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Highly Accurate Discretization of the Navier-Stokes Equations in Streamfunction Formulation

D. Fishelov, M. Ben-Artzi and J.-P. Croisille

Dedicated to the memory of Professor David Gottlieb for his Wisdom and Generosity

Abstract A discrete version of the pure streamfunction formulation of the Navier-Stokes equation is presented. The proposed scheme is fourth order in both two and three spatial dimensions.

1 Fourth order scheme for the Navier-Stokes equations in two dimensions

We consider the Navier-Stokes equations in pure streamfunction form, which in the two-dimensional case leads to the scalar equation

$$\begin{cases} \partial_t \Delta \psi + \nabla^\perp \psi \cdot \nabla \Delta \psi - \nu \Delta^2 \psi = f(x, y, t), \\ \psi(x, y, t) = \psi_0(x, y). \end{cases} \quad (1)$$

Recall that $\nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$ is the velocity vector. The no-slip boundary condition associated with this formulation is

$$\psi = \frac{\partial \psi}{\partial n} = 0, \quad (x, y) \in \partial\Omega, \quad t > 0 \quad (2)$$

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and the initial condition is

$$\psi(x, y, 0) = \psi_0(x, y), \quad (x, y) \in \Omega. \quad (3)$$

The spatial derivatives in Equation (1) are discretized as we describe next. The fourth order discrete Laplacian $\tilde{\Delta}_h \psi$ and biharmonic $\tilde{\Delta}_h^2 \psi$ operators introduced in [3] are perturbations of the second order operators $\Delta_h \psi = (\delta_x^2 + \delta_y^2) \psi$ and $\Delta_h^2 \psi = (\delta_x^4 + \delta_y^4 + 2\delta_x^2 \delta_y^2) \psi$. They are designed as follows.

$$\tilde{\Delta}_h \psi_{i,j} = 2\Delta_h \psi_{i,j} - (\delta_x(\psi_x)_{i,j} + \delta_y(\psi_y)_{i,j}) = (\Delta \psi)_{i,j} + O(h^4). \quad (4)$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to $\partial_x \psi, \partial_y \psi$ described as

$$\begin{cases} \sigma_x \psi_x = \frac{1}{6}(\psi_x)_{i-1,j} + \frac{2}{3}(\psi_x)_{i,j} + \frac{1}{6}(\psi_x)_{i+1,j} = \delta_x \psi_{i,j} & , \quad 1 \leq i, j \leq N-1 \\ \sigma_y \psi_y = \frac{1}{6}(\psi_y)_{i,j-1} + \frac{2}{3}(\psi_y)_{i,j} + \frac{1}{6}(\psi_y)_{i,j+1} = \delta_y \psi_{i,j} & , \quad 1 \leq i, j \leq N-1. \end{cases} \quad (5)$$

We use the standard central difference operators $\delta_x, \delta_y, \delta_x^2, \delta_y^2$.

The fourth-order approximation to the biharmonic operator $\Delta^2 \psi$ is

$$\tilde{\Delta}_h^2 \psi = \delta_x^4 \psi + \delta_y^4 \psi + 2\delta_x^2 \delta_y^2 \psi - \frac{h^2}{6}(\delta_x^4 \delta_y^2 \psi + \delta_y^4 \delta_x^2 \psi) = \Delta^2 \psi + O(h^4), \quad (6)$$

where δ_x^4 and δ_y^4 are the compact approximations of ∂_x^4 and ∂_y^4 , respectively.

$$\delta_x^4 \psi_{i,j} = \frac{12}{h^2} ((\delta_x \psi_x)_{i,j} - \delta_x^2 \psi_{i,j}) \quad , \quad \delta_x^4 \psi = \partial_x^4 \psi - \frac{1}{720} h^4 \partial_x^8 \psi + O(h^6), \quad (7)$$

$$\delta_y^4 \psi_{i,j} = \frac{12}{h^2} ((\delta_y \psi_y)_{i,j} - \delta_y^2 \psi_{i,j}) \quad , \quad \delta_y^4 \psi = \partial_y^4 \psi - \frac{1}{720} h^4 \partial_y^8 \psi + O(h^6). \quad (8)$$

