big((t-\tau)^\alpha\lambda\big)}{E_\alpha\big((s-\tau)^\alpha\lambda\big)}v, \quad \forall t\geq\tau. \tag{5.5}$$

(ii) *If $E_\alpha((s-\tau)^\alpha\lambda)=0$ and $v\neq 0$ then* (5.4) *has no strong solution.*

Proof. In view of Proposition 4.2, it is enough to consider
$$\mathbf{D}^\alpha_{0,t}u=\lambda u, \quad u(s)=v, \tag{5.6}$$
where $s\geq 0$.

(i) If $E_\alpha(s^\alpha\lambda)\neq 0$ then by Proposition 5.1,
$$t\mapsto\frac{E_\alpha(t^\alpha\lambda)}{E_\alpha(s^\alpha\lambda)}v$$
is a solution to (5.6). Regarding uniqueness, let us consider two solutions u and U of (5.6). According to Proposition 5.1,
$$u(t)=E_\alpha(t^\alpha\lambda)u(0)$$

and

$$U(t) = E_\alpha(t^\alpha \lambda)U(0), \quad \forall t \geq 0.$$

Since $E_\alpha(s^\alpha \lambda) \neq 0$ and $u(s) = U(s)$, we infer that $u(0) = U(0)$ and, finally, $u = U$.

(ii) We assume that $E_\alpha(s^\alpha \lambda) = 0$. Let $u(\cdot)$ be a solution to (5.6). Using Proposition 5.1 once again, we get

$$u(t) = E_\alpha(t^\alpha \lambda)u(0).$$

For $t = s$,

$$v = E_\alpha(s^\alpha \lambda)u(0) = 0.$$

This complete the proof of the proposition. □

Let us state a consequence of the above result for solution operators corresponding to (5.4). For this, we set

$$\mathcal{D}_1 := \big\{(t, s, \tau) \in \mathbb{R}^3 \mid t \geq \tau, \ s \geq \tau\big\}. \tag{5.7}$$

Corollary 5.3. *Let us assume that for some fixed $\lambda \in \mathbb{C}$, we have*

$$E_\alpha(s\lambda) \neq 0, \quad \forall s > 0. \tag{5.8}$$

Then the following assertions hold true.

(i) *For $s \geq \tau$, Problem (5.4) has a unique strong solution u_{s} on $[\tau, \infty)$.*

(ii) *The solution operator $\boldsymbol{\mathcal{T}} : \mathcal{D}_1 \to \mathcal{F}(\mathbb{C})$ defined for each $(t, s, \tau) \in \mathcal{D}_1$ and $v \in \mathbb{C}$ by*

$$\boldsymbol{\mathcal{T}}(t, s, \tau)v = u_{\mathrm{s}}(t) = \frac{E_\alpha\big((t-\tau)^\alpha \lambda\big)}{E_\alpha\big((s-\tau)^\alpha \lambda\big)}v$$

is translation invariant.

(iii) *For each τ_0 in $\mathbb{R}$, $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is a propagator. For $\lambda \neq 0$, it is not positively translation invariant.*

(iv) *The solution operator $T_{\boldsymbol{\mathcal{T}},id}$ is positively translation invariant. For $\lambda \neq 0$, it is not a propagator.*

Remark 5.1. (5.8) holds for instance if $\lambda \in \mathbb{R}$. That is a consequence of Proposition 3.1. Also by (3.4), the set

$$\big\{z \in \mathbb{C} \mid E_\alpha(z) = 0, \ \mathrm{Re}(z) \leq 0\big\}$$

is bounded. Besides, since E_α is analytic the above set is finite. Thus (5.8) holds for almost every λ in

$$\big\{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\big\}.$$

Proof of Corollary 5.3. By Theorem 4.3, we obtain assertions (i), (ii) and that $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is a propagator and, also, that $T_{\boldsymbol{\mathcal{T}},id}$ is positively translation invariant.

If $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is positively translation invariant then by using Theorem 2.1 with

$$f(t) = \tau_0, \quad \forall t \in \mathbb{R},$$

we deduce that $S_{\boldsymbol{\mathcal{T}}}$ is additive. However,

$$S_{\boldsymbol{\mathcal{T}}}(t) = \boldsymbol{\mathcal{T}}(t, 0, 0) = E_\alpha\big(t^\alpha \lambda\big), \quad \forall t \geq 0.$$

Thus $\lambda = 0$ by Proposition 3.2.

