Distortions of Kolmogorov spectra in convectively driven systems: Insights from numerical experiments
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Numerical experiments with 1D toy models find significant distortions of the normative Kolmogorov spectrum. Convective instabilities feed in additional energy at intermediate wavenumbers. Furthermore, subtle differences in derivative couplings can spoil the usual intuitive picture of the Kolmogorov-Richardson cascade. Compressible flows as well as incompressible flows in thin layers violate a condition apparently necessary for the cascade and might be expected to equipartition at lower wavenumbers.

Turbulent flows are commonly said to obey the Kolmogorov spectrum, which describes the partition of kinetic energy among eddy modes of different spatial frequencies. The power-law roll-off can be derived via arguments of self-similarity or via naive dimensional analysis. A key premise of the derivation is that power is supplied at the lowest possible wavenumber (i.e., spatial frequency) but dissipated (by viscosity or heat diffusion) at very high wavenumbers.

One may picture a cascade of kinetic energy from large-scale down to small-scale eddies, but this picture breaks down if power is supplied at some intermediate wavenumber. We might ask whether the resulting energy spectrum approaches equipartition upstream or else peaks at a strongly driven wavenumber.

In the theory of heat transport, Bénard’s analysis of convective instability suggests that the normal modes of lowest wavenumber are not strongly driven. If dissipative phenomena (namely viscosity and heat diffusion) are neglected, the exponential growth rate of convection modes is simply proportional to $k / \sqrt{k^2 + (\pi / Z)^2}$, where $k$ denotes wavenumber, and $Z$ the vertical thickness of the convective layer.

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2 Bénard solved the linearized Navier-Stokes and heat transport equations (with diffusion terms) against a background temperature gradient to find the threshold of convective instability. A uniform temperature gradient is realistic for heat conduction in stable systems, but the gradient is altered by convection. The conclusion about exponential growth rates nevertheless remains qualitatively valid. Growth rates increase linearly with wavenumber, then plateau.
MATHEMATICAL FRAMEWORK

The Navier-Stokes equation belongs to a class of nonlinear dynamics systems that is more fully discussed in an earlier paper (Barrois 2016), short portions of which are duplicated here for context.

Let us set the problem in a more abstract and general framework. First consider a Hamiltonian system of nonlinear ODEs rigged to conserve the sum of squares.

\[
\frac{d}{dt} X^i = \frac{1}{2} \sum_{jk} f^i_{jk} X^j X^k; \quad f^k_{ij} = f^k_{ji}
\]

\[
f^i_{jk} + f^j_{ki} + f^k_{ij} = 0 \Rightarrow \frac{d}{dt} \sum_i (X^i)^2 = 0
\]

\[
(\forall j) \sum_i f^i_{kj} = 0 \Rightarrow \sum_i \partial X^i / \partial X^i = 0
\]

The trajectories of this system never leave a hypersphere, and thanks to the Liouville condition, non-periodic trajectories almost always cover the hypersphere fully and uniformly in ergodic fashion. We may say that such systems equipartition, in the sense that \( \langle X^m X^a \rangle = E \delta^{mn} \). Higher moments such as \( \langle A^{2M} B^{2N} \rangle \) can be calculated exactly and factorize cleanly in the limit of infinite dimensionality: \( \langle A^2 B^2 \rangle = \langle A^2 \rangle \langle B^2 \rangle \).

If the state variables are complex, then the system is rigged to conserve \( \sum X^* X \), and the Liouville condition is satisfied automatically, as if by magic.

\[
\frac{d}{dt} X^i = \frac{i}{2} \sum_{jk} f^i_{jk} X^j X^k
\]

\[
\frac{\partial \text{Re} X^i}{\partial \text{Re} X^j} = - \frac{\partial \text{Im} X^i}{\partial \text{Im} X^j} = \text{Re} \sum_k f^i_{jk} X^k
\]

We now proceed to ask what will happen when we add an extra term to drive some variables and damp others. Experience teaches that the orbits are likely to approach an attractor set with fractal character.

\[
\frac{d}{dt} X^i = \frac{1}{2} \sum_{jk} f^i_{jk} X^j X^k + \sum_m g^i_m X^m
\]

As discussed in (Barrois 2016), there is no fully satisfactory way to predict the equilibrium distribution of energy from the coefficients. Numerical simulation proves indispensable.
Fluid Flow in Flatland

Consider incompressible flow in two dimensions. The Navier-Stokes equation may be reformulated as a complex system in wavenumber (k) space, with normalized basis elements having the form \( \mathbf{V}_k = (+ik_y, -ik_x) \exp(i\mathbf{k} \cdot \mathbf{x}) / k \). Given a composite flow \( \mathbf{V} = \mathbf{U} + \mathbf{U}' \), the coupling follows from the cross-term:

\[
\frac{d}{dt} \mathbf{V} = -(\mathbf{U} \cdot \nabla) \mathbf{U}' - (\mathbf{U}' \cdot \nabla) \mathbf{U} - \nabla P.
\]