The convective term in (1) is $C(\psi) = -\partial_y \psi \Delta(\partial_x \psi) + \partial_x \psi \Delta(\partial_y \psi)$. Its fourth-order approximation needs special care. The mixed derivative $\partial_x \partial_y^2 \psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$ using a suitable combination of lower order approximations.

$$\tilde{\psi}_{yyx} = \delta_y^2 \psi_x + \delta_x \delta_y^2 \psi - \delta_x \delta_y \psi_y = \partial_x \partial_y^2 \psi + O(h^4). \quad (9)$$

For the pure third order derivative $\partial_x^3 \psi$ we note that if ψ is smooth then

$$\psi_{xxx} = \frac{3}{2h^2} (10\delta_x \psi - h^2 \delta_x^2 \partial_x \psi - 10\partial_x \psi)_{i,j} + O(h^4). \quad (10)$$

One needs to approximate $\partial_x \psi$ to sixth-order accuracy in order to obtain from (10) a fourth-order approximation for $\partial_x^3 \psi$. Denoting this approximation by $\tilde{\psi}_x$, we invoke the Pade formulation [2], having the following form.

$$\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9} \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} + \frac{1}{9} \frac{\psi_{i+2,j} - \psi_{i-2,j}}{4h}. \quad (11)$$

At near-boundary points we apply a special treatment as in [2]. Carrying out the same procedure for $\partial_y \psi$, which yields the approximate value $\tilde{\psi}_y$, and combining with all other mixed derivatives, a fourth order approximation of the convective term is

$$\begin{aligned} \tilde{C}_h(\psi) &= -\psi_y(\Delta_h \tilde{\psi}_x + \frac{5}{2}(6\frac{\delta_x \psi - \tilde{\psi}_x}{h^2} - \delta_x^2 \tilde{\psi}_x) + \delta_x \delta_y^2 \psi - \delta_x \delta_y \tilde{\psi}_y) \\ &\quad + \psi_x(\Delta_h \tilde{\psi}_y + \frac{5}{2}(6\frac{\delta_y \psi - \tilde{\psi}_y}{h^2} - \delta_y^2 \tilde{\psi}_y) + \delta_y \delta_x^2 \psi - \delta_y \delta_x \tilde{\psi}_x) \\ &= C(\psi) + O(h^4). \end{aligned} \quad (12)$$

Our implicit-explicit time-stepping scheme is of the Crank-Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h \psi^{(n)} + \frac{\nu}{2} [\tilde{\Delta}_h^2 \psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j}^n] \quad (13)$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t} = -\tilde{C}_h \psi^{(n+1/2)} + \frac{\nu}{2} [\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j}^n]. \quad (14)$$

Due to stability reasons we have chosen an Explicit-Implicit time stepping scheme. It is possible however to use an explicit time-stepping scheme if one can afford a small time step in order to advance the solution in time. The set of linear equations is solved via a FFT solver using the Sherman-Morrison formula (see [4]). This solver is of $O(N^2 \log N)$ operations, where N is the number of grid points in each spatial direction. For the application of the pure streamfunction formulation on an irregular domain see [5].

2 The pure streamfunction formulation in three dimensions

Let Ω be a bounded domain in R^3 . The three-dimensional Navier-Stokes equations in vorticity-velocity formulation is

$$\begin{aligned} \omega_t + \nabla \times (\omega \times \mathbf{u}) - \nu \Delta \omega &= \nabla \times \mathbf{f}, \quad \text{in } \Omega \\ \omega &= \nabla \times \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \partial\Omega \\ \omega(\mathbf{x}, 0) &= \omega_0(\mathbf{x}) := \nabla \times \mathbf{u}_0, \quad \text{in } \Omega. \end{aligned} \quad (15)$$

where $\omega = \nabla \times \mathbf{u}$ and the no-slip boundary condition has been imposed. The pure streamfunction formulation for this system is obtained by introducing a streamfunction $\psi(\mathbf{x}, t) \in R^3$, such that

$$\mathbf{u} = -\nabla \times \psi. \quad (16)$$

This is always possible since $\nabla \cdot \mathbf{u} = 0$. Thus,

$$\omega = \nabla \times \mathbf{u} = \Delta \psi - \nabla(\nabla \cdot \psi). \quad (17)$$