If $T_{\boldsymbol{\mathcal{T}},id}$ is a propagator then by using Theorem 2.1 with

$$f(t) = t, \quad \forall t \in \mathbb{R}, \quad \tau_0 = 0,$$

we deduce that $S_{\mathcal{T}}$ is additive. Hence $\lambda = 0$ as above. □

6. Time fractional Equations with constant coefficients differential operators

Let $d \geq 1$, $M \geq 0$ be integers and

$$P : \mathbb{R}^d \to \mathbb{C}, \quad \xi \mapsto \sum_{|\beta| \leq M} p_\beta \xi^\beta$$

be a complex polynomial.

We will use Hormander's notation for partial derivatives. More precisely, for $k = 1, \ldots, d$, we put

$$D_k := \frac{1}{\mathrm{i}} \partial_{x_k}, \quad D := (D_1, , \ldots, D_d),$$

where $\mathrm{i}^2 = -1$ and ∂_{x_k} denotes the usual partial derivative in the direction x_k.

Then

$$P(D) = \sum_{|\beta| \leq M} p_\beta D^\beta$$

is the linear partial differential operator whose symbol is P. The advantage of this notation is that the Fourier transform $\mathcal{F}$ of $P(D)$ is P provided $\mathcal{F}$ is properly normalise, namely that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{-\mathrm{i}\xi \cdot x} dx.$$

With this normalisation, we have

$$\|\mathcal{F}^{-1}(f)\|_2 = c_{\mathcal{F},d} \|f\|_2, \quad \forall f \in L^2(\mathbb{R}^d)$$

where $\| \cdot \|_2$ is the standard L^2-norm on $\mathbb{R}^d$ and $c_{\mathcal{F},d}$ is a constant depending on d and on the above normalisation.

It is well known that $P(D)$ is a closed densely defined linear operator on $L^2(\mathbb{R}^d)$ with domain

$$D(P) = \{f \in L^2(\mathbb{R}^d) \mid P(D)f \in L^2(\mathbb{R}^d)\} \tag{6.1}$$

$$= \{f \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |P(\xi)\hat{f}(\xi)|^2 d\xi < \infty\}. \tag{6.2}$$

For $0 < \alpha < 1$, $\tau \leq s$ and $v \in L^2(\mathbb{R}^d)$, let us consider the following fractional problem

$$\mathbf{D}^\alpha_{\tau,t} u = P(D)u, \quad u(s) = v. \tag{6.3}$$

As in the one dimensional case (see Section 5), the study of (6.3) differs whether s is equal to τ or not. We will start with the case $\tau < s$. In this setting, we must be sure that certain denominators do not vanish. This is why we make some additional assumptions on the polynomial P.

Theorem 6.1. *Let $\alpha \in (0, 1)$, $\tau < s$ and $v \in D(P)$. Assume that*

$$P(\xi) \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^d; \tag{6.4}$$

$$M_P := \sup_{\xi \in \mathbb{R}^d} P(\xi) < \infty. \tag{6.5}$$

Then (6.3) *has a unique solution (in the sense of Definition 4.1). Moreover, this solution denoted by* u *has the following representation.*

$$u(t) = \mathscr{F}^{-1}\Big(\frac{E_\alpha\big((t-\tau)^\alpha P(\xi)\big)}{E_\alpha\big((s-\tau)^\alpha P(\xi)\big)}\hat{v}(\xi)\Big), \quad \forall t \geq \tau. \tag{6.6}$$

Remark 6.1. Since E_α has no real zero,

$$E_\alpha\big((s-\tau)^\alpha P(\xi)\big) \neq 0.$$

(6.6) means that, for all $t \geq \tau$, $u(t)$ is the inverse Fourier transform of the function

$$\mathbb{R}^d \to \mathbb{C}, \quad \xi \mapsto \frac{E_\alpha\big((t-\tau)^\alpha P(\xi)\big)}{E_\alpha\big((s-\tau)^\alpha P(\xi)\big)}\hat{v}(\xi).$$

