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HAL Id: hal-01282496
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Submitted on 7 Mar 2016

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Lp HARMONIC ANALYSIS FOR DIFFERENTIAL-REFLECTION OPERATORS

SALEM BEN SAID, ASMA BOUSSEN & MOHAMED SIFI

ABSTRACT. We introduce and study differential-reflection operators \(\Lambda_{A,\varepsilon}\) acting on smooth functions defined on \(\mathbb{R}\). Here \(A\) is a Sturm-Liouville function with additional hypotheses and \(\varepsilon \in \mathbb{R}\). For special pairs \((A, \varepsilon)\), we recover Dunkl’s, Heckman’s and Cherednik’s operators (in one dimension).

As, by construction, the operators \(\Lambda_{A,\varepsilon}\) are mixture of \(d/dx\) and reflection operators, we prove the existence of an operator \(V_{A,\varepsilon}\) so that \(\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ d/dx\). The positivity of the intertwining operator \(V_{A,\varepsilon}\) is also established.

Via the eigenfunctions of \(\Lambda_{A,\varepsilon}\), we introduce a generalized Fourier transform \(\mathcal{F}_{A,\varepsilon}\). For \(-1 \leq \varepsilon \leq 1\) and \(0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}\), we develop an \(L^p\)-Fourier analysis for \(\mathcal{F}_{A,\varepsilon}\), and then we prove an \(L^p\)-Schwartz space isomorphism theorem for \(\mathcal{F}_{A,\varepsilon}\).

Details of this paper will be given in another article [3].

RESUMÉ. Nous introduisons et étudions des opérateurs différentiels aux différences \(\Lambda_{A,\varepsilon}\) agissant sur les fonctions régulières définies sur \(\mathbb{R}\). Ici \(A\) est une fonction de Sturm-Liouville avec des hypothèses supplémentaires et \(\varepsilon \in \mathbb{R}\). Pour des cas particuliers de paires \((A, \varepsilon)\), nous obtenons les opérateurs de Dunkl, de Heckman et de Cherednik (unidimensionnels).

Comme, par construction, les opérateurs \(\Lambda_{A,\varepsilon}\) entremêlent \(d/dx\) et des opérateurs de réflexion, nous prouvons qu’il existe un opérateur \(V_{A,\varepsilon}\) tel que \(\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ d/dx\). La positivité de l’opérateur \(V_{A,\varepsilon}\) a été établie.

A l’aide des fonctions propres de \(\Lambda_{A,\varepsilon}\), nous introduisons une transformée de Fourier généralisée \(\mathcal{F}_{A,\varepsilon}\). Nous développons de l’analyse de Fourier de type \(L^p\) pour \(\mathcal{F}_{A,\varepsilon}\) quand \(0 < p \leq \frac{2}{1+\sqrt{1-\varepsilon^2}}\) et \(-1 \leq \varepsilon \leq 1\), et nous caractérisons l’image des \(p\)-espaces de Schwartz par \(\mathcal{F}_{A,\varepsilon}\).

Les détails seront publiés dans un autre article [3].

1. A FAMILY OF DIFFERENTIAL-REFLECTION OPERATORS

It became apparent long ago that radial Fourier analysis on real rank one symmetric spaces is closely connected to certain classes of special functions in one variable:

– Bessel functions in connection with radial Fourier analysis on Euclidean spaces,
– Jacobi functions in connection with radial Fourier analysis on hyperbolic spaces.

We refer to [11] for a detailed exposition.

In the late 80’s/early 90’s Dunkl [9] found a remarkable family of commuting operators that now bear his name. In one dimension this reads

\[
D_{\alpha} f(x) = f'(x) + \frac{2\alpha + 1}{x} \left( \frac{f(x) - f(-x)}{2} \right) \quad \alpha \geq -1/2.
\] (1.1)

2000 Mathematics Subject Classification. Primary 34B25, 43A32, 43A15.

Keywords. Differential-reflection operators, intertwining operators, Fourier transform, \(L^p\)-harmonic analysis.
The eigenfunctions of Dunkl’s operators, known as the Dunkl kernel, are the nonsymmetric version of Bessel functions.