Imposing a gauge condition

$$\nabla \cdot \psi = 0, \quad (18)$$

yields

$$\omega = \Delta \psi. \quad (19)$$

The system (15) can now be rewritten as

$$\frac{\partial \Delta \psi}{\partial t} - \nabla \times (\Delta \psi \times (\nabla \times \psi)) = \nu \Delta^2 \psi + \nabla \times \mathbf{f}, \quad \text{in } \Omega. \quad (20)$$

The boundary conditions $\mathbf{u} = 0$ translates to $\nabla \times \psi = 0$ on $\partial\Omega$. We require that

$$\mathbf{n} \times \psi = \mathbf{0}, \quad \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \quad \text{on } \partial\Omega. \quad (21)$$

The condition $\mathbf{n} \times \psi = \mathbf{0}$ means that ψ is parallel to $\mathbf{n}$, hence the normal component of the velocity vector is zero on the boundary. Adding the condition $\mathbf{n} \times (\nabla \times \psi) = \mathbf{0}$ ensures that the full velocity vector vanishes on the boundary. The requirements in (21) are equivalent to four scalar conditions, namely the vanishing of the two tangential components of ψ and $\nabla \times \psi$.

Turning now to the gauge condition $\nabla \cdot \psi = 0$, we add the condition

$$\frac{\partial(\psi \cdot \mathbf{n})}{\partial n} = 0, \quad \text{on } \partial\Omega. \quad (22)$$

Together with the vanishing of the tangential components of ψ , it implies that $\nabla \cdot \psi = 0$ on $\partial\Omega$.

Equations (21)-(22) consist of five scalar conditions for ψ on the boundary. We can still add one more scalar boundary condition, as the equations for the 3- component streamfunction ψ contain the fourth order biharmonic operator. The sixth scalar boundary condition that we choose to add is

$$\Delta(\nabla \cdot \psi) = 0, \quad \text{on } \partial\Omega. \quad (23)$$

We thus obtain

$$\nabla \cdot \psi = 0, \quad \Delta(\nabla \cdot \psi) = 0, \quad \text{on } \partial\Omega. \quad (24)$$

We assume that the initial value $\psi(\mathbf{x}, 0)$ satisfies $(\nabla \cdot \psi)(\mathbf{x}, 0) = 0$. Taking the divergence of (20) we obtain an evolution equation for $\nabla \cdot \psi$.

$$\frac{\partial \Delta(\nabla \cdot \psi)}{\partial t} = \nu \Delta^2(\nabla \cdot \psi), \quad \text{in } \Omega. \quad (25)$$

Equations (24)-(25) together with the assumption that $\nabla \cdot \psi = 0$ initially ensure that $\nabla \cdot \psi = 0$ for all $t > 0$. See also [1], [6] and [7]. Finally, we have the following three-dimensional pure streamfunction formulation

$$\begin{cases} \frac{\partial \Delta \psi}{\partial t} - \nabla \times (\Delta \psi \times (\nabla \times \psi)) = \nu \Delta^2 \psi + \nabla \times \mathbf{f}, & \text{in } \Omega \\ \mathbf{n} \times \psi = \mathbf{0}, \frac{\partial(\psi \cdot \mathbf{n})}{\partial n} = 0, & \text{on } \partial\Omega \\ \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \Delta(\nabla \cdot \psi) = 0, & \text{on } \partial\Omega. \end{cases} \quad (26)$$

3 The Numerical Scheme

Our numerical scheme is based on the approximation of the following equation

$$\frac{\partial \Delta \psi}{\partial t} - ((\nabla \times \psi) \cdot \nabla) \Delta \psi + (\Delta \psi \cdot \nabla)(\nabla \times \psi) - \nu \Delta^2 \psi = \nabla \times \mathbf{f}, \quad \text{in } \Omega, \quad (27)$$

assuming that $\psi \in H_0^2(\Omega)$. For the vector function ψ we construct a fourth-order approximation to the biharmonic operator as follows. The pure fourth-order derivatives are approximated by $\delta_x^4, \delta_y^4, \delta_z^4$ as in (7)-(8).