Proof of Theorem 6.1. In view of Theorem 4.3, we may assume that $\tau = 0$. Thus it is enough to consider for each positive s, the problem

$$\mathbf{D}^\alpha_{0,t} u = P(D)u, \quad u(s) = v. \tag{6.7}$$

Let us show that (6.6) provides a solution. We will first prove that for $t \geq 0$, $u(t)$ belongs to $L^2(\mathbb{R}^d)$. Indeed, from (6.4), (6.5) and (3.4), we infer

$$\frac{1}{E_\alpha\big(s^\alpha P(\xi)\big)} \leq C\big(1 + s^\alpha |P(\xi)|\big), \quad \forall \xi \in \mathbb{R}^d, \tag{6.8}$$

where $C \geq 0$ is independent of ξ. Thus (see (6.5))

$$\frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}|\hat{v}(\xi)| \leq C E_\alpha(t^\alpha M_P)\big(1+s^\alpha|P(\xi)|\big)|\hat{v}(\xi)|, \quad \forall t \geq 0, \ \text{ a.e. } \ \xi \in \mathbb{R}^d. \tag{6.9}$$

Since $v \in D(P)$, we deduce that

$$\xi \mapsto \frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{v}(\xi)$$

belongs to $L^2(\mathbb{R}^d)$. Hence $u(t)$ given by (6.6) is well defined and lies in $L^2(\mathbb{R}^d)$.

For τ_0, T satisfying $0 < \tau_0 < s < T$, let us show that

$$g_{1-\alpha} *_0 \big(u - u(0)\big) \in C^1([\tau_0, T], L^2(\mathbb{R}^d)).$$

By Proposition 5.2, for a.e. $\xi \in \mathbb{R}^d$, the function

$$t \mapsto \hat{u}(t,\xi) = \frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{v}(\xi)$$

is the solution to

$$\mathbf{D}^\alpha_{0,t}\hat{u} = P(\xi)\hat{u}, \quad \hat{u}(s) = \hat{v}(\xi).$$

Thus for a.e. $\xi \in \mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{g_{1-\alpha} *_0 \frac{E_\alpha\big(t^\alpha P(\xi)\big) - 1}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{v}(\xi)\Big\} = \frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}P(\xi)\hat{v}(\xi). \tag{6.10}$$

Next by (3.4), for $x < 0$, one has

$$\frac{E_\alpha(t^\alpha x)}{E_\alpha(s^\alpha x)} \sim \Big(\frac{s}{t}\Big)^\alpha \qquad \text{as } \ x \to -\infty.$$

Hence, with (6.4), there exists $R = R(\tau_0) > 0$ such that for each $t \geq \tau_0$,

$$\frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)} \leq 2\big(\frac{s}{t}\big)^\alpha \leq 2\big(\frac{s}{\tau_0}\big)^\alpha, \quad \forall P(\xi) < -R(\tau_0). \tag{6.11}$$

If $P(\xi) \geq -R(\tau_0)$ then by (6.5) and the continuity of the Mittag-Leffler function,

$$\sup_{t\in[\tau_0,T]} \frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)} \leq C(T, s).$$

Thus there exists a constant C such that for all $t \in [\tau_0, T]$,

$$\frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}|P(\xi)\hat{v}(\xi)| \leq C|P(\xi)\hat{v}(\xi)|.$$

Recalling that v belongs to $D(P)$, we obtain by Lebesgue's differentiation Theorem that

$$t \mapsto \mathcal{F}^{-1}\Big(g_{1-\alpha} *_0 \frac{E_\alpha\big(t^\alpha P(\xi)\big) - 1}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{v}(\xi)\Big) = g_{1-\alpha} *_0 \big(u(t) - u(0)\big)$$

is in $C^1([\tau_0, T], L^2(\mathbb{R}^d))$ and (see (6.10)),

$$\begin{aligned} \mathbf{D}^\alpha_{0,t} u &= \mathcal{F}^{-1}\Big(\frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}P(\xi)\hat{v}(\xi)\Big) \\ &= P(D)u \qquad \text{in} \;\; C([\tau_0, T], L^2(\mathbb{R}^d)), \end{aligned} \tag{6.12}$$

since $\mathcal{F}(P(D)w)(\xi) = P(\xi)\hat{w}(\xi)$ for w in $D(P)$. As a by-product,

$$u \in C((0, \infty), D(P)).$$

Let us show that u is continuous at $t = 0$ in $L^2(\mathbb{R}^d)$. For each $t \in (0, 1]$, we have

$$\|u(t) - u(0)\|_2 = c_{\mathcal{F},d}\left\|\frac{E_\alpha\big(t^\alpha P(\xi)\big) - 1}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{v}\right\|_2.$$