Some years after [9], in [7] Cherednik wrote down a trigonometric variant of the Dunkl operator. In one dimension this reads

\[ T_{\alpha,\beta}f(x) = f'(x) + \left((2\alpha + 1)\coth x + (2\beta + 1)\tanh x\right)\left(\frac{f(x) - f(-x)}{2}\right) - \varepsilon f(-x), \quad (1.2) \]

where \( \alpha \geq \beta \geq -1/2, \alpha \neq -1/2, \) and \( \varepsilon = \alpha + \beta + 1. \) The eigenfunctions of Cherednik’s operators, known as the Opdam functions [13], are the nonsymmetric version of Jacobi functions. We mention that the trigonometric Dunkl operators were originally introduced by Heckman [10] in a different form. In one dimension his operator reads

\[ S_{\alpha,\beta}f(x) = f'(x) + \left((2\alpha + 1)\coth x + (2\beta + 1)\tanh x\right)\left(\frac{f(x) - f(-x)}{2}\right). \]

This paper gives some aspects of harmonic analysis associated with the following family of one dimensional \((A, \varepsilon)\)-operators

\[ \Lambda_{A,\varepsilon}f(x) = f'(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x) - f(-x)}{2}\right) - \varepsilon f(-x), \]

where \( \varepsilon \in \mathbb{R} \) and \( A : \mathbb{R} \to \mathbb{R}^+ \) satisfies the following conditions (cf. [4],[5],[15]):

(C1) \( A(x) = |x|^{2\alpha+1}B(x), \) where \( \alpha > -\frac{1}{2} \) and \( B \in \mathcal{C}^\infty(\mathbb{R}) \) is even, positive, and \( B(0) = 1. \)

(C2) On \( \mathbb{R}^+ \setminus \{0\}, \) \( A \) is increasing, whereas \( A'/A \) is decreasing. This condition implies that the limit \( \lim_{x \to +\infty} A'(x)/2A(x) \geq 0 \) exists.

(C3) There exists a constant \( \delta > 0 \) such that for \( x \gg 0, \)

\[ \frac{A'(x)}{A(x)} = \begin{cases} 2\varepsilon + e^{-\delta x}D(x) & \text{if } \varepsilon > 0, \\ 2\alpha + 1 + e^{-\delta x}D(x) & \text{if } \varepsilon = 0, \end{cases} \quad (1.3) \]

with \(|D^{(k)}(x)| \leq c_k \) for all \( x \gg 0 \) and \( k \in \mathbb{N}. \)

The function \( A \) and the real number \( \varepsilon \) are the deformations parameters giving back the above three operators (as special examples) when:

1) \( A(x) = A_{\varepsilon}(x) = |x|^{2\alpha+1} \) and \( \varepsilon \) arbitrary (Dunkl’s operators \( D_{\varepsilon} \)),

2) \( A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1} \) and \( \varepsilon = 0 \) (Heckman’s operators \( S_{\alpha,\beta} \)),

3) \( A(x) = A_{\alpha,\beta}(x) = |\sinh x|^{2\alpha+1}(\cosh x)^{2\beta+1} \) and \( \varepsilon = 1 \) (Cherednik’s operators \( T_{\alpha,\beta} \)).

Let \( \lambda \in \mathbb{C} \) and consider the initial data problem

\[ \Lambda_{A,\varepsilon}f(x) = i\lambda f(x), \quad f(0) = 1, \quad (1.4) \]

where \( f : \mathbb{R} \to \mathbb{C}. \) We prove that:

**Theorem 1.1.**  
1) For \( \lambda \in \mathbb{C}, \) there exists a unique solution \( \Psi_{A,\varepsilon}(\lambda, \cdot) \) to the problem (1.4). Further, for every \( x \in \mathbb{R}, \) the function \( \lambda \mapsto \Psi_{A,\varepsilon}(\lambda, x) \) is analytic on \( \mathbb{C}. \)

2) Under the restriction \(-1 \leq \varepsilon \leq 1, \) for all \( x \in \mathbb{R} \) we have:

