A Paley-Wiener theorem about the spectral parameter, and a support theorem for general types of Dunkl spherical means

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A PALEY-WIENER THEOREM ABOUT THE SPECTRAL PARAMETER, AND A SUPPORT THEOREM FOR GENERAL TYPES OF DUNKL SPHERICAL MEANS

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Abstract. For $s \in \mathbb{R}$, denote by $\mathscr{P}_k^s$ the "projections" of a function f in $\mathcal{D}(\mathbb{R}^d)$ into the eigenspaces of the Dunkl Laplacian Δ_k corresponding to the eigenvalue $-s^2$. The parameter k comes from Dunkl's theory of differential-difference operators. Wa shall characterize the range of $\mathscr{P}_k^s$ on the space of functions $f \in \mathcal{D}(\mathbb{R}^d)$ supported inside the closed ball $\overline{B(O,R)}$. As a first application of this Paley-Wiener type theorem, we provide a spectral version of de Jeu's Paley-Wiener theorem for the Dunkl transform. The second application concerns a support theorem for general types of Dunkl spherical means.

1. Introduction

Analysis of the Dunkl Laplacian operator Δ_k on $\mathbb{R}^d$ commenced in the early 90's, inspired by numerous results in the Euclidean setting, as well as some extensions of this to flat symmetric spaces. Here the parameter k comes from Dunkl's theory of differential-difference operators [9]. In recent years, there have been increasing interests in the study of problems involving the Dunkl Laplacian and have received a lot of attention, see [5], [6], [29], [11] and references therein. The purpose of this paper is to study a family of eigenfunctions for the Dunkl Laplacian derived through the use of the inversion formula for the Dunkl transform. Our main result may be interpreted as a contribution to the spectral theory of the Dunkl Laplacian. Here we understand the term "spectral theory" to mean any analysis related to eigenfunctions of a given Laplacian operator, and how they can bring to light other objects.

In order to state the main result, we need to introduced some notation. Writing the inversion formula for the Dunkl transform in polar coordinates, we obtain

$$f(x) = \int_0^\infty f_k^s(x)\,ds, \quad f \in \mathcal{D}(\mathbb{R}^d),$$

where f_k^s are "projections" of f into the eigenspaces of Δ_k corresponding to the eigenvalue $-s^2$. We may also write the projection operators $f \mapsto f_k^s$ as Dunkl-convolution with a normalized Bessel function of the first kind (see (3.7)). In this paper we discuss on $\mathcal{D}(\mathbb{R}^d)$ how properties of f are related into properties of the eigenfunctions f_k^s. Essentially, we prove a Paley-Wiener type theorem characterizing f_k^s for $f \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{supp}(f) \subset \overline{B(O,R)}$, involving analytic continuation to $s \in \mathbb{C}$ and growth estimates of type

$$|f_k^s(x)| \le C_{k,N}(\|x\|)\,(1+|s|)^{-N} e^{(R+\|x\|)\,|\mathrm{Im}\, s|}, \quad x \in \mathbb{R}^d$$

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for all $N \in \mathbb{N}$, where $C_{k,N}$ is a positive continuous increasing function on $\mathbb{R}_+$ (see Theorem 3.4). Several contributions have been dedicated to this subject, see for instance the papers [24], [3] for the Euclidean Fourier transform, [4], [13], [14] for the Laplace-Beltrami operator on Riemannian symmetric spaces of non-compact types and rank-one, [20] for the Heisenberg group, and [18] for homogeneous trees.

The main result of this paper has a number of applications. The first one concerns the "usual" Paley-Wiener theorem for the Dunkl transform proved in [16] by de Jeu, which characterizes the image of the space $\mathcal{D}_R(\mathbb{R}^d)$ of compactly supported smooth functions with support in $\overline{B(O,R)}$ under the Dunkl transform. By means of our result, we prove a spectral version of de Jeu's Paley-Wiener theorem (see Theorem 3.5).

The second application concerns a support theorem for the operator $f \mapsto M^f_{k,m}$ defined on $\mathcal{D}(\mathbb{R}^d)$ by

$$M^f_{k,m}(x,r) = \int_0^\infty j_{\lambda_{k,m}}(rs) f^s_k(x)ds,$$

with $1 \leq m \leq d$, j_α is the normalized Bessel function of the first kind, and the index $\lambda_{k,m}$ is a parameter which depends on k and m. In the case $m = d$, the operator $f \mapsto M^f_{k,d}$ reduces to the so-called Dunkl spherical mean operator introduced first in [19] and further studied in [23]. When the integers d and m have the same parity, we show in Theorem 4.2 that $M^f_{k,m}(x,r)$ can be written as the x-Dunkl-convolution product of f with a radial distribution H^m_r with support on the sphere of radius r centered at the origin. An expression for the distribution H^m_r is also given. This statement enlighten on the compactly supported probability measure $\sigma^k_{x,r}$ in [23, Theorem 4.1] which represent the Dunkl spherical mean operator $M^f_{k,d}$ in the form

$$M^f_{k,d}(x,r) = \int_{\mathbb{R}^d} f d\sigma^k_{x,r}.$$

Moreover, always under the condition that d and m have the same parity, if $f \in \mathcal{D}_R(\mathbb{R}^d)$ then $M^f_{k,m}(x,r) = 0$ whenever $r > R + \|x\|$. As a second application of the main result of this paper, we establish in Theorem 4.4 a support theorem for $M^f_{k,m}$ stating, under an additional condition on f, that if $M^f_{k,m}(x,r) = 0$ for $r > R + \|x\|$ then $f = 0$ outside the closed ball of radius R.

2. Background

For $x, y \in \mathbb{R}^d$ we let $\langle x, y\rangle$ denote the usual Euclidean inner product of $\mathbb{R}^d$ and $\|x\| := \sqrt{\langle x, x\rangle}$ the Euclidean norm. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$. We denote by $d\sigma$ the Lebesgue surface measure on $\mathbb{S}^{d-1}$.

For a nonzero vector $\alpha \in \mathbb{R}^d$ define the reflection r_α by

$$r_\alpha(x) := x - 2(\langle \alpha, x\rangle/\|\alpha\|^2)\,\alpha, \quad x \in \mathbb{R}^d.$$

A root system is a finite set $\mathscr{R}$ of nonzero vectors in $\mathbb{R}^d$ such that $\alpha, \beta \in \mathscr{R}$ implies $r_\alpha(\beta) \in \mathscr{R}$. If, in addition, $\alpha, \beta \in \mathscr{R}$ and $\alpha = c\beta$ for some scalar c implies $c = \pm 1$, then $\mathscr{R}$ is called reduced. Henceforth we will assume that $\mathscr{R}$ is a reduced root system. Fix a set of positive roots $\mathscr{R}^+$, so that $\mathscr{R} = \mathscr{R}^+ \cup (-\mathscr{R}^+)$.

The finite reflection group G generated by the root system $\mathscr{R}$ is the subgroup of the orthogonal group $O(d)$ generated by the reflections $\{r_\alpha \ : \ \alpha \in \mathscr{R}^+\}$.

For a given root system $\mathscr{R}$, a multiplicity function $k : \mathscr{R} \to \mathbb{R}_+; \alpha \mapsto k_\alpha$ is a nonnegative G-invariant function defined on $\mathscr{R}$.

Given a reduced root system $\mathscr{R}$ on $\mathbb{R}^d$ and a multiplicity function $k = (k_\alpha)_{\alpha\in\mathscr{R}}$, we define the weight function ϑ_k by

$$\vartheta_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha}, \quad x \in \mathbb{R}^d.$$

Then ϑ_k is a positively homogeneous G-invariant function of degree $2\langle k\rangle$, where

$$\langle k\rangle := \sum_{\alpha\in\mathscr{R}^+} k_\alpha. \tag{2.1}$$

The main ingredient of the Dunkl theory is a family of commuting first-order differential-difference operators, $T_\xi(k)$ (called the Dunkl operators [9]), defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \frac{f(x) - f(r_\alpha x)}{\langle\alpha, x\rangle}\langle\alpha, \xi\rangle, \quad \xi \in \mathbb{R}^d,$$

where ∂_ξ is the ordinary partial derivative with respect to ξ. The Dunkl operators are akin to the partial derivatives and they can be used to define the Dunkl Laplacian Δ_k, which plays the role similar to that of the ordinary Laplacian,

$$\Delta_k f(x) := \sum_{i=1}^d T_{\xi_i}(k)^2 f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \frac{\langle\nabla f(x), \alpha\rangle}{\langle\alpha, x\rangle} - \sum_{\alpha\in\mathscr{R}^+} k_\alpha \frac{f(x) - f(r_\alpha x)}{\langle\alpha, x\rangle^2}\|\alpha\|^2,$$

where $\{\xi_1, \ldots, \xi_d\}$ is an orthonormal basis of $(\mathbb{R}^d, \langle\cdot,\cdot\rangle)$. The above explicit expression of Δ_k has been proved in [10].