Moreover, for each $\xi \in \mathbb{R}^d$,

$$\frac{E_\alpha\big(t^\alpha P(\xi)\big) - 1}{E_\alpha\big(s^\alpha P(\xi)\big)} \to 0 \qquad \text{as} \;\; t \to 0$$

and (see (6.9)),

$$\Big|\frac{E_\alpha\big(t^\alpha P(\xi)\big) - 1}{E_\alpha\big(s^\alpha P(\xi)\big)}\Big| \leq CE_\alpha(M_P)(1 + s^\alpha|P(\xi)|).$$

Thus Lebesgue's dominated convergence Theorem implies the desirated continuity result.

There remains to prove that

$$\mathbf{D}^\alpha_{0,t} u \in L^1_{\text{loc}}([0, 1], L^2(\mathbb{R}^d)). \tag{6.13}$$

For, let

$$w : [0, 1] \to L^2(\mathbb{R}^d), \; t \mapsto \frac{E_\alpha\big(t^\alpha P(\cdot)\big)}{E_\alpha\big(s^\alpha P(\cdot)\big)}P(\cdot)\hat{v}.$$

By (6.12), (6.13) is equivalent to

$$w \in L^1_{\mathrm{loc}}([0,1], L^2(\mathbb{R}^d)). \tag{6.14}$$

Then, due to (6.11),

$$\|w(t)\|_2 \leq \big(2\frac{s^\alpha}{t^\alpha} + C(s)\big)\|P(\cdot)\hat{v}\|_2.$$

Thus

$$\int_0^1 \|w(t)\|_2 \,\mathrm{d}t \leq \big(2\frac{s^\alpha}{1-\alpha} + C(s)\big)\|v\|_{D(P)}.$$

That proves (6.14) and then (6.13).

Let us consider the uniqueness for (6.7). By linearity, we may assume that $v = 0$. Since $s > 0$ and P satisfies (6.4) and (6.5) , we have by Proposition 3.1,

$$\sup_{\xi\in\mathbb{R}^d} \big|E_\alpha\big(s^\alpha P(\xi)\big)P(\xi)\big| < \infty.$$

Thus, since, by Definition 4.1, $u(0) \in L^2(\mathbb{R}^d)$, we deduce that the function

$$f := \mathscr{F}^{-1}\Big(E_\alpha\big(s^\alpha P(\xi)\big)\hat{u}(0)\Big) \tag{6.15}$$

belongs to $D(P)$. For each $t \geq 0$, we put

$$u_1(t) := \mathscr{F}^{-1}\Big(\frac{E_\alpha\big(t^\alpha P(\xi)\big)}{E_\alpha\big(s^\alpha P(\xi)\big)}\hat{f}\Big). \tag{6.16}$$

According to the existence part of the proof, u_1 is solution to

$$\mathbf{D}^\alpha_{0,t}u_1 = P(D)u_1, \quad u_1(s) = f.$$

We will show that $u = u_1$. That equality leads to the uniqueness for (6.7). Indeed, one has

$$\hat{u}(t) = \hat{u}_1(t) = E_\alpha\big(t^\alpha P(\cdot)\big)\hat{u}(0). \tag{6.17}$$

Evaluating at $t = s$, we get

$$0 = E_\alpha\big(s^\alpha P(\cdot)\big)\hat{u}(0).$$

Since our hypothesis imply $E_\alpha\big(s^\alpha P(\xi)\big) \neq 0$, we get $\hat{u}(0) = 0$ and then, going back to (6.17), $u = 0$. That is our desired uniqueness result.

So, let us show that $u = u_1$. For, the function

$$w = u - u_1$$

solves

$$\mathbf{D}^\alpha_{0,t}w = P(D)w, \quad w(0) = 0$$

in the sense of Definition 4.1 (see (6.15), (6.16)). Thus, by Proposition 4.4,

$$w = g_\alpha * P(D)w \quad \text{ in } C\big([0,\infty), L^2(\mathbb{R}^d)\big)$$

and

$$\hat{w} = g_\alpha * P(\cdot)\hat{w} \quad \text{ in } C\big([0,\infty), L^2(\mathbb{R}^d)\big).$$