   1) For \( \lambda \in \mathbb{R} \) we have \( |\Psi_{A,\varepsilon}(\lambda, x)| \leq \sqrt{2}. \)

   2) For \( \lambda \in i\mathbb{R} \) we have \( \Psi_{A,\varepsilon}(\lambda, x) > 0. \)
3) Assume that $\lambda \in \mathbb{C}$ and $|x| \geq x_0$ with $x_0 > 0$. Then
\[
|\frac{\partial^N \Psi_{A,\epsilon}(\lambda, x)}{\partial x^N}| \leq c(|\lambda| + 1)^N(|x| + 1)e^{(|\text{Im}\lambda| - \varepsilon(1 - \sqrt{1 - x^2}))|x|}.
\]

4) Assume that $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Then
\[
|\frac{\partial^M \Psi_{A,\epsilon}(\lambda, x)}{\partial \lambda^M}| \leq c|x|^M(|x| + 1)e^{(|\text{Im}\lambda| - \varepsilon(1 - \sqrt{1 - x^2}))|x|}.
\]

**Sketch of Proof.**

I) The proof is based on the following facts:

Fact 1) Under the conditions (C1) and (C2), the Cauchy problem

\[
\begin{align*}
\frac{h''(x)}{A(x)^2} + \frac{A'(x)}{A(x)}h'(x) &= -\mu h(x) \\
h(0) &= 1, \quad h'(0) = 0,
\end{align*}
\]

with $\mu \in \mathbb{C}$, admits a unique solution which we denote by $\varphi_{\mu}$ (see [5, 6]).

Fact 2) Define $\mu_\varphi$ so that $\mu_\varphi^2 = A^2 + (\varepsilon^2 - 1)\varphi^2$. For $i\lambda \neq \varepsilon\varphi$, the function
\[
\Psi_{A,\epsilon}(\lambda, x) := \varphi_{\mu_\varphi}(x) + \frac{1}{i\lambda - \varepsilon\varphi}\varphi_{\mu_\varphi}'(x).
\]

satisfies the problem (1.4).

Fact 3) We may rewrite (1.6) as
\[
\Psi_{A,\epsilon}(\lambda, x) = \varphi_{\mu_\varphi}(x) + (i\lambda + \varepsilon\varphi)\frac{sg(x)}{A(x)} \int_0^{|x|} \varphi_{\mu_\varphi}(t)A(t)dt,
\]

which implies that $\lambda \mapsto \Psi_{A,\epsilon}(\lambda, x)$ is analytic, and therefore the restriction on $\lambda$ can be dropped. The uniqueness follows by standard arguments.

II.1) The proof is inspired by Opdam’s proof of Proposition 6.1 in [3]. Using the fact that $\Psi_{A,\epsilon}$ satisfies
\[
\Psi_{A,\epsilon}'(\lambda, x) = \frac{A'(x)}{2A(x)} \left(\Psi_{A,\epsilon}(\lambda, x) - \Psi_{A,\epsilon}(\lambda, -x)\right) + \varepsilon\varphi \Psi_{A,\epsilon}(\lambda, -x) + i\lambda \Psi_{A,\epsilon}(\lambda, x),
\]

we prove that for all $x \in \mathbb{R}^+$, the derivative $|\Psi_{A,\epsilon}(\lambda, -x)|^2 + |\Psi_{A,\epsilon}'(\lambda, x)|^2 \leq 0$. This implies that for $x \in \mathbb{R}^+$, we have $|\Psi_{A,\epsilon}(\lambda, -x)|^2 + |\Psi_{A,\epsilon}'(\lambda, x)|^2 \leq |\Psi_{A,\epsilon}(\lambda, 0)|^2 + |\Psi_{A,\epsilon}'(\lambda, 0)|^2 = 2$.

II.2) Assume that $\Psi_{A,\epsilon}(\lambda, \cdot)$ is not strictly positive. Since $\Psi_{A,\epsilon}(\lambda, 0) = 1 > 0$, it follows that $\Psi_{A,\epsilon}(\lambda, \cdot)$ vanishes. Let $x_0$ be a zero of $\Psi_{A,\epsilon}(\lambda, \cdot)$ so that $|x_0| = \inf \{ |x| : \Psi_{A,\epsilon}(\lambda, x) = 0\}$.