For arbitrary finite reflection group G, and for any nonnegative multiplicity function k, there is a unique linear operator V_k on the space of algebraic polynomials on $\mathbb{R}^d$ that intertwines between the Dunkl operators and the partial derivatives,

$$T_\xi(k)V_k = V_k\partial_\xi, \quad \forall\xi \in \mathbb{R}^d, \quad V_k 1 = 1.$$

It has been proved in [22, Theorem 1.2] that V_k has a Laplace type representation which allows to extend V_k to larger function spaces. In fact, V_k induces a homeomorphism of $C(\mathbb{R}^d)$ and also that of $C^\infty(\mathbb{R}^d)$; see [16, Theorem 5.1] or [26].

For $x, y \in \mathbb{R}^d$, define

$$E_k(x, y) := V_k(e^{\langle\cdot,y\rangle})(x). \tag{2.2}$$

For fixed y, the function $E_k(\cdot, y)$ is the unique real-analytic solution of $T_\xi(k)f(x) = \langle y, \xi\rangle f(x)$ with $f(0) = 1$ (see [8, 21]). Further, the (Dunkl) kernel E_k has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies the following properties:

Fact 2.1 (see, for instance, [15]).

1) *For all* $z, w \in \mathbb{C}^d$ *and* $\lambda \in \mathbb{C}$, *we have* $E_k(z, w) = E_k(w, z)$ *and* $E_k(\lambda z, w) = E_k(z, \lambda w)$.
2) *For all* $\nu \in \mathbb{N}^d$, *we have* $|\partial_z^\nu E_k(x, z)| \le \|x\|^{|\nu|} e^{\|x\|\,\|\mathrm{Re}\, z\|}$. *In particular,* $|E_k(x, iy)| \le 1$ *for all* $x, y \in \mathbb{R}^d$.

For $f \in L^1(\mathbb{R}^d, \vartheta_k(x)dx)$, the Dunkl transform is defined by

$$\mathscr{F}_k f(\xi) := c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(x, -i\xi) \vartheta_k(x) dx, \quad \xi \in \mathbb{R}^d, \tag{2.3}$$

where c_k is the constant

$$c_k := \int_{\mathbb{R}^d} e^{-\|x\|^2/2} \vartheta_k(x) dx. \tag{2.4}$$

The closed form of c_k is known for every reflection group G; see [12]. The Dunkl transform was introduced in [7] where the L^2-isometry (or the Plancherel theorem) was proved, while the main results of the L^1-theory were established in [15]. In particular, it has been proved that if f and $\mathscr{F}_k f$ are in $L^1(\mathbb{R}^d, \vartheta_k(x)dx)$, then for almost every $x \in \mathbb{R}^d$,

$$f(x) = c_k^{-1} \int_{\mathbb{R}^d} \mathscr{F}_k f(\xi) E_k(\xi, ix) \vartheta_k(\xi) d\xi, \quad \xi \in \mathbb{R}^d. \tag{2.5}$$

It is worth mentioning that the Dunkl transform is a homeomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Further, according to [11, Proposition 5.7.8], for $f \in L^1(\mathbb{R}^d, \vartheta_k(x)dx)$ such that $f(x) = f_0(\|x\|)$ with $f_0 : \mathbb{R}_+ \to \mathbb{C}$, we have

$$\mathscr{F}_k f(\xi) = \mathscr{H}_{\lambda_k} f_0(\|\xi\|), \tag{2.6}$$

where

$$\lambda_k := \langle k \rangle + \frac{d-2}{2}, \tag{2.7}$$

and $\mathscr{H}_\alpha$ is the Hankel transform of index α on $L^1(\mathbb{R}_+, r^{2\alpha+1}dr)$, given by

$$\mathscr{H}_\alpha g(s) = \frac{1}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty g(r) \jmath_\alpha(rs) r^{2\alpha+1} dr. \tag{2.8}$$

Here $\jmath_\alpha$ is the normalized Bessel function defined by

$$\jmath_\alpha(z) := \Gamma(\alpha+1)\Big(\frac{z}{2}\Big)^{-\alpha} \mathbb{J}_\alpha(z), \quad \text{where} \quad \mathbb{J}_\alpha(z) = \Big(\frac{z}{2}\Big)^\alpha \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(\alpha+n+1)}. \tag{2.9}$$

We refer the reader to (for instance) the book [2] for more details on Bessel functions and their properties.

The next integral formula is needed for later use.

Fact 2.2 (see [1, (11.59)])**.** *For $\alpha > -\frac{1}{2}$ and $r, s > 0$, we have*

$$\int_0^\infty \mathbb{J}_\alpha(rt) \mathbb{J}_\alpha(st)\, t\, dt = \frac{1}{r} \delta(r-s).$$

Let $y \in \mathbb{R}^d$ be given. For $f \in \mathcal{S}(\mathbb{R}^d)$, the generalized translation operator is defined by

$$\tau_y f(x) := c_k^{-1} \int_{\mathbb{R}^d} \mathscr{F}_k f(\xi) E_k(ix, \xi) E_k(iy, \xi) \vartheta_k(\xi) d\xi, \quad x \in \mathbb{R}^d.$$

We collect some properties of the translation operator, which are important for our purpose:

Fact 2.3 (see [26])**.** *The translation operator has the following properties:*

1) *For all $x, y \in \mathbb{R}^d$, $\tau_y f(x) = \tau_x f(y)$.*
2) *For fixed $y \in \mathbb{R}^d$, τ_y is a continuous linear mapping from $C^\infty(\mathbb{R}^d)$ into itself.*

3) *If f is supported in* $\{x \in \mathbb{R}^d \ : \ \|x\| \le R\}$, *then* $\tau_y f$ *is supported in* $\{x \in \mathbb{R}^d \ : \ \|x\| \le R + \|y\|\}$.
4) *If* $f, g \in \mathcal{S}(\mathbb{R}^d)$ *and* $y \in \mathbb{R}^d$, *then*

$$\int_{\mathbb{R}^d} \tau_y f(x) g(x) \vartheta_k(x) dx = \int_{\mathbb{R}^d} f(x) \tau_{-y} g(x) \vartheta_k(x) dx.$$

The generalized translation operator is used to define a convolution structure: For $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$f *_k g(x) := c_k^{-1} \int_{\mathbb{R}^d} f(y) \tau_x \breve{g}(y) \vartheta_k(y) dy,$$

where $\breve{g}(x) := g(-x)$. We can also write the convolution $*_k$ as

$$f *_k g(x) = c_k^{-1} \int_{\mathbb{R}^d} \mathscr{F}_k f(\xi) \mathscr{F}_k g(\xi) E_k(ix, \xi) \vartheta_k(\xi) d\xi. \tag{2.10}$$

We refer the reader to [25] for more details on the convolution product $*_k$.

For $n \in \mathbb{N}$, let $\mathcal{H}_k^n$ be the space of k-harmonic polynomials of degree n on $\mathbb{R}^d$,

$$\mathcal{H}_k^n = \operatorname{Ker} \Delta_k \cap \mathcal{P}_n(\mathbb{R}^d),$$

where Δ_k is the Dunkl Laplacian and $\mathcal{P}_n(\mathbb{R}^d)$ denotes the space of homogeneous polynomials of degree n on $\mathbb{R}^d$. The restriction of elements in $\mathcal{H}_k^n$ on the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ are the so-called spherical k-harmonics. We shall not distinguish between $Y_k^n \in \mathcal{H}_k^n$ and its restriction to $\mathbb{S}^{d-1}$. The space $\mathcal{H}_k^n$ has a reproducing kernel $P_k^n(\cdot, \cdot)$ in the sense that

$$f(x) = d_k^{-1} \int_{\mathbb{S}^{d-1}} f(y) P_k^n(x, y) \vartheta_k(y) d\sigma(y), \qquad \forall f \in \mathcal{H}_k^n, \quad \|x\| \le 1.$$

Here d_k is the constant

$$d_k := \int_{\mathbb{S}^{d-1}} \vartheta_k(x) d\sigma(x) = \frac{c_k}{2^{\lambda_k} \Gamma(\lambda_k + 1)}, \tag{2.11}$$

where c_k and λ_k are as defined in (2.4) and (2.7), respectively. According to [30, Theorem 3.2], for $x, y \neq 0$, the kernel P_k^n can be written as

$$P_k^n(x, y) = (\|x\| \, \|y\|)^n \, \frac{n + \lambda_k}{\lambda_k} \, V_k\Big[C_n^{\lambda_k}\Big(\Big\langle \cdot \, , \frac{y}{\|y\|}\Big\rangle\Big)\Big]\Big(\frac{x}{\|x\|}\Big), \tag{2.12}$$

where V_k is the Dunkl intertwining operator, and C_n^α is the Gegenbauer polynomial of degree n,

$$C_n^\alpha(x) = \frac{(2\alpha)_n}{n!} {}_2F_1\Big(-n, n + 2\alpha, \alpha + 1/2; \frac{1 - x}{2}\Big)$$

for $\alpha > 0$, with ${}_2F_1$ is the hypergeometric function.