Moreover, since, for each $T > 0$, $w \in L^1((0,T), L^2(\mathbb{R}^d))$, we deduce that for a.e. $\xi \in \mathbb{R}^d$, $w(\cdot,\xi)$ belongs to $L^1(0,T)$. Thus, for a.e. $\xi \in \mathbb{R}^d$ and a.e. t in $[0,T]$,

$$\hat{w}(t,\xi) = P(\xi)\int_0^t g_\alpha(t-y)\hat{w}(y,\xi)\mathrm{d}y. \tag{6.18}$$

Without los of generality, we may assume $P(\xi) \neq 0$. Let

$$t_0 := \sup\big\{t \geq 0 \mid \hat{w}(\cdot,\xi) = 0 \ \text{ a.e. on } \ [0,t]\big\}.$$

If t_0 is finite then there exists $T > t_0$ such that

$$|P(\xi)| \int_0^{T-t_0} g_\alpha(t)\,\mathrm{d}t = \frac{1}{2} \tag{6.19}$$

and

$$\int_{t_0}^{T} |\hat{w}(t,\xi)|\,\mathrm{d}t \neq 0. \tag{6.20}$$

Then by integrating (6.18) over $[t_0, T]$, we get

$$\begin{aligned}
\int_{t_0}^{T} |\hat{w}(t,\xi)|\,\mathrm{d}t &\leq |P(\xi)| \int_{t_0}^{T} \mathrm{d}t \int_{t_0}^{t} g_\alpha(t-y)|\hat{w}(y,\xi)|\mathrm{d}y \\
&\leq |P(\xi)| \int_{t_0}^{T} |\hat{w}(y,\xi)|\mathrm{d}y \int_{y}^{T} g_\alpha(t-y)\,\mathrm{d}t \quad \text{(by Fubbini's Theorem)} \\
&\leq |P(\xi)| \int_{t_0}^{T} |\hat{w}(y,\xi)|\mathrm{d}y \int_{0}^{T-y} g_\alpha(t)\,\mathrm{d}t \\
&\leq |P(\xi)| \int_{t_0}^{T} |\hat{w}(y,\xi)|\mathrm{d}y \int_{0}^{T-t_0} g_\alpha(t)\,\mathrm{d}t \quad \text{(since } t_0 \leq y) \\
&\leq \frac{1}{2} \int_{t_0}^{T} |\hat{w}(y,\xi)|\mathrm{d}y \qquad \text{(by (6.19))}.
\end{aligned}$$

That contradicts (6.20). Thus $t_0 = \infty$ and uniqueness holds. □

The idea for proving $\hat{w}(\cdot,\xi) = 0$ relies on [SKM93, Proof of Theorems 42.1-42.6].

Remark 6.2. In view of (6.6), we claim that

$$u(\tau) = \mathscr{F}^{-1}\Big(\frac{1}{E_\alpha\big((s-\tau)^\alpha P(\xi)\big)}\hat{v}(\xi)\Big)$$

does not belong to $D(P)$ in general. Indeed, assuming that $D(P^2) \neq D(P)$, let

$$v \in D(P) \setminus D(P^2).$$

We may assume $\tau = 0$ and $s > 0$. Then the solution to

$$\mathbf{D}_{0,t}^{\alpha} u = P(D)u, \quad u(s) = v$$

belongs to $C([0,\infty), L^2(\mathbb{R}^d))$ and

$$\hat{u}(0,\xi) = \frac{\hat{v}(\xi)}{E_\alpha\big(s^\alpha P(\xi)\big)}, \qquad \text{a.e. } \ \xi \in \mathbb{R}^d.$$

We want to prove that

$$u(0,\cdot) \notin D(P). \tag{6.21}$$

Indeed,

$$\big|P(\xi)\hat{u}(0,\xi)\big| = \Big|\frac{P(\xi)\hat{v}(\xi)}{E_\alpha\big(s^\alpha P(\xi)\big)}\Big| \sim C s^\alpha |P(\xi)|^2 |v(\xi)| \qquad \text{for } \ |P(\xi)| \to \infty.$$

Thus $P(\cdot)u(0,\cdot)$ does not belong to $L^2(\mathbb{R}^d)$, which proves (6.21).

Going back to the original problem (6.3), we conclude that its solution u given by (6.6) has a singularity in $D(P)$ when $t \to \tau^+$.