We prove that $\Psi_{A,\epsilon}(\lambda, \pm x_0) = 0$ and $\Psi_{A,\epsilon}'(\lambda, \pm x_0) = 0$. Differentiating (1.8), we see that the second derivative of $\Psi_{A,\epsilon}(\lambda, \cdot)$ vanishes at $\pm x_0$. Repeating the same argument over and over again to get $\Psi_{A,\epsilon}(\lambda, \pm kx_0) = 0$ for all $k \in \mathbb{N}$. Since $\Psi_{A,\epsilon}(\lambda, \cdot)$ is a real analytic function, we deduce that $\Psi_{A,\epsilon}(\lambda, x) = 0$ for all $x \in \mathbb{R}$. This contradicts $\Psi_{A,\epsilon}(\lambda, 0) = 1$.

II.3) If $N = 0$ we show that for $\lambda \in \mathbb{C}$ we have
\[
|\Psi_{A,\epsilon}(\lambda, x)| \leq \Psi_{A,\epsilon}(0, x) e^{\text{Im}\lambda|\lambda||x|},
\]

where $\Psi_{A,\epsilon}(0, x) = 1$ for $\varepsilon = 0$, and $\Psi_{A,\epsilon}(0, x) \leq c_\epsilon(|x| + 1)e^{-\varepsilon(1 - \sqrt{1 - x^2})|x|}$ for $\varepsilon \neq 0$. So assume $N \geq 1$. The identity (1.8) allows us to express the derivatives of $\Psi_{A,\epsilon}(\lambda, \cdot)$ in terms of lower order derivatives. On the other hand, since $A'/2A$ satisfies the condition (C3), it follows that
\[
\left|\left(\frac{A'(x)}{2A(x)}\right)^{(N)}\right| \leq C, \quad \forall |x| \geq x_0 \text{ with } x_0 > 0.
\]
II.4) If \( M = 0 \) this is just (1.9). So assume \( M \geq 1 \). If \( x = 0 \), the statement follows from Liouville’s theorem. If \( x \neq 0 \), apply Cauchy’s integral formula for \( \Psi_{A,\varepsilon}(\lambda, x) \) over a circle with radius proportional to \( \frac{1}{|x|} \), centered at \( \lambda \) in the complex plane. \( \square \)

2. The existence and the positivity of an intertwining operator

Recall from the (sketch of) proof of Theorem 1.1 the function \( \varphi_\mu \) which is the unique solution to the Cauchy problem (1.5). By [5] we have the following Laplace type representation

\[
\varphi_\mu(x) = \int_0^{[x]} K(|x|, y) \cos(\mu y)dy \quad x \in \mathbb{R}^*, \quad (2.1)
\]

where \( K(|x|, \cdot) \) is a non-negative even continuous function supported in \([-|x|, |x|]\). Using a Delsarte type operator introduced in [14, Proposition 2.1] (see also Theorem 5.1 in [12]), we prove that the integral representation (2.1) can be rewritten as

\[
\varphi_{\mu,\varepsilon}(x) = \int_0^{[x]} K_{\varepsilon}(|x|, y) \cos(\lambda y)dy \quad x \in \mathbb{R}^*, \quad (2.2)
\]

where the relationship between \( \mu,\varepsilon \) and \( \lambda \) is given by \( \mu^2 = \lambda^2 + (\varepsilon^2 - 1)\rho^2 \). Here \( K_{\varepsilon}(|x|, \cdot) \) is even, continuous and supported in \([-|x|, |x|]\). Now, in view of the expression (1.7) of the eigenfunction \( \Psi_{A,\varepsilon}(\lambda, x) \), we deduce that

\[
\Psi_{A,\varepsilon}(\lambda, x) = \int_{|y|<|x|} \mathbb{K}_{\varepsilon}(x, y) e^{i\lambda y}dy \quad x \in \mathbb{R}^*, \quad (2.3)
\]