The following analogue of the Funk-Hecke formula for k-spherical harmonics will be used later on; for the proof, the reader is referred to [31, Theorem 2.1]. Let h be a continuous function on $[-1, 1]$. Then for any $Y_k^n \in \mathcal{H}_k^n$,

$$\frac{1}{d_k} \int_{\mathbb{S}^{d-1}} V_k[h(\langle x, \cdot \rangle)](y) Y_k^n(y) \vartheta_k(y) d\sigma(y) = \Lambda_n(h) Y_k^n(x), \quad x \in \mathbb{S}^d, \tag{2.13}$$

where $\Lambda_n(h)$ is a constant defined by

$$\Lambda_n(h) := \frac{\Gamma(\lambda_k+1)}{\sqrt{\pi}\Gamma(\lambda_k+1/2)}\frac{n!}{(2\lambda_k)_n}\int_{-1}^{1} h(t)C_n^{\lambda_k}(t)(1-t^2)^{\lambda_k-1/2}dt.$$

We summarize some basic properties of Gegenbauer polynomials in a way that we shall use later.

Fact 2.4 (see [6, (1.2.10)], [32, (3.32.3)])**.** *For $\lambda \in \mathbb{C}$ such that* Re $\lambda > 0$, *the following two integral formulas hold:*

1) $\displaystyle\int_{-1}^{1}(1-t^2)^{\lambda-\frac{1}{2}}C_m^{\lambda}(t)C_n^{\lambda}(t)dt = \frac{\lambda}{n+\lambda}\frac{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})\,\Gamma(2\lambda+n)}{n!\,\Gamma(\lambda+1)\,\Gamma(2\lambda)}\delta_{m,n}.$

2) $\displaystyle\int_{-1}^{1}e^{izt}(1-t^2)^{\lambda-\frac{1}{2}}C_n^{\lambda}(t)dt = \frac{\sqrt{\pi}\,i^n\,\Gamma(\lambda+\frac{1}{2})\,\Gamma(2\lambda+n)}{n!\,\Gamma(2\lambda)\Gamma(\lambda+n+1)}\Big(\frac{z}{2}\Big)^n \mathcal{J}_{\lambda+n}(z).$

Let $\mathcal{D}_R(\mathbb{R})^e$ denote the space of even compactly supported smooth functions with support in $[-R,R]$, where $R>0$. The Paley-Wiener theorem for the Hankel transform $\mathscr{H}_\alpha$ (see (2.8)) states that $\mathscr{H}_\alpha$ maps $\mathcal{D}_R(\mathbb{R})^e$ bijectively onto the space $\mathcal{H}_R(\mathbb{C})^e$ of even entire functions g satisfying, for all $N\in\mathbb{N}$,

$$|g(z)| \leq C_N(1+|z|)^{-N}e^{R|\mathrm{Im}\,z|}, \quad \forall z\in\mathbb{C},$$

for some positive constant C_N; see for instance [17, Theorem 3.1].

This result has been generalized by de Jeu [16] to the Dunkl transform. To state the Paley-Wiener theorem for $\mathscr{F}_k$ we introduce the following notation: For $R>0$ let $\mathcal{H}_R(\mathbb{C}^d)$ be the space of entire functions F on $\mathbb{C}^d$ with the property that for all $N\in\mathbb{N}$ there exists a constant $C_N>0$ such that

$$|F(z)| \leq C_N(1+\|z\|)^{-N}e^{R\|\mathrm{Im}\,z\|}, \quad \forall z\in\mathbb{C}^d.$$

We let $\mathcal{D}_R(\mathbb{R}^d)$ denote the space of smooth compactly supported functions with support contained in the closed ball $\overline{B(O,R)}\subset\mathbb{R}^d$ with radius $R>0$ and the origin as center.

Fact 2.5 (see [16, Theorem 4.10])**.** *The Dunkl transform $\mathscr{F}_k$ is a linear isomorphism between $\mathcal{D}_R(\mathbb{R}^d)$ and $\mathcal{H}_R(\mathbb{C}^d)$, for all $R>0$.*

An immediate consequence of the above Paley-Wiener theorems can be stated as:

Lemma 2.6. *Let $F_0\in C^\infty(\mathbb{R})^e$. Then $F_0(\|\xi\|)=\mathscr{F}_kf(\xi)$ for some radial function f in $\mathcal{D}_R(\mathbb{R}^d)$ if and only if F_0 extends to an entire function on $\mathbb{C}$ satisfying the estimate*

$$|F_0(z)| \leq C_N(1+|z|)^{-N}e^{R|\mathrm{Im}\,z|}, \quad \forall z\in\mathbb{C}$$

for all $N\in\mathbb{N}$.

Proof. The statement follows from the fact that $\mathscr{F}_kf(\xi)=\mathscr{H}_{\lambda_k}f_0(\|\xi\|)$ whenever f is a radial function with $f(x)=f_0(\|x\|)$ (see (2.6)), together with the Paley-Wiener theorems stated above for the Hankel and the Dunkl transforms. □

A compactly supported distribution $T\in\mathcal{E}'(\mathbb{R}^d)$ is called radial if for all orthogonal transformations $A\in O(d)$ (that is, for all rotations on $\mathbb{R}^d$) we have $\langle T,\varphi\rangle=\langle T,\varphi\circ A\rangle$ for all $\varphi\in\mathcal{E}(\mathbb{R}^d)$. Using the distributional forms of the Paley-Wiener theorems for the Hankel and the Dunkl transforms (see [28, Theorem 5.VII.1] and [27, Theorem 5.2], respectively) enable one to state the following version of Lemma 2.6 on $\mathcal{E}'(\mathbb{R}^d)$.

Lemma 2.7. *Let $F_0 \in \overline{C^\infty(\mathbb{R})^e}$. Then F_0 is the Dunkl transform of a radial distribution $T \in \mathcal{E}'(\mathbb{R}^d)$ with* $\operatorname{supp}(T) \subset \overline{B(O,R)}$ *if and only if F_0 extends to an entire function on $\mathbb{C}$ satisfying the estimate*

$$|F_0(z)| \leq C_N(1+|z|)^N e^{R|\operatorname{Im} z|}, \quad \forall z \in \mathbb{C}$$

for some $N \in \mathbb{N}$.

3. A Paley-Wiener theorem about the spectral parameter

Recall from (2.3) that the Dunkl transform of $f \in \mathcal{D}(\mathbb{R}^d)$ is defined by

$$\mathscr{F}_k f(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(x, -i\xi) \vartheta_k(x) dx. \tag{3.1}$$

Using polar coordinates, the Dunkl inversion formula (2.5) becomes

$$\begin{aligned} f(x) &= c_k^{-1} \int_0^\infty s^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\eta) E_k(ix, s\eta) \vartheta_k(\eta) d\sigma(\eta) ds \\ &= \int_0^\infty \mathscr{P}_k^s f(x) ds, \end{aligned} \tag{3.2}$$

where

$$\mathscr{P}_k^s f(x) := c_k^{-1} s^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\eta) E_k(ix, s\eta) \vartheta_k(\eta) d\sigma(\eta). \tag{3.3}$$

Notice that $\Delta_k^x \mathscr{P}_k^s f(x) = -s^2 \mathscr{P}_k^s f(x)$, and we have obtained f as a superposition of such eigenfunctions of the Dunkl Laplacian. The decomposition (3.2) is reminiscent of the spectral theorem applied to Δ_k although $\mathscr{P}_k^s f$ is not a projection operator.