The mixed terms ψ_{xxyy}, ψ_{yyzz} and ψ_{zzxx} are approximated by

$$\begin{cases} \tilde{\delta}_{xy}^2 \psi_{i,j,k} = 3\delta_x^2 \delta_y^2 \psi_{i,j,k} - \delta_x^2 \delta_y \psi_{y,i,j,k} - \delta_y^2 \delta_x \psi_{x,i,j,k} = \partial_x^2 \partial_y^2 \psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{yz}^2 \psi_{i,j,k} = 3\delta_y^2 \delta_z^2 \psi_{i,j,k} - \delta_y^2 \delta_z \psi_{z,i,j,k} - \delta_z^2 \delta_y \psi_{y,i,j,k} = \partial_y^2 \partial_z^2 \psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{zx}^2 \psi_{i,j,k} = 3\delta_z^2 \delta_x^2 \psi_{i,j,k} - \delta_z^2 \delta_x \psi_{x,i,j,k} - \delta_x^2 \delta_z \psi_{z,i,j,k} = \partial_z^2 \partial_x^2 \psi_{i,j,k} + O(h^4). \end{cases} \quad (28)$$

A fourth order approximation of the biharmonic operator is then obtained as

$$\tilde{\Delta}_h^2 \psi = \delta_x^4 \psi + \delta_y^4 \psi + \delta_z^4 \psi + 2\tilde{\delta}_{xy}^2 \psi + 2\tilde{\delta}_{yz}^2 \psi + 2\tilde{\delta}_{zx}^2 \psi. \quad (29)$$

The approximate derivatives ψ_x, ψ_y and ψ_z are related to ψ via the Hermitian derivatives as in (5).

Equation (29) provides a fourth order compact operator for $\Delta^2 \psi$, which involves values of ψ, ψ_x, ψ_y and ψ_z at (i, j, k) and at its twenty six nearest neighbors. The Laplacian operator is approximated by a fourth order operator via

$$\tilde{\Delta}_h \psi = 2\Delta_h \psi - (\delta_x \psi_x + \delta_y \psi_y + \delta_z \psi_z). \quad (30)$$

The nonlinear part in (27) consists of two terms, the convective term and the stretching term. We design a fourth-order scheme which approximates the convective term. The convective term in the three-dimensional case is

$$C(\psi) = -((\nabla \times \psi) \cdot \nabla) \Delta \psi = u \Delta \partial_x \psi + v \Delta \partial_z \psi + w \Delta \partial_z \psi. \quad (31)$$

Here $(u, v, w) = \mathbf{u} = -\nabla \times \psi$ is the velocity vector, whose components contain first order derivatives of the streamfunction, and thus may be approximated to fourth-order accuracy. The terms $\Delta \partial_x \psi, \Delta \partial_z \psi, \Delta \partial_z \psi$ may be approximated as in the two-dimensional case. The term $\Delta \partial_x \psi$, for example, may be written as

$$\Delta \partial_x \psi = \partial_x^3 \psi + \partial_x \partial_y^2 \psi + \partial_x \partial_z^2 \psi. \quad (32)$$

Here, the pure and mixed type derivatives may be approximated as in the two-dimensional Navier-Stokes equations (see (10), (9)). We denote the approximation to the convective term by $\tilde{C}_h(\psi)$.