Let us consider the case $s = \tau$; that is to say, we consider the problem

$$\mathbf{D}^{\alpha}_{s,t} u = P(D)u, \quad u(s) = v. \tag{6.22}$$

That case is simpler since the bad term of (6.6), namely

$$E_\alpha\big((s-\tau)^\alpha P(\xi)\big),$$

disappears. Consequently, we may relax our assumptions on the polynomial P.

Theorem 6.2. *Let $\alpha \in (0,1)$, $s \in \mathbb{R}$ and $v \in D(P)$. Assume that*

$$\sup_{\xi \in \mathbb{R}^d} \operatorname{Re}\big(P(\xi)\big) < \infty. \tag{6.23}$$

Then (6.22) *has a unique strong solution u and*

$$u(t) = \mathscr{F}^{-1}\Big(E_\alpha\big((t-s)^\alpha P(\xi)\big)\hat{v}(\xi)\Big), \quad \forall t \geq s. \tag{6.24}$$

When $s = 0$, the representation (6.24) can be found in [Kos14, Section 1]. That paper focuses on abstract fractional equation with $1 < \alpha < 2$.

The proof relies on the two lemmas below.

Lemma 6.3. *Let z, w in $\mathbb{C} \setminus \{0\}$. Assume*

$$\operatorname{Re}(z) \leq \operatorname{Re}(w), \quad |w| \leq |z|, \quad 0 < \arg(w) < \frac{\pi}{2}.$$

Then

$$|\arg(z)| \geq \arg(w).$$

The proof of the above geometric lemma is elementary so we omit it.

Lemma 6.4. *Let $0 < \alpha < 1$ and* (6.23) *be valid. Then for all $T > 0$, there exists a constant $C = C(T, \alpha)$ such that*

$$\sup_{t \in [0,T]} \sup_{\xi \in \mathbb{R}^d} |E_\alpha\big(P(\xi)t^\alpha\big)| < \infty. \tag{6.25}$$

Proof. Let us choose μ such that

$$\mu \in (\frac{\pi}{2}\alpha, \frac{\pi}{2}).$$

For all $\xi \in \mathbb{R}^d$ and $t \in [0,T]$, we have by (6.23)

$$\operatorname{Re}\big(P(\xi)t^\alpha\big) \leq CT^\alpha.$$

Notice that constants depending (or not) on α are generically denoted by C. Let

$$R := \frac{CT^\alpha}{\cos(\mu)}, \quad w := R\exp(\mathrm{i}\mu).$$

If $|P(\xi)t^\alpha| \geq R$ then, since $\operatorname{Re}(w) = CT^\alpha$, we deduce from Lemma 6.3 that

$$|\arg(P(\xi)t^\alpha)| \geq \mu.$$

Then, by Proposition 3.1, there exists some $R_1 \geq R$ such that for $|P(\xi)t^\alpha| \geq R_1$, one has

$$\big|E_\alpha\big(P(\xi)t^\alpha\big)\big| \leq \frac{C}{|P(\xi)t^\alpha|} \leq \frac{C}{R_1}. \tag{6.26}$$

Besides, if $|P(\xi)t^\alpha| \leq R_1$ then by continuity of the Mittag-Leffler function, we get

$$|E_\alpha\big(P(\xi)t^\alpha\big)| \leq C(R_1).$$

And (6.25) follows. □

Proof of Theorem 6.2. It follows the lines of the proof of Theorem 6.1. We may consider the problem

$$\mathbf{D}^\alpha_{0,t}u = P(D)u, \quad u(0) = v.$$

Let T be a positive number. In view of Lemma 6.4, the function u defined by

$$u(t) := \mathscr{F}^{-1}\Big(E_\alpha\big(t^\alpha P(\xi)\big)\hat{v}(\xi)\Big), \quad \forall t \geq 0,$$

belongs to $C([0,T], D(P))$. Moreover, by Proposition 5.2, for a.e. $\xi \in \mathbb{R}^d$, the function

$$t \mapsto \hat{u}(t,\xi)$$

is the solution to

$$\mathbf{D}^\alpha_{0,t}\hat{u} = P(\xi)\hat{u}, \quad \hat{u}(0) = \hat{v}(\xi).$$

Thus for a.e. $\xi \in \mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{g_{1-\alpha} *_0 \Big(E_\alpha\big(t^\alpha P(\xi)\big) - 1\Big)\hat{v}(\xi)\Big\} = E_\alpha\big(t^\alpha P(\xi)\big)P(\xi)\hat{v}(\xi).$$