From (3.3) we may derive a second formula for $\mathscr{P}_k^s f$. Indeed, substituting (3.1) into (3.3) we obtain

$$\begin{aligned} \mathscr{P}_k^s f(x) &= c_k^{-2} s^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} \Big(\int_{\mathbb{R}^d} f(y) E_k(-is\eta, y) \vartheta_k(y) dy \Big) E_k(ix, s\eta) \vartheta_k(\eta) d\sigma(\eta) \\ &= c_k^{-2} s^{2\lambda_k+1} \int_{\mathbb{R}^d} f(y) \underbrace{\Big(\int_{\mathbb{S}^{d-1}} E_k(\eta, -isy) E_k(isx, \eta) \vartheta_k(\eta) d\sigma(\eta) \Big)}_{:= I_k(x,y;s)} \vartheta_k(y) dy. \end{aligned} \tag{3.4}$$

According to [23, page 2424], the inner integral is equal to

$$I_k(x,y;s) = d_k \sum_{n=0}^\infty \Big(\frac{\Gamma(\lambda_k+1)}{2^n \Gamma(\lambda_k+n+1)} \Big)^2 j_{n+\lambda_k}(s\|x\|) j_{n+\lambda_k}(s\|y\|) P_k^n(isx, -isy), \tag{3.5}$$

where d_k is the constant (2.11), $P_k^n(\cdot,\cdot)$ is the reproducing kernel (2.12), and j_α is the normalized Bessel function (2.9). By the addition formula for Bessel functions ([2, p. 215]), the series expansion in (3.5) reduces to

$$I_k(x,y;s) = d_k V_k\Big(j_{\lambda_k}(s\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \cdot\rangle}) \Big)(y),$$

where V_k is the Dunkl intertwining operator. If we let $\mathring{j}_{s,\lambda_k}(y) := j_{\lambda_k}(s\|y\|)$, then, by [23, p. 2429], we have

$$V_k\Big(j_{\lambda_k}(s\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, \cdot\rangle}) \Big)(y) = \tau_y \mathring{j}_{s,\lambda_k}(-x) = \tau_{-x} \mathring{j}_{s,\lambda_k}(y).$$

Consequently, the eigenfunction $\mathscr{P}_k^s f$ can be rewritten as

$$\begin{aligned} \mathscr{P}_k^s f(x) &= d_k c_k^{-2} s^{2\lambda_k+1} \int_{\mathbb{R}^d} \tau_x f(y) j_{\lambda_k}(s\|y\|)\vartheta_k(y)dy && (3.6)\\ &= 2^{-\lambda_k}\Gamma(\lambda_k+1)^{-1} s^{2\lambda_k+1}(f *_k \mathring{j}_{s,\lambda_k})(y). && (3.7)\end{aligned}$$

Above we have used some of the properties of the generalized translation operator listed in Fact 2.3.

Henceforth we will assume that $2\lambda_k + 1 \in \mathbb{N}$, where $\lambda_k = \langle k\rangle + \frac{d-2}{2}$ (see (2.7)).

Proposition 3.1. *Assume that $f \in \mathcal{D}_R(\mathbb{R}^d)$ and let $\mathscr{P}_k^s f(x)$ defined either by* (3.3) *or* (3.6)*, with $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Then the following hold:*

(1) *$\mathscr{P}_k^s f(x)$ is a smooth function on $\mathbb{R}\times\mathbb{R}^d$.*
(2) *$\Delta_k^x \mathscr{P}_k^s f(x) = -s^2 \mathscr{P}_k^s f(x)$, where the upper index x indicates the relevant variable.*
(3) *For $x \in \mathbb{R}^d$ be given, $\mathscr{P}_k^s f(x)$ extends to an entire function of $s \in \mathbb{C}$ with the same parity as $2\lambda_k + 1$.*
(4) *For every $N \in \mathbb{N}$ there exists a constant $C_{k,N} > 0$ such that*

$$|\mathscr{P}_k^s f(x)| \le C_{k,N}(1+|s|)^{-N} e^{(R+\|x\|)\,|\mathrm{Im}\, s|}, \quad \forall s \in \mathbb{C}. \tag{3.8}$$

(5) *For any k-spherical harmonic Y_k^ℓ of degree ℓ and for every $r > 0$, the map*

$$s \mapsto s^{-(2\lambda_k+2\ell+1)} \int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\omega) Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega)$$

is entire on $\mathbb{C}$.

Proof. 1) In view of properties of the translation operator τ_x and the normalized Bessel function j_α, the first statement follows from the representation (3.6) of $\mathscr{P}_k^s f(x)$.
2) The second property is immediate from (3.3), since $\Delta_k^x E_k(x, is\eta) = -s^2 E_k(x, is\eta)$.
3) Recall that $2\lambda_k + 1 \in \mathbb{N}$ and that j_α is even. Now, by (3.6), the map $s \mapsto \mathscr{P}_k^s f$ is certainly extends to en entire function on $\mathbb{C}$ with the same parity as $2\lambda_k + 1$.
4) Since $f \in \mathcal{D}_R(\mathbb{R}^d)$, it follows from the Paley-Wiener theorem for the Dunkl transform that $\mathscr{F}_k f$ extends to an entire function on $\mathbb{C}^d$ satisfying the estimate

$$|\mathscr{F}_k f(\xi)| \le C_{k,M}(1+\|\xi\|)^{-M} e^{R\,\|\mathrm{Im}\,\xi\|}$$

for all $M \in \mathbb{N}$; see Fact 2.5. Consequently, by (3.3) we obtain

$$|\mathscr{P}_k^s f(x)| \le C'_{k,M}(1+|s|)^{-M}|s|^{2\lambda_k+1} e^{(R+\|x\|)\,|\mathrm{Im}\, s|}$$

for all $M \in \mathbb{N}$, where we used the estimate $|E_k(x,z)| \le e^{\|x\|\,\|\mathrm{Re}\, z\|}$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$. This finishes the proof of the estimate (3.8).
5) Let Y_k^ℓ be a k-spherical harmonic of degree ℓ. By the Fubini theorem and (3.4) we have

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\omega) Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega)\\ &\qquad = c_k^{-2} s^{2\lambda_k+1} \int_{\mathbb{R}^d} f(y)\Big(\int_{\mathbb{S}^{d-1}} I_k(r\omega, y; s) Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega)\Big)\vartheta_k(y)dy,\end{aligned}$$

where $I_k(r\omega, y; s)$ is as in (3.5),

$$I_k(r\omega, y; s) = d_k \sum_{n=0}^{\infty} \frac{n+\lambda_k}{\lambda_k}\Big(\frac{\Gamma(\lambda_k+1)}{2^n\Gamma(\lambda_k+n+1)}\Big)^2 (rs^2\|y\|)^n$$
$$J_{n+\lambda_k}(rs)J_{n+\lambda_k}(s\|y\|)V_k\Big[C_n^{\lambda_k}\Big(\Big\langle\cdot, \frac{y}{\|y\|}\Big\rangle\Big)\Big](\omega).$$

We now apply the Funk-Hecke formula (2.13) to deduce that

$$\int_{\mathbb{S}^{d-1}} V_k\Big[C_n^{\lambda_k}\Big(\Big\langle\cdot, \frac{y}{\|y\|}\Big\rangle\Big)\Big](\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega) = d_k\Lambda_\ell(C_n^{\lambda_k})Y_k^\ell\Big(\frac{y}{\|y\|}\Big),$$

where, by Fact 2.4.1, we have

$$\begin{aligned}\Lambda_\ell(C_n^{\lambda_k}) &= \frac{\Gamma(\lambda_k+1)}{\sqrt{\pi}\Gamma(\lambda_k+1/2)}\frac{\ell!}{(2\lambda_k)_\ell}\int_{-1}^{1} C_n^{\lambda_k}(t)C_\ell^{\lambda_k}(t)(1-t^2)^{\lambda_k-1/2}dt\\ &= \frac{\lambda_k}{\lambda_k+\ell}\,\delta_{\ell,n}.\end{aligned}$$

Using the above identities, it follows that

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega)\\ &= \frac{1}{2^{2\lambda_k+2\ell}\Gamma(\lambda_k+\ell+1)^2}s^{2\lambda_k+1}\int_{\mathbb{R}^d} f(y)(rs^2\|y\|)^\ell J_{\ell+\lambda_k}(rs)J_{\ell+\lambda_k}(s\|y\|)Y_k^\ell\Big(\frac{y}{\|y\|}\Big)\vartheta_k(y)dy\\ &= s^{2\lambda_k+1}\varphi_{\ell+\lambda_k}(rs)\int_0^\infty \varphi_{\ell+\lambda_k}(st)f_\ell(t)t^{2\lambda_k+1}dt,\end{aligned}$$

where $\varphi_{\ell+\lambda_k}(z) := \frac{1}{2^{\lambda_k+\ell}\Gamma(\lambda_k+\ell+1)}\, z^\ell\, J_{\ell+\lambda_k}(z)$ and

$$f_\ell(t) := \int_{\mathbb{S}^{d-1}} f(t\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta).$$

Above we have used the fact that $d_k = c_k/\{2^{\lambda_k}\Gamma(\lambda_k+1)\}$; see (2.11). In conclusion,

$$s^{-(2\ell+2\lambda_k+1)}\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega) = s^{-\ell}\varphi_{\ell+\lambda_k}(rs)\int_0^\infty s^{-\ell}\varphi_{\ell+\lambda_k}(st)f_\ell(t)t^{2\lambda_k+1}dt.$$

The desired result now follows from the fact that $s^{-\ell}\varphi_{\ell+\lambda_k}(sz)$ is an entire function of $s \in \mathbb{C}$. ◻

For the converse of the above proposition we need the following lemma. We could not find this statement in the literature, and so we give its proof here.