Now, we construct a fourth-order approximation to the stretching term $S = (\omega \cdot \nabla) \mathbf{u} = -(\Delta \psi \cdot \nabla)(\nabla \times \psi)$. Note that the stretching term contains $\Delta \psi$ and mixed second order derivatives of the streamfunction. The Laplacian of ψ may be approximated to fourth-order accuracy, as in (30). The second order mixed terms, such as $\partial_x \partial_y \psi$, may be approximated using a Hermitian approximation of the type

$$(\sigma_x \sigma_y)(\psi_{xy})_{i,j,k} = \delta_x \delta_y \psi_{i,j,k}. \quad (33)$$

Hence,

$$(I + \frac{h^2}{6} \delta_x^2)(I + \frac{h^2}{6} \delta_y^2)(\psi_{xy})_{i,j,k} = \delta_x \delta_y \psi_{i,j,k}, \quad 1 \leq i, j, k \leq N-1 \quad (34)$$

is an implicit equation for ψ_{xy} . We denote the approximation of the stretching term by $\tilde{S}_h(\psi)$. For the approximation in time, we apply a Crank-Nicholson scheme (see the comment after (13)-(14)).

We obtain the following scheme

$$\frac{(\tilde{\Delta}_h \psi_{i,j,k})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j,k})^n}{\Delta t/2} = -\tilde{C}_h \psi_{i,j,k}^{(n)} + \tilde{S}_h \psi_{i,j,k}^{(n)} + \frac{\nu}{2} [\tilde{\Delta}_h^2 \psi_{i,j,k}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j,k}^n] \quad (35)$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j,k})^{n+1} - (\tilde{\Delta}_h \psi_{i,j,k})^n}{\Delta t} = -\tilde{C}_h \psi_{i,j,k}^{(n+1/2)} + \tilde{S}_h \psi_{i,j,k}^{(n+1/2)} + \frac{\nu}{2} [\tilde{\Delta}_h^2 \psi_{i,j,k}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j,k}^n]. \quad (36)$$

At present, a direct solver is invoked to solve the linear set of equations (35)-(36).

Some preliminary MATLAB computations with coarse grids confirm the fourth order accuracy of the scheme. We first show numerical results for the time-dependent Stokes equations

$$\frac{\partial \Delta \psi}{\partial t} = \nu \Delta^2 \psi + \mathbf{f}, \quad \text{in } \Omega. \quad (37)$$

We have picked the exact solution ψ

$$\psi^T(\mathbf{x}, t) = -\frac{1}{4} e^{-t} (z^4, x^4, y^4) \quad (38)$$

in the cube $\Omega = (0, 1)^3$. Here, $\mathbf{f}$ is chosen such that ψ in (38) satisfied (37) exactly. In the numerical results shown here we have chosen the time step Δt of order h^2 in order to retain the overall fourth-order accuracy of the scheme. In practice, if we are interested mainly in the steady state solution, a larger time step, which is independent of h , may be used. In Table 1 we show results for the Stokes problem with $\Delta t = 0.1h^2$ and $t = 0.00625$. Here e is the error in the l_h^2 norm, i.e.

$$e^2 = \sum_i \sum_j \sum_k (\psi_3(x_i, y_j, z_k) - \tilde{\psi}_3(x_i, y_j, z_k))^2 h^3,$$

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	2.5460(-9)	3.82	1.8017(-10)	3.98	1.1443(-11)
e_y	7.7417(-9)	3.73	5.8037(-10)	3.96	3.7391(-11)
$\text{div}(\psi)$	1.3409(-8)	3.74	1.0052(-9)	3.96	6.4621(-11)

Table 1 Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$.

where ψ_3 is the z component of the exact solution and $\tilde{\psi}_3$ is the z component of the approximate solution. e_y is the l_h^2 in the y derivative of ψ_3 . In Table 2 we display the results for $t = 0.0625$ using $\Delta t = h^2$.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	9.6461(-7)	4.41	4.5309(-8)	4.00	2.8291(-9)
e_y	3.0293(-6)	4.33	1.5049(-7)	3.99	9.4269(-9)
$\text{div}(\psi)$	5.2470(-6)	4.33	2.6066(-7)	4.00	1.6328(-8)

Table 2 Stokes equations with $\Delta t = h^2$ for $t = 0.0625$.