By Lemma 6.4,

$$|E_\alpha\big(t^\alpha P(\xi)\big)P(\xi)\hat{v}(\xi)| \leq C|P(\xi)\hat{v}(\xi)|.$$

Thus arguing as in the proof of Theorem 6.1, we get

$$\mathbf{D}^\alpha_{0,t}u = P(D)u \qquad \text{in} \quad C\big([0,T], L^2(\mathbb{R}^d)\big).$$

This completes the proof of the theorem. □

Corollary 6.5. *Let $\mathcal{D}_0$ be the subset of $\mathbb{R}^3$ defined by* (4.9) *and P be a polynomial satisfying* (6.4), (6.5)*. For each $(t,s,\tau) \in \mathcal{D}_0$, we set*

$$\boldsymbol{\mathcal{T}}(t,s,\tau)v = \mathscr{F}^{-1}\Big(\frac{E_\alpha\big((t-\tau)^\alpha P(\xi)\big)}{E_\alpha\big((s-\tau)^\alpha P(\xi)\big)}\hat{v}(\xi)\Big).$$

Then the following properties hold.

(i) *$\boldsymbol{\mathcal{T}}$ maps $\mathcal{D}_0$ into $\mathcal{F}(D(P))$.*
(ii) *$\boldsymbol{\mathcal{T}}$ is translation invariant.*
(iii) *For each $\tau_0 \in \mathbb{R}$, the solution operator $T_{\boldsymbol{\mathcal{T}},\tau_0}$, defined by* (4.11) *and* (4.12)*, is a propagator. If $P \not\equiv 0$ then $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is not positively translation invariant.*
(iv) *The solution operator*

$$T_{\boldsymbol{\mathcal{T}},id} : \big\{(t,s) \in \mathbb{R}^2 \mid t \geq s\big\} \to \mathcal{F}(D(P))$$
$$(t,s) \mapsto \boldsymbol{\mathcal{T}}(t,s,s)$$

is the only positively translation invariant operator satisfying

$$T_{\boldsymbol{\mathcal{T}},id}(t,0) = \boldsymbol{\mathcal{T}}(t,0,0), \quad \forall t \geq 0.$$

(v) *If $P \not\equiv 0$ then $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is not a propagator.*

Proof. The proof relies on Theorems 4.3, 6.1 and 6.2. Besides, if $P \not\equiv 0$ then $\boldsymbol{\mathcal{T}}(\cdot, 0, 0)$ is not additive by Proposition 3.2. Hence $T_{\boldsymbol{\mathcal{T}},\tau_0}$ is not positively translation invariant and $T_{\boldsymbol{\mathcal{T}},id}$ is not a propagator according to Theorem 2.1. This completes the proof of the corollary. □

The theory featured in this paper applies to the *time fractional heat equation*, namely

$$\mathbf{D}^{\alpha}_{\tau,t} u = \Delta u. \tag{6.27}$$

Indeed, setting

$$P(\xi) := Q_2(\xi) = -\sum_{k=0}^{d} \xi_k^2, \quad \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d, \tag{6.28}$$

we have

$$P(D) = \Delta, \quad D(P) = H^2(\mathbb{R}^d).$$

Moreover, P is real valued and

$$\sup_{\xi \in \mathbb{R}^d} P(\xi) < \infty.$$

Then the above results apply to (6.27).

Another example is

$$\mathbf{D}^{\alpha}_{\tau,t} u = -\Delta^2 u + p_2 \Delta u + p_0 u, \tag{6.29}$$

where p_0, p_2 are real numbers. In the case $p_2 = 2$, we recover in the right hand side of (6.29), the differential operator of the linearised Swift-Hohenberg equation. That equation modelizes pattern formations in physical systems (see for instance [CH93]).

Then, by setting

$$P(\xi) := -\big(Q_2(\xi)\big)^2 + p_2 Q_2(\xi) + p_0,$$

we have

$$P(D) = -\Delta^2 + p_2 \Delta + p_0, \quad D(P) = H^4(\mathbb{R}^d)$$

and

$$\sup_{\xi \in \mathbb{R}^d} P(\xi) \le p_2^2 + p_0.$$

Hence the above results apply to (6.29).

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