Lemma 3.2. *Consider the linear differential equation*

$$z^2u''(z) + bzu'(z) + (cz^2 - d^2 + (1-b)d)u(z) = 0, \tag{3.9}$$

where $b, c, d \in \mathbb{C}$. The function

$$u_p(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma\Big(n+d+\frac{b+1}{2}\Big)}\Big(\frac{z}{2}\Big)^{2n+d}, \quad z \in \mathbb{C},$$

is a particular solution for (3.9).

Proof. By using the Frobenius method we can seek the solution of equation (3.9) in the following form $u(z) = z^d \sum_{n\geq 0} a_n z^n$. From the differential equation we obtain a recurrence relation for the coefficients:

$$n(n+2d+b-1)a_n = -ca_{n-2}, \quad n \geq 2.$$

Accordingly, a particular solution of (3.9) for all $z \in \mathbb{C}$ is of the form

$$\begin{aligned} u(z) &= a_0(d) \sum_{n\geq 0} \frac{(-c/4)^n z^{2n+d}}{n! \prod_{m=1}^n \left(m + d + \frac{b-1}{2}\right)} \\ &= a_0(d) \sum_{n\geq 0} \frac{(-c/4)^n}{n!} \frac{\Gamma\left(d + \frac{b+1}{2}\right)}{\Gamma\left(n + p + \frac{b+1}{2}\right)} z^{2n+d}, \end{aligned}$$

where $a_0(d) \neq 0$. If $a_0(d) = 2^{-d}\Gamma\left(d + \frac{b+1}{2}\right)^{-1}$, we obtain

$$u(z) = \sum_{n\geq 0} \frac{(-c)^n}{n!\Gamma\left(n + d + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+d}, \quad z \in \mathbb{C}.$$

The radius of convergence of the series $u(z)$ is infinity, an thus $u(z)$ converges for all $b, c, d, z \in \mathbb{C}$. □

In the converse of Proposition 3.1 some of the conditions are more tolerant.

Proposition 3.3. *For $s \in \mathbb{R}$ and $x \in \mathbb{R}^d$, let $f_s(x)$ be a function satisfying the following conditions:*

1. *$f_s(x)$ is continuous on $\mathbb{R} \times \mathbb{R}^d$.*
2. *$f_s(x)$ is an eigenfunction of the Dunkl Laplacian with eigenvalue $-s^2$.*
3. *The mapping $s \mapsto f_s(x)$ extends to an entire function on $\mathbb{C}$ with the same parity as $2\lambda_k + 1$.*
4. *For every $N \in \mathbb{N}$ there exists a positive continuous increasing function $C_{k,N}$ on $\mathbb{R}_+$ such that*
$$|f_s(x)| \leq C_{k,N}(\|x\|)(1+|s|)^{-N} e^{(R+\|x\|)\,|\mathrm{Im}\, s|}, \quad \forall s \in \mathbb{C}. \tag{3.10}$$
5. *For every $r > 0$ and every k-spherical harmonic Y_k^ℓ of degree ℓ, the mapping*
$$s \mapsto s^{-(2\lambda_k+2\ell+1)} \int_{\mathbb{S}^{d-1}} f_s(r\omega) Y_k^\ell(\omega) \vartheta_k(\omega) d\sigma(\omega)$$
is entire on $\mathbb{C}$.

Then there exists $f \in \mathcal{D}_R(\mathbb{R}^d)$ such that $f_s(x) = \mathscr{P}_k^s f(x)$.

Proof. For $m \in \mathbb{N}$, define the following function

$$F_m(x) := (-1)^m \int_0^\infty f_s(x) s^{2m} ds \tag{3.11}$$

and set $F_0 = f$. The estimate (3.10) shows that the integrals converge absolutely, and therefore, by the assumption (1), F_m is continuous on $\mathbb{R}^d$ for all $m \in \mathbb{N}$. Further, one may check that $\langle \Delta_k^m f, \psi \rangle = \langle F_m, \psi \rangle$ for all $\psi \in \mathcal{D}(\mathbb{R}^d)$. Hence, the hypoelliplicity of Δ_k (see [19]) yields that $f \in C^\infty(\mathbb{R}^d)$.

We now turn to the proof of $f(x) = 0$ for all $\|x\| > R$. Let $\{Y_{\ell,j} \,:\, 1 \le j \le \dim(\mathcal{H}_k^\ell)\}$ be an orthonormal basis of $\mathcal{H}_k^\ell$. By (3.11) with $m = 0$, we have

$$f_{\ell,j}(r) = \int_0^\infty f_{s,\ell,j}(r)ds, \quad r > 0, \tag{3.12}$$

where $f_{\ell,j}(r) := \langle f(r\,\cdot\,), Y_{\ell,j}\rangle_k$ and $f_{s,\ell,j}(r) := \langle f_s(r\,\cdot\,), Y_{\ell,j}\rangle_k$ are the spherical k-harmonic coefficients of f and f_s, respectively. Here $\langle\cdot,\cdot\rangle_k$ stands for the inner product in $L^2(\mathbb{S}^{d-1}, \vartheta_k(\eta)d\sigma(\eta))$. To show that $f(x) = 0$ for all $\|x\| > R$, it is enough to prove that $f_{\ell,j}(r) = 0$ for all $r > R$. The fact that $f_{s,\ell,j}(r)Y_{\ell,j}(\eta)$ are eigenfunctions of Δ_k with eigenvalue $-s^2$ implies

$$\left\{\frac{d^2}{dr^2} + \frac{2\lambda_k+1}{r}\frac{d}{dr} + \left(s^2 - \frac{\ell(\ell+2\lambda_k)}{r^2}\right)\right\} f_{s,\ell,j}(r) = 0. \tag{3.13}$$

Above we have used the fact that in polar coordinates, the Dunkl Laplacian can be expressed as

$$\Delta_k = \frac{d^2}{dr^2} + \frac{2\lambda_k+1}{r}\frac{d}{dr} + \frac{1}{r^2}\Delta_{k,\mathbb{S}^{d-1}},$$

where $\Delta_{k,\mathbb{S}^{d-1}}$ is the analogue of the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$, which, in particular, has k-spherical harmonics as eigenfunctions,

$$\Delta_{k,\mathbb{S}^{d-1}} Y_k^\ell(\eta) = -\ell(\ell+2\lambda_k)Y_k^\ell(\eta).$$

We refer the reader to [29] for more details on $\Delta_{k,\mathbb{S}^{d-1}}$ and its properties.

By Lemma 3.2, the solution $f_{s,\ell,j}(r)$ of equation (3.13) is in the following form

$$f_{s,\ell,j}(r) = c_{\ell,j}(s)\, r^\ell J_{\ell+\lambda_k}(rs) = \tilde{c}_{\ell,j}(s)\, r^\ell s^{2\ell+2\lambda_k+1} \jmath_{\ell+\lambda_k}(rs), \tag{3.14}$$

where $c_{\ell,j}$ is a function which depends only on s, and $\tilde{c}_{\ell,j}(s) := s^{-(2\ell+2\lambda_k+1)}c_{\ell,j}(s)$. Because of the condition (5) on $f_s(x)$, the mapping $s \mapsto \tilde{c}_{\ell,j}(s)$ extends to an entire function of $s \in \mathbb{C}$.

On the other hand, it is known that the normalized Bessel function $\jmath_\alpha(z)$ has infinitely many positive zeros $0 < \varrho_\alpha^{(1)} < \varrho_\alpha^{(2)} < \cdots$. Let $0 < r_0 < \varrho_{\ell+\lambda_k}^{(1)}$ and define

$$\iota(r_0) := \inf_{z\in\mathbb{C},\,|z|=r_0} |\jmath_{\ell+\lambda_k}(z)|.$$

For $s \in \mathbb{C}$ such that $|s| > 0$, the identity (3.14) and the assumption (4) on $f_s(x)$ imply, for $r = r_0/|s|$,

$$|\tilde{c}_{\ell,j}(s)| \le C_{k,N}\Big(\frac{r_0}{|s|}\Big)\,\iota(r_0)^{-1} r_0^\ell\, |s|^{-\ell}\,(1+|s|)^{-N} e^{(R+\frac{r_0}{|s|})\,|\mathrm{Im}\, s|} \tag{3.15}$$

for all $N \in \mathbb{N}$. In particular, if $|s| \ge 1$ we have

$$|\tilde{c}_{\ell,j}(s)| \le C_{k,M,r_0}(1+|s|)^{-M} e^{(R+r_0)\,|\mathrm{Im}\, s|} \tag{3.16}$$

for all $M \in \mathbb{N}$ (recall that $C_{k,N}$ is an increasing function). For the compact domain $0 \le |s| \le 1$, the estimate (3.16) holds true with a different constant. Moreover, by (3.14) and the condition (3) on $f_s(x)$, the map $s \mapsto \tilde{c}_{\ell,j}(s)$ is even. Applying Lemma 2.6, there exists a radial function $\psi \in \mathcal{D}_{R+r_0}(\mathbb{R}^{d+2\ell})$ such that $\tilde{c}_{\ell,j}(s) = \mathscr{H}_{\langle k\rangle+\frac{d+2\ell-2}{2}}\psi_0(\|\xi\|)$ with $\|\xi\| = s$, where $\psi(x) = \psi_0(\|x\|)$, and $\mathscr{H}_\alpha$ is the Hankel transform. Here we have used the fact that the Dunkl transform of radial functions at ξ is a Hankel transform at $\|\xi\|$ (see (2.6)). Now letting $r_0 \to 0$ shows $\mathrm{supp}(\psi) \subset \overline{B(O,R)} \subset \mathbb{R}^{d+2\ell}$.