The pressure gradient term merely serves to enforce incompressibility and drops out when projected onto any divergence-free pattern.

\[
f_{pq}^k \equiv f(p, q \rightarrow k) = \frac{p \times q}{pqk} (p^2 - q^2) \delta^2(k + p + q)
\]

\[
\frac{d}{dt} X^*(k) = \frac{1}{2} \sum_{pq} f_{pq}^k X(p)X(q) - \chi k^2 X^*(k)
\]

It is easy to verify that the coupling coefficients obey the cyclic sum condition that guarantees conservation of kinetic energy until viscosity goes to work. Wavenumbers are conserved, as required by translational invariance.

The fact that velocities are real in position space requires the amplitudes to obey \( X(-k) = X^*(k) \). However, cross-correlations between amplitudes with wavenumbers of unequal magnitude must vanish, thanks to translational invariance. The wavenumber basis serves to diagonalize the covariance matrix \( E_{ij} = \langle X(i)X^*(j) \rangle \).

We should also take note of two scaling rules involving the coefficients:

- Homogeneous scaling: \( f(\lambda p, \lambda q \rightarrow \lambda k) = |\lambda| f(p, q \rightarrow k) \)
- Inhomogenous scaling: If the three wavenumbers are grossly unequal, e.g., \( |p| \approx |q| \gg |k| \), then the coefficient will scale with the final wavenumber \( k \).

The second scaling rule follows from incompressibility and translational invariance, unless boundary conditions disrupt the invariance and invalidate the result:

\[
U \sim \exp(ip \cdot x); \quad V \sim \exp(iq \cdot x); \quad W \sim \exp(ik \cdot x)
\]

\[
k + p + q = 0; \quad U \cdot p = V \cdot q = W \cdot k = 0; \quad |U| = |V| = |W| = 1
\]

\[
f_{UV}^W = -(U \cdot q)(V \cdot W) - (V \cdot p)(U \cdot W) = +(U \cdot k)(V \cdot W) + (V \cdot k)(U \cdot W) \leq |2k|
\]

**Kolmogorov Spectrum for Fluid Flow**

Kolmogorov theory derives the energy spectrum by naive dimensional analysis from a single parameter \( \mathcal{W} \) representing power supplied (and ultimately dissipated) per unit mass. It has units of \( L^2 / T^3 \), whereas \( \int d^Dk \ E(k) \) has units of \( L^2 / T^2 \). If wavenumber...
cutoffs (set by the physical length scale of the system and by viscous dissipation) are irrelevant, then the dimensional argument suggests the following power laws for the energy per mode \( E(k) \) and fluctuation bandwidth \( G(k) \):

\[
E(k) \sim W^{2/3}k^{-D-2/3}
\]

\[
G(k) \sim W^{1/3}k^{+2/3}
\]

The Kolmogorov-Richardson cascade may be pictured as quasi-local diffusion in \( k \)-space, given a diffusion coefficient that scales as \( W^{1/3}k^{8/3} \sim f^2G \). (Since the resulting spectrum is IR-divergent, low-wavenumber modes move \( k \) up or downhill in small steps.) Conservation of cascading energy requires that \( W^{1/3}k^{8/3}k^{D-1} \frac{\partial}{\partial k} E(k) = W \), but deviations from the normative power laws may be expected in convectively driven systems, because instabilities feed in additional energy at intermediate wavenumbers. Such deviations are modest because \( g(k) \approx g_0 \ll G(k) \sim k^{+2/3} \) at high \( k \).

\[
\frac{\partial}{\partial k} k^{8/3}k^{D-1} \frac{\partial}{\partial k} E(k) = k^{D-1} g(k) E(k)
\]

The inhomogeneous scaling rule is key to the Kolmogorov-Richardson cascade. Given any three coupled modes, energy is redistributed in the fixed ratio \( f_{pq}^* \cdot f_{lp}^* \cdot f_{kp} \). Since \( f_{pq}^* \to 0 \) as \( k \to 0 \), low-\( k \) modes act as bystanders that catalyze the cascade but contribute little of their own energy. But if the scaling rule is violated, the energy in the low-\( k \) modes is easily transferred to high-\( k \) modes, thereby promoting equipartition over some portion of the inertial regime. (Inhomogenous scaling is not always sufficient for the low-\( k \) modes to be catalytic bystanders. Appendix A discusses a counterexample.)