Next we show results for the Navier-Stokes Equations

$$\frac{\partial \Delta \psi}{\partial t} - ((\nabla \times \psi) \cdot \nabla) \Delta \psi + (\Delta \psi \cdot \nabla)(\nabla \times \psi) - \nu \Delta^2 \psi = \nabla \times \mathbf{f}, \quad \text{in } \Omega \quad (39)$$

in the cube $\Omega = (0,1)^3$. Here, the source term $\mathbf{g} = \nabla \times \mathbf{f}$ is chosen such that $\psi^T(\mathbf{x}, t) = -\frac{1}{4}e^{-t}(z^4, x^4, y^4)$ is an exact solution of (39). In Table 3 we present results for $t = 0.00625$ using $\Delta t = 0.1h^2$.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	2.4497(-9)	3.86	1.6924(-10)	4.01	1.0473(-11)
e_y	7.6486(-9)	3.75	5.6845(-10)	3.98	3.5917(-11)
$\text{div}(\psi)$	1.2294(-8)	3.71	9.3619(-10)	3.92	6.1700(-11)

Table 3 Navier-Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$.

In Table 4 we show results for the Navier-Stokes Equations with $\Delta t = h^2$ for $t = 0.0625$. In Figures 1(a) and 1(b) we display the errors for Navier-Stokes equations in ψ_3 and $(\psi_3)_y$ at $t = 0.0625$ with $dt = h^2$ and a 17^3 grid.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	9.4418(-7)	4.46	4.2709(-8)	4.04	2.5934(-9)
e_y	2.9836(-6)	4.38	1.4334(-7)	4.03	8.7800(-9)
$\text{div}(\psi)$	5.0471(-6)	4.40	2.3944(-7)	4.02	1.4778(-8)

Table 4 Navier-Stokes equations for $t = 0.0625$ using $\Delta t = h^2$.

Fig. 1 Navier-Stokes : Errors in (a) ψ_3 and (b) $(\psi_3)_y$ for $N = 17$, $t = 0.0625$, $dt = h^2$.

References

1. M. Ben-Artzi, *Planar Navier-Stokes equations, vorticity approach*, Handbook of Mathematical Fluid Dynamics, Chapter 5, Vol. II, (2003).
2. M. H. Carpenter and D. Gottlieb and S. Abarbanel *The stability of numerical boundary treatments for compact high-order schemes finite difference schemes*, J. Comput. Phys., Vol. 108, pp. 272–295, (1993).
3. M. Ben-Artzi, J.-P. Croisille and D. Fishelov, *A High Order Compact Scheme for the Pure-Streamfunction Formulation of the Navier-Stokes Equations*, J. Sci. Computing, Vol. 42 (2), pp. 216–250 (2010).
4. M. Ben-Artzi, J.-P. Croisille and D. Fishelov, *A fast direct solver for the biharmonic problem in a rectangular grid*, SIAM J. Sci. Computing, Vol. 31 (1), pp. 303–333 (2008).
5. M. Ben-Artzi, I. Chorev, J.-P. Croisille and D. Fishelov, *A compact difference scheme for the biharmonic equation in planar irregular domains*, SIAM J. Numer. Anal., Vol. 47 (4), pp. 3087–3108 (2009).
6. V. Ruas and L. Quartapelle, *Uncouples finite element solutions of biharmonic problems for vector potentials*, Inter. J. Numer. Methods in Fluids, Vol. 11, pp. 811–822, (1990).
7. A. Rubel and G. Volpe, *Biharmonic vector stream function formulation and multigrid solutions for a three-dimensional driven-cavity Stokes flow*, AIAA Computational Fluid Dynamics Conference, 9th, Buffalo, NY, June 13–15, 1989, AIAA, pp. 380–388 (1989). Inter. J. Numer. Methods in Fluids, Vol. 11, pp. 811–822, (1990)