Using again (3.14) and the fact that $\tilde{c}_{\ell,j}(s) = \mathscr{H}_{\langle k\rangle+\frac{d+2\ell-2}{2}}\psi_0(s)$, the integral (3.12) becomes

$$\begin{aligned} f_{\ell,j}(r) &= r^\ell \int_0^\infty \tilde{c}_{\ell,j}(s) \jmath_{\ell+\lambda_k}(rs) s^{2\ell+2\lambda_k+1} ds \\ &= r^\ell \int_0^\infty \mathscr{H}_{\langle k\rangle+\frac{d+2\ell-2}{2}}\psi_0(s) \jmath_{\ell+\lambda_k}(rs) s^{2\ell+2\lambda_k+1} ds \\ &= r^\ell \int_0^\infty \mathscr{H}_{\lambda_k+\ell}\psi_0(s) \jmath_{\ell+\lambda_k}(rs) s^{2\ell+2\lambda_k+1} ds \\ &= 2^{\lambda_k+\ell}\Gamma(\lambda_k+\ell) r^\ell \psi_0(r). \end{aligned}$$

That is

$$\psi(x) = 2^{-(\lambda_k+\ell)}\Gamma(\lambda_k+\ell)^{-1}\|x\|^{-\ell} f_{\ell,j}(\|x\|), \quad x \in \mathbb{R}^{d+2\ell},$$

which implies that $f_{\ell,j}(\|x\|) = 0$ for $\|x\| > R$ as desired.

It will follow that $f_s(x) = \mathscr{P}_k^s f(x)$ with $f(x) = F_0(x) = \int_0^\infty f_s(x)ds$ provided we prove that if $h_s(x)$ satisfies the assumptions (1)–(5) of Proposition 3.3 such that $\int_0^\infty h_s(x)ds = 0$ then $h_s(x) = 0$ for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$. To do so, it suffices to show that $\int_0^\infty h_{s,\ell,j}(x)ds = 0$ implies $h_{s,\ell,j}(x) = 0$ for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$. However, this follows by mimicking the proof given above and the injectivity of the Hankel transform on $\mathbb{R}_+$. □

We can now state the main result of this paper by putting the above propositions together.

Theorem 3.4. *Let $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$. There exists a smooth compactly supported function f with support contained in $\overline{B(O,R)}$ such that $\mathscr{P}_k^s f(x) = f_s(x)$ if and only if $f_s(x)$ satisfies the conditions listed in Proposition 3.3.*

The above theorem has several applications. The following statement illustrates one of these, which we may think of it as a spectral-reformulation of the Paley-Wiener theorem for the Dunkl transform (see Fact 2.5).

Theorem 3.5. *Let φ be a smooth function on $\mathbb{R} \times \mathbb{S}^{d-1}$. Then $\varphi(s,\eta) = \mathscr{F}_k f(s\eta)$ for some $f \in \mathcal{D}_R(\mathbb{R}^d)$ if and only if the following two conditions hold:*

(1) *For each $\eta \in \mathbb{S}^{d-1}$, the map $s \mapsto \varphi(s,\eta)$ has entire extension with the property that for all $N \in \mathbb{N}$ there exists a constant $C_{k,N} > 0$ such that*

$$|\varphi(s,\eta)| \leq C_{k,N}(1+|s|)^{-N} e^{R|\mathrm{Im}\, s|}.$$

(2) *For an arbitrary k-spherical harmonic Y_k^ℓ of degree ℓ, the map*

$$s \mapsto s^{-\ell} \int_{\mathbb{S}^{d-1}} \varphi(s,\eta) Y_k^\ell(\eta)\vartheta_k(\eta) d\sigma(\eta)$$

has even entire extension.

Proof. Recall from (2.2) the Dunkl kernel $E_k(x,y) = V_k(e^{\langle\cdot,y\rangle})(x)$. Following [27], let ${}^{\mathrm{t}}V_k$ be the transpose of the Dunkl intertwining operator V_k which satisfies on $\mathcal{D}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f(x)\, V_k h(x)\vartheta_k(x)dx = \int_{\mathbb{R}^d} {}^{\mathrm{t}}V_k f(x)\, h(x)dx.$$

By [27, Theorem 4.1], $\mathrm{supp}(f) \subset \overline{B(O,R)}$ if and only if $\mathrm{supp}({}^{\mathrm{t}}V_k f) \subset \overline{B(O,R)}$.

We now proceed towards the proof of the direct part. Assume that $\varphi(s,\eta) = \mathscr{F}_k f(s\eta)$ for some $f \in \mathcal{D}_R(\mathbb{R}^d)$. Then

$$\begin{aligned}\varphi(s,\eta) &= \int_{\mathbb{R}^d} f(x)V_k(e^{-is\langle\eta,\cdot\rangle})(x)\vartheta_k(x)dx\\ &= \int_{\mathbb{R}} g_\eta(p)e^{-isp}dp,\end{aligned}$$

where $g_\eta(p) := \int_{\langle\eta,y\rangle=p} {}^{t}V_k f(y)dy$. Since $\mathrm{supp}(f) \subset \overline{B(O,R)}$, it follows that $g_\eta \in \mathcal{D}_R(\mathbb{R})$. This together with the Paley-Wiener theorem for the Euclidean Fourier transform on $\mathbb{R}$ completes the proof of the property (1).

Next we turn our attention to the property (2). For $r > 0$ and $s \in \mathbb{C}$, we have

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta)\\ =\ & c_k^{-1}s^{2\lambda_k+1}\int_{\mathbb{S}^{d-1}} Y_k^\ell(\eta)\Big(\int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\omega)E_k(\omega, irs\eta)\vartheta_k(\omega)d\sigma(\omega)\Big)\vartheta_k(\eta)d\sigma(\eta)\\ =\ & c_k^{-1}s^{2\lambda_k+1}\int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\omega)\Big(\int_{\mathbb{S}^{d-1}} E_k(irs\omega,\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta)\Big)\vartheta_k(\omega)d\sigma(\omega).\end{aligned}$$

Using the Funk-Hecke formula (2.13) to deduce that

$$\int_{\mathbb{S}^{d-1}} E_k(irs\omega,\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta) = d_k\Lambda_\ell(e^{irs\,\cdot})Y_k^\ell(\omega), \tag{3.17}$$

where, by Fact 2.4.2,

$$\begin{aligned}\Lambda_n(e^{irs\,\cdot}) &= \frac{\Gamma(\lambda_k+1)}{\sqrt{\pi}\Gamma(\lambda_k+1/2)}\frac{\ell!}{(2\lambda_k)_\ell}\int_{-1}^{1} e^{irst}C_\ell^{\lambda_k}(t)(1-t^2)^{\lambda_k-1/2}dt\\ &= \frac{\Gamma(\lambda_k+1)}{2^\ell\Gamma(\lambda_k+\ell+1)}(irs)^\ell \jmath_{\ell+\lambda_k}(rs).\end{aligned}$$

This shows that

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta)\\ &\qquad = \frac{1}{2^{\lambda_k+\ell}\Gamma(\lambda_k+\ell+1)}(irs)^\ell \jmath_{\ell+\lambda_k}(rs)s^{2\lambda_k+1}\int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega).\end{aligned}$$

That is

$$\begin{aligned}&s^{-(2\ell+2\lambda_k+1)}\int_{\mathbb{S}^{d-1}} \mathscr{P}_k^s f(r\eta)Y_k^\ell(\eta)\vartheta_k(\eta)d\sigma(\eta)\\ &\qquad = \frac{1}{2^{\lambda_k+\ell}\Gamma(\lambda_k+\ell+1)}(ir)^\ell \jmath_{\ell+\lambda_k}(rs)s^{-\ell}\int_{\mathbb{S}^{d-1}} \varphi(s,\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega).\end{aligned}$$

Since the mapping $s \mapsto \mathscr{P}_k^s f$ has the same parity as $2\lambda_k + 1$, it follows that the map $s \mapsto s^{-\ell}\int_{\mathbb{S}^{d-1}} \varphi(s,\omega)Y_k^\ell(\omega)\vartheta_k(\omega)d\sigma(\omega)$ is even. The analycity of the latter map is immediate from Theorem 3.4.