**NUMERICAL EXPERIMENTS**

It is computationally burdensome to simulate fluid flow, even in 2D, but it is possible to formulate toy systems in 1D with similar physical premises.

- The variables can be real or complex, and \( X(+k) \) can be independent of \( X^*(-k) \). Little is lost by keeping \( X(k) \) purely real; it merely implies that the Fourier transform obeys \( V(-x) = V^*(+x) \).

- The wavenumber indices \( k,p,q \) are integers that run between \( \pm \Lambda \). The computational workload per time step scales as \( \Lambda^2 \) in 1D.

- Spatial frequencies above 75% of cutoff are damped to terminate the inertial regime. The driver can be made to peak at the lowest spatial frequencies for consistency with the usual premise of Kolmogorov theory:

\[
\text{“Kolmogorov”} \quad g(k) = g_0 \frac{K^2}{k^2 + K^2}
\]
• Alternatively, the driver can be made to plateau, so as to simulate convective instability:

\[ g(k) = g_0 \sqrt{\frac{k^2}{k^2 + K^2}} \]

• The coupling coefficients are faithful to fluid dynamics, except that they flip sign under spatial inversion, and \( X(-k) \) is independent of \( X(k) \).

\[
\begin{align*}
&f(X(p)X(q) \to X(k)) = k \\
&f(X(q)X(k) \to X(p)) = p \\
&f(X(k)X(p) \to X(q)) = q
\end{align*}
\]

• Behemoth is seen to be a complex parody of the Navier-Stokes equation:

\[
\frac{d}{dt}V^\#(x) = 2i V \nabla V + (\text{linear}) = i \nabla (VV) + (\text{linear})
\]

Figures 1a and 1b show the result of multiple simulations, plotted on the same graph. \( \log E(k) \) is plotted against wavenumber, and compared to the normative trend. The green and white curves have been normalized to equal total energy.

![Figure 1a. Energy spectra of Behemoth (Kolmogorov premise: Driven at K<20)](image)

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3 This artificial system is named after the mythical spirit of chaos. Its horrid offspring will be named after three-headed creatures of legend.
The identity $\nabla(VV) = 2V(\nabla V)$ suggests two alternative generalizations of Behemoth to three-variable systems, but they are found to behave very differently:

- $f(...)$ scales with the outgoing wavenumber: (Figure 2a and 2b)
  - "Kerberos"
    $$\frac{\partial}{\partial t} A^* = \nabla (BC) + \text{(linear)}$$
    $$\frac{\partial}{\partial t} B^* = \nabla (CA) + \text{(linear)}$$
    $$\frac{\partial}{\partial t} C^* = \nabla (BA) + \text{(linear)}$$

- $f(...)$ scales with one of the incoming wavenumbers: (Figure 3)
  - "Hecate"
    $$\frac{\partial}{\partial t} A^* = B \nabla C + \text{(linear)}$$
    $$\frac{\partial}{\partial t} B^* = C \nabla A + \text{(linear)}$$
    $$\frac{\partial}{\partial t} C^* = A \nabla B + \text{(linear)}$$
The subtle difference in inhomogeneous scaling somehow leads to Hecate’s flat spectrum. The picture of quasi-local diffusion breaks down because the scaling rule $f(B(p)C(q) \to A(k)) \sim q$ vice $\sim k$ favors large steps. In a field-theoretic treatment of energy redistribution (Barrois 2016), one-loop diagrams become UV-divergent.

It is unclear how such a difference would affect the renormalization group analysis by Yakhot & Orsag (Yakhot 1986), which assumed away the influence of low-wavenumber modes on rescaling of $k$ relative to $\Lambda$, the UV cutoff.

We might also consider systems that are constructed not just-so, but just-anyhow. The coefficients can be random numbers, chosen subject to the cyclic sum condition, then appropriately rescaled. For each triplet with $k + p + q = 0$ and $k \neq p \neq q \neq k$, random numbers $a, b, c$ are drawn from [0,1] and used to set the coefficients:
Leviathan’s scaling rule \( f(p, q \rightarrow k) \sim \sqrt[3]{pqk} \) is closer to that of Kerberos, but its spectrum is as flat as Hecate’s. Alternative rules that have couplings scale with the largest or smallest wavenumber behave similarly. (See Figure 4.)