where \( \mathbb{K}_{\varepsilon}(x, \cdot) \) is a continuous function supported in \([-|x|, |x|]\). This integral representation of \( \Psi_{A,\varepsilon}(\lambda, x) \) is the starting point for obtaining an intertwining operator between the operator \( \Lambda_{A,\varepsilon} \) and the ordinary derivative \( d/dx \). More precisely, for \( f \in C^\infty(\mathbb{R}) \) we define \( V_{A,\varepsilon}f \) by

\[
V_{A,\varepsilon}f(x) = \begin{cases} 
\int_{|y|<|x|} \mathbb{K}_{\varepsilon}(x, y) f(y)dy & x \neq 0 \\
 f(0) & x = 0
\end{cases} \quad (2.4)
\]

where the kernel \( \mathbb{K}_{\varepsilon}(x, y) \) is as in (2.3).

Theorem 2.1. 1) The operator \( V_{A,\varepsilon} \) is the unique automorphism of \( C^\infty(\mathbb{R}) \) such that

\[
\Lambda_{A,\varepsilon} \circ V_{A,\varepsilon} = V_{A,\varepsilon} \circ \frac{d}{dx}. \quad (2.5)
\]

2) For all \( (x, y) \in \mathbb{R}^* \times \mathbb{R} \), the kernel \( \mathbb{K}_{\varepsilon}(x, y) \) is positive.

The positivity of \( V_{A,\varepsilon} \) played a fundamental role in [2] in establishing an analogue of Beurling’s theorem, and its relatives such as theorems of type Gelfand-Shilov, Morgan’s, Hardy’s, and Cowling-Price in the setting of this paper.

For \( \varepsilon = 0 \) and 1, the positivity of \( \mathbb{K}_{\varepsilon}(x, y) \) can be found in [16] and [17].

Sketch of Proof of Theorem 2.1. 1) Write \( f \) as the superposition \( f = f_e + f_o \) of an even function \( f_e \) and an odd function \( f_o \). We prove that \( V_{A,\varepsilon} \) can be expressed as

\[
V_{A,\varepsilon}f(x) = (\text{id} + \varepsilon \mathcal{Q}) \circ \Lambda_{A,\varepsilon} f_e(x) + \mathbb{M} \circ \Lambda_{A,\varepsilon} f_o(x), \quad (2.6)
\]
where
\[ \mathcal{M}h(x) := \frac{sg(x)}{A(x)} \int_0^{|x|} h(t)A(t)dt \]
and
\[ \mathcal{A}_x f(x) := \frac{1}{2} \int_{|y|<|x|} K_\varepsilon(|y|, y)f(y)dy, \]
with \( K_\varepsilon(|x|, y) \) is as in (2.2). The transform \( \mathcal{M} \) is an isomorphism from \( C_c^\infty(\mathbb{R}) \) to \( C_0^\infty(\mathbb{R}) \) and its inverse is given by \( \mathcal{M}^{-1} = \frac{d}{dx} + \frac{1}{A(x)} \) id, while \( \mathcal{A}_x \) is an automorphism of \( C_c^\infty(\mathbb{R}) \). Further, \( (d^2/dx^2 + (A'/A)(x) d/dx) \circ \mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon \circ (d^2/dx^2 - \varepsilon^2 g^2) \) and \( \Lambda_{A,\varepsilon} \circ \mathcal{M} = \text{id} + \varepsilon \mathcal{M} \).

Now, the first statement follows from (2.6). The uniqueness of \( V_{A,\varepsilon} \) is due to the fact that the unique solution \( \Psi_{A,\varepsilon} \) to the problem (1.4) can be written as \( \Psi_{A,\varepsilon}(\lambda, x) = V_{A,\varepsilon}(e^{\lambda t})g(x) \) (see (2.3)).

2) For a linear operator \( L \) on \( \mathcal{D}(\mathbb{R}) \) we denote by \( \langle L \rangle \) its dual operator in the sense that
\[ \int_{\mathbb{R}} Lf(x)g(x)A(x)dx = \int_{\mathbb{R}} f(y)\langle L \rangle g(y)dy. \]

It is more convenient to deal with the dual operator \( L^{\infty} \) than with \( V_{A,\varepsilon} \). For \( g \in \mathcal{