For the converse part, define

$$f_s(x) := c_k^{-1} s^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} \varphi(s,\eta) E_k(ix, s\eta)\vartheta_k(\eta)d\sigma(\eta), \qquad s \in \mathbb{R},\ x \in \mathbb{R}^d.$$

By the assumptions (1) and (2) on $\varphi(s,\eta)$, the function f_s satisfies the hypothesis of Theorem 3.4. Thus there exists $f \in \mathcal{D}_R(\mathbb{R}^d)$ such that $f_s(x) = \mathscr{P}_k^s f(x)$. That is

$$c_k^{-1} s^{2\lambda_k+1} \int_{\mathbb{S}^{d-1}} \Big\{\varphi(s,\eta) - \mathscr{F}_k f(s\eta)\Big\} E_k(ix, s\eta)\vartheta_k(\eta)d\sigma(\eta) = 0,$$

which implies

$$c_k^{-1} \int_0^\infty \int_{\mathbb{S}^{d-1}} \Big\{\varphi(s,\eta) - \mathscr{F}_k f(s\eta)\Big\} E_k(ix, s\eta)\vartheta_k(\eta)d\sigma(\eta) s^{2\lambda_k+1} ds = 0.$$

This finishes the proof of the converse part. □

4. A support theorem for general types of Dunkl spherical means

For an integer $1 \le m \le d$, we define the operator $f \mapsto M_{k,m}^f$ on $\mathcal{D}(\mathbb{R}^d)$ by

$$M_{k,m}^f(x,r) := \int_0^\infty j_{\lambda_{k,m}}(rs) \mathscr{P}_k^s f(x) ds, \qquad x \in \mathbb{R}^d,\ r > 0,$$

where

$$\lambda_{k,m} := \langle k \rangle + \frac{m-2}{2},$$

and $j_{\lambda_{k,m}}$ is the normalized Bessel function (2.9) of order $\lambda_{k,m}$. Notice that $\lambda_{k,d} = \langle k \rangle + \frac{d-2}{2}$ which is nothing but λ_k defined in (2.7).

Lemma 4.1. *In case $m = d$, the operator $f \mapsto M_{k,d}^f$ reduces to the so-called Dunkl spherical mean operator,*

$$M_{k,d}^f(x,r) = \frac{1}{d_k} \int_{\mathbb{S}^{d-1}} \tau_{r\eta} f(x) \vartheta_k(\eta) d\sigma(\eta). \tag{4.1}$$

The Dunkl spherical mean operator was first introduced in [19] *and further studied in* [23].

Proof. By the definition of $M_{k,m}^f$ together with formula (3.6) of $\mathscr{P}_k^s f(x)$ and Fact 2.2, we have

$$\begin{aligned}
M_{k,d}^f(x,r) &= d_k c_k^{-2} \int_0^\infty j_{\lambda_k}(rs) \Big(s^{2\lambda_k+1} \int_{\mathbb{R}^d} \tau_x f(y) j_{\lambda_k}(s\|y\|)\vartheta_k(y)dy\Big)ds \\
&= d_k c_k^{-2} \int_{\mathbb{R}^d} \tau_x f(y) \Big(\int_0^\infty j_{\lambda_k}(rs) j_{\lambda_k}(s\|y\|) s^{2\lambda_k+1} ds\Big)\vartheta_k(y)dy \\
&= d_k c_k^{-2} \Gamma(\lambda_k+1)^2 \Big(\frac{r}{2}\Big)^{-\lambda_k} \int_{\mathbb{R}^d} \tau_x f(y) \Big(\frac{\|y\|}{2}\Big)^{-\lambda_k} \Big(\int_0^\infty \mathbb{J}_{\lambda_k}(rs)\mathbb{J}_{\lambda_k}(s\|y\|) s ds\Big)\vartheta_k(y)dy \\
&= d_k c_k^{-2} \Gamma(\lambda_k+1)^2 \Big(\frac{r}{2}\Big)^{-\lambda_k} \int_{\mathbb{R}^d} \tau_x f(y) \Big(\frac{\|y\|}{2}\Big)^{-\lambda_k} \frac{\delta(r-\|y\|)}{r} \vartheta_k(y)dy \\
&= d_k c_k^{-2} 2^{2\lambda_k} \Gamma(\lambda_k+1)^2 \int_{\mathbb{S}^{d-1}} \tau_x f(r\eta)\vartheta_k(\eta)d\sigma(\eta) \\
&= d_k c_k^{-2} 2^{2\lambda_k} \Gamma(\lambda_k+1)^2 \int_{\mathbb{S}^{d-1}} \tau_{r\eta} f(x)\vartheta_k(\eta)d\sigma(\eta).
\end{aligned}$$

Now, the fact that $d_k = c_k/\{2^{\lambda_k}\Gamma(\lambda_k+1)\}$ finishes the proof. $\square$

For fixed $r > 0$, the map $s \mapsto j_{\lambda_{k,m}}(rs)$ is an even entire function satisfying the estimate $|j_{\lambda_{k,m}}(rs)| \leq ce^{r|\mathrm{Im}\, s|}$. Therefore, by Lemma 2.7, $j_{\lambda_{k,m}}(r\,\cdot)$ is the Dunkl transform of a radial distribution H_r^m supported inside the closed ball in $\mathbb{R}^d$ with radius r and the origin as center. We have then

$$\begin{aligned}
M_{k,m}^f(x,r) &= \int_0^\infty j_{\lambda_{k,m}}(rs)\mathscr{P}_k^s f(x)ds \\
&= c_k^{-1}\int_0^\infty j_{\lambda_{k,m}}(rs)\Big(\int_{\mathbb{S}^{d-1}} \mathscr{F}_k f(s\eta)E_k(s\eta, ix)\vartheta_k(\eta)d\sigma(\eta)\Big)s^{2\lambda_k+1}ds \\
&= c_k^{-1}\int_{\mathbb{R}^d} \mathscr{F}_k f(\xi)\mathscr{F}_k(H_m^r)(\xi)E_k(\xi, ix)\vartheta_k(\xi)d\xi \\
&= (f *_k H_r^m)(x); \qquad (4.2)
\end{aligned}$$

see (2.10). This formula allows to extend the operator $f \mapsto M_{k,m}^f$ to various larger function spaces, including $C(\mathbb{R}^d)$.

Let $\delta_r(x)$ be the delta function supported on the sphere $\partial B(O,r) = \{x \in \mathbb{R}^d \,:\, \|x\| = r\}$.

Theorem 4.2. *Assume that the integers d and m have the same parity, i.e. $d = m + 2\ell$ for some nonnegative integer ℓ. Then the distribution H_r^m is given by*

$$H_r^m = 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)r^{-(2\langle k\rangle+m-2)}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^\ell\Big\{\frac{\delta_r}{r}\Big\}.$$

In particular, the distribution H_r^m is supported on the sphere of radius r centered at the origin.

Proof. According to (3.17) (with $\ell = 0$), the Dunkl transform of $\delta_r(x)$ is

$$\mathscr{F}_k(\delta_r)(\xi) = 2^{-\lambda_k}\Gamma(\lambda_k+1)^{-1}j_{\lambda_k}(r\|\xi\|)r^{2\lambda_k+1},$$

where $\lambda_k = \langle k\rangle + \frac{d-2}{2}$. Here we have used the fact that $d_k = c_k/\{2^{\lambda_k}\Gamma(\lambda_k+1)\}$. Recall that $\mathscr{F}_k(H_r^m)(\xi) = j_{\lambda_{k,m}}(r\|\xi\|)$, where $\lambda_{k,m} = \langle k\rangle + \frac{m-2}{2}$ and $m = d - 2\ell$. We will denote $\mathscr{F}_k(\delta_r)(\xi)$ and $\mathscr{F}_k(H_r^m)(\xi)$ by $\mathscr{F}_k(\delta_r)(s)$ and $\mathscr{F}_k(H_r^m)(s)$, where $\|\xi\| = s$. Using the well known formula $\Big(\frac{1}{z}\frac{d}{dz}\Big)^n(z^\alpha\mathbb{J}_\alpha(z)) = z^{\alpha-n}\mathbb{J}_{\alpha-n}(z)$, we get

$$\begin{aligned}
\mathscr{F}_k(H_r^m)(s) &= \Gamma(\lambda_{k,m}+1)\Big(\frac{rs}{2}\Big)^{-\lambda_{k,m}}\mathbb{J}_{\lambda_{k,m}}(rs) \\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)(rs)^{-(2\langle k\rangle+d-2\ell-2)}\,(rs)^{(\langle k\rangle+\frac{d}{2}-\ell-1)}\mathbb{J}_{\langle k\rangle+\frac{d}{2}-\ell-1}(rs) \\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)(rs)^{-(2\langle k\rangle+d-2\ell-2)}\,s^{-2\ell}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^\ell\Big\{(rs)^{\langle k\rangle+\frac{d}{2}-1}\mathbb{J}_{\langle k\rangle+\frac{d}{2}-1}(rs)\Big\} \\
&= \frac{2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)}{2^{\lambda_k}\Gamma(\lambda_k+1)}r^{-(2\langle k\rangle+d-2\ell-2)}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^\ell\Big\{r^{2\langle k\rangle+d-2}j_{\langle k\rangle+\frac{d}{2}-1}(rs)\Big\} \\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)r^{-(2\langle k\rangle+d-2\ell-2)}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big)^\ell\Big\{\frac{\mathscr{F}_k(\delta_r)(s)}{r}\Big\}.
\end{aligned}$$