**HYDRO- AND MAGNETO-DYNAMICS IN A THIN LAYER**

Fluid dynamics in a thin layer (as opposed to free space) might be suspected of similar ill-behavior because not all couplings scale with final wavenumber. It is essential to distinguish between “S” modes characterized by vertical plumes that produce irrotational divergence zones as they approach the hard boundaries, and “T” modes characterized by purely horizontal swirling patterns that follow contours of a stream function. (S&T modes correspond to poloidal and toroidal modes in spherical geometry.) Vertical flows dead-end at the boundaries, whereas horizontal flows slip. (Figure 5)
The allowable couplings may be calculated in a thin flat layer with symmetric boundary conditions. Given a composite flow \( \mathbf{V} = \mathbf{U} + \mathbf{U}' \), the couplings may be derived from the cross term, \( \frac{d}{dt} \mathbf{V} = -(\mathbf{U} \cdot \nabla)\mathbf{U} - (\mathbf{U}' \cdot \nabla)\mathbf{U} - \nabla P \):

\[
\begin{align*}
 f(VT(p) : VT(q) \to VT(k)) &= \left( \frac{p \cdot q}{pqk} \right) (p^2 - q^2) \sim k \\
 f(VS(p) : VS(q) \to VT(k)) &= \left( \frac{p \cdot q}{pqk} \right) (p^2 - q^2) \sim k \\
 f(VS(p) : VT(q) \to VS(k)) &= \left( \frac{p \cdot q}{pqk} \right) (k^2 - q^2) \sim p \\
 f(VT(p) : VS(q) \to VS(k)) &= \left( \frac{p \cdot q}{pqk} \right) (k^2 - p^2) \sim q
\end{align*}
\]

ST→T, TS→T, SS→S, and TT→S processes are forbidden by a parity principle, T-modes being even, S-modes odd under interchange of the upper and lower boundaries.

MHD couplings may be derived from \( \frac{d}{dt} \mathbf{B} = -(\mathbf{V} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{V} \), assuming \( \text{div}(\mathbf{V}) = 0 \). They are more complicated when the boundary conditions are not symmetric and the magnetic field patterns lack definite parity. (Figure 6)

![Figure 6. Boundary conditions for magnetic fields](image)

\[
\begin{align*}
 f(VT(p) : BT(q) \to BT(k)) &= \left( \frac{1}{pqk} \right) (k^2) \sim k^2 / p \\
 f(VS(p) : BS(q) \to BT(k)) &= \left( \frac{1}{pqk} \right) (-k^2) \sim k^2 / p \\
 f(VS(p) : BT(q) \to BS(k)) &= \left( \frac{1}{pqk} \right) (p^2) \sim p \\
 f(VT(p) : BS(q) \to BS(k)) &= \left( \frac{1}{pqk} \right) (p^2) \sim q \\
 f(VT(p) : BT(q) \to BS(k)) &= 0 \\
 f(VS(p) : BS(q) \to BS(k)) &\sim k \\
 f(VS(p) : BT(q) \to BT(k)) &\sim q \\
 f(VT(p) : BS(q) \to BT(k)) &\sim p
\end{align*}
\]

There will also be reactive force \( \mathbf{J} \times \mathbf{B} \) terms. Given a composite field \( \mathbf{B} + \mathbf{B}' \), the couplings may be derived from the cross-term, which happens to resemble its hydrodynamic counterpart:

\[
\frac{d}{dt} \mathbf{V} = \text{curl}(\mathbf{B}) \times \mathbf{B}' + \text{curl}(\mathbf{B}') \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B}' + (\mathbf{B}' \cdot \nabla)\mathbf{B} - \nabla(\mathbf{B} \cdot \mathbf{B}')
\]
APPENDIX A. INSUFFICIENCY OF INHOMOGENOUS SCALING

Inhomogeneous scaling does not quite guarantee that low-\(k\) modes will be catalytic bystanders. Consider the following example in two dimensions:

\[
\frac{d}{dt} A^* = \alpha (\nabla B) \times (\nabla C) \\
\frac{d}{dt} B^* = \beta (\nabla C) \times (\nabla A) \\
\frac{d}{dt} C^* = \gamma (\nabla A) \times (\nabla B) \\
\alpha + \beta + \gamma = 0
\]

Although \(f^{k}_{pq} \to 0\) as \(k \to 0\), energy redistributes in the fixed ratio \(\alpha : \beta : \gamma\) in the purely real case. If \(\alpha, \beta, \gamma\) are the complex cube-roots of unity, then energy redistributes in the variable ratio \(\text{Re}(\alpha ABC) : \text{Re}(\beta ABC) : \text{Re}(\gamma ABC)\).

Dimensional scaling arguments predict a minor modification of the Kolmogorov spectrum. The variables \(A, B, C\) now have units of \(L^2 / T\), the analogue of power \(W\) has units of \(L^4 / T^3\), and \(\int d^D k \ E(k)\) has units of \(L^4 / T^2\), hence \(E(k) \sim W^{4/3} k^{-2/3} \). Simulations show the steep trend in the low-\(k\) regime, but equipartition in an intermediate regime.

References

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