The theorem follows from the injectivity of the Dunkl transform. $\square$

Remark 4.3. *Recall from* (4.1) *the Dunkl spherical mean operator* $M^f_{k,d}$*. In* [23, Theorem 4.1] *it has been proved that there exists a unique compactly supported probability measure* $\sigma^k_{x,r}$ *such that* $M^f_{k,d}(x,r) = \int_{\mathbb{R}^d} f(y)d\sigma^k_{x,r}(y)$*. By the convolution product* (4.2)*, the above theorem, with* $m = d$*, enlighten on the measure* $\sigma^k_{x,r}$*.*

Since $f \in \mathcal{D}(\mathbb{R}^d)$, then $M^f_{k,m}(x,r)$ is a smooth function in x and r. One may also check that for $f \in \mathcal{D}(\mathbb{R}^d)$ we have $\lim_{r\to 0} M^f_{k,m}(x,r) = f(x)$, since $\jmath_{\lambda_{k,m}}(0) = 1$. Moreover, in view of Fact 2.3.3 and that $M^f_{k,m}(x,r) = (H^m_r *_k f)(x)$, it follows that if $f \in \mathcal{D}_R(\mathbb{R}^d)$ then $M^f_{k,m}(x,r) = 0$ whenever $r > R + \|x\|$ and m and d have the same parity. This fact does not hold if d and m have different parity.

For the converse direction we have:

Theorem 4.4. *Let* $1 \leq m \leq d$ *be an integer such that* m *and* d *have the same parity. Let* $f \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \vartheta_k(x)dx)$ *such that, for every k-spherical harmonic* Y^ℓ_k *of degree* ℓ *and for every* $r > 0$*, the function*

$$s \mapsto s^{-(2\lambda_k+2\ell+1)} \int_{\mathbb{S}^{d-1}} \mathscr{P}^s_k f(r\eta) Y^\ell_k(\eta)\vartheta_k(\eta)d\sigma(\eta) \tag{4.3}$$

is entire on $\mathbb{C}$*. Then* $M^f_{k,m}(x,r) = 0$ *for* $r > R + \|x\|$ *implies* $\operatorname{supp}(f) \subset \overline{B(O,R)}$*.*

Proof. Let $x \in \mathbb{R}^d$, $y \in \mathbb{R}^m$, and put $s := \|y\|$. On $\mathbb{R}^m$, we will consider multiplicity functions $k^{(m)}$ so that $\langle k^{(m)}\rangle = \langle k\rangle$, and we will denote the corresponding Dunkl transform on $\mathbb{R}^m$ by $\mathscr{F}_{k^{(m)}}$. Observe that $\lambda_{k^{(m)}} = \langle k^{(m)}\rangle + \frac{m-2}{2} = \lambda_{k,m}$. The analogues of the constants defined by (2.4) and (2.11) satisfy $d_{k^{(m)}} = c_{k^{(m)}}/\{2^{\lambda_{k^{(m)}}}\Gamma(\lambda_{k^{(m)}}+1)\} = c_{k^{(m)}}/\{2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)\}$. For simplicity, we shall write $M^{f,x}_{k,m}(y)$ for $M^f_{k,m}(x,\|y\|)$ in the proof below. By (3.17) and Fact 2.2, we have

$$\begin{aligned}
&\mathscr{F}_{k^{(m)}} M^{f,x}_{k,m}(\xi)\\
&= c^{-1}_{k^{(m)}} \int_{\mathbb{R}^m} M^f_{k,m}(x,\|y\|)E_{k^{(m)}}(-i\xi,y)\vartheta_{k^{(m)}}(y)dy\\
&= c^{-1}_{k^{(m)}} \int_0^\infty \int_{\mathbb{S}^{m-1}} M^f_{k,m}(x,s)E_{k^{(m)}}(-i\xi,s\eta)\vartheta_{k^{(m)}}(\eta)d\sigma(\eta)s^{2\lambda_{k,m}+1}ds \qquad (\lambda_{k^{(m)}} = \lambda_{k,m})\\
&= d_{k^{(m)}}c^{-1}_{k^{(m)}} \int_0^\infty M^f_{k,m}(x,s)\jmath_{\lambda_{k,m}}(s\|\xi\|)s^{2\lambda_{k,m}+1}ds\\
&= d_{k^{(m)}}c^{-1}_{k^{(m)}} \int_0^\infty \int_0^\infty \jmath_{\lambda_{k,m}}(rs)\jmath_{\lambda_{k,m}}(s\|\xi\|)\mathscr{P}^r_k f(x)s^{2\lambda_{k,m}+1}dsdr\\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)\|\xi\|^{-\langle k\rangle-\frac{m}{2}+1} \int_0^\infty \mathscr{P}^r_k f(x)\Big(\int_0^\infty \mathbb{J}_{\langle k\rangle+\frac{m}{2}-1}(s\|\xi\|)\mathbb{J}_{\langle k\rangle+\frac{m}{2}-1}(rs)sds\Big)r^{-\langle k\rangle-\frac{m}{2}+1}dr\\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)\|\xi\|^{-\langle k\rangle-\frac{m}{2}+1} \int_0^\infty \mathscr{P}^r_k f(x)\frac{\delta(\|\xi\|-r)}{r}r^{-\langle k\rangle-\frac{m}{2}+1}dr\\
&= 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)\|\xi\|^{-2\langle k\rangle-m+1}\mathscr{P}^{\|\xi\|}_k f(x).
\end{aligned}$$

Since d and m have the same parity, we are permitted to write the above conclusion as $\mathscr{F}_{k^{(m)}} M^{f,x}_{k,m}(\mu) = 2^{\lambda_{k,m}}\Gamma(\lambda_{k,m}+1)\mu^{-(2\langle k\rangle+m-1)}\mathscr{P}^\mu_k f(x)$.

Since the function $y \mapsto M^f_{k,m}(x, \|y\|)$ is compactly supported in $\overline{B(O, R + \|x\|)}$ for every fixed $x \in \mathbb{R}^d$, it follows that its Dunkl transform $\mathscr{F}_{k^{(m)}} M^{f,x}_{k,m}(\mu)$ is an entire function on $\mathbb{C}$ with the property that for all $M \in \mathbb{N}$ there exists a positive $C_{k,M}$ such that

$$|\mathscr{F}_{k^{(m)}} M^{f,x}_{k,m}(\mu)| \leq C_{k,M}(R + \|x\|)^{2\langle k\rangle+m-1}(1 + |\mu|)^{-M} e^{(R+\|x\|)\,|\mathrm{Im}\,\mu|}, \quad \mu \in \mathbb{C}.$$

Making use of the fact that $\mathscr{F}_{k^{(m)}} M^{f,x}_{k,m}(\mu) = 2^{-\lambda_{k,m}}\Gamma(\lambda_{k,m} + 1)^{-1}\mu^{-(2\langle k\rangle+m-1)} \mathscr{P}^\mu_k f(x)$, we deduce that the mapping $\mu \mapsto \mathscr{P}^\mu_k f(x)$ is an entire function on $\mathbb{C}$ satisfying the estimate

$$|\mathscr{P}^\mu_k f(x)| \leq C_{k,N}(R + \|x\|)^{2\langle k\rangle+m-1}(1 + |\mu|)^{-N} e^{(R+\|x\|)\,|\mathrm{Im}\,\mu|}, \quad \mu \in \mathbb{C}$$

for all $N \in \mathbb{N}$.

In view of the hypothesis (4.3) we can now apply Theorem 3.4 to deduce that $f \in \mathcal{D}_R(\mathbb{R}^d)$ as desired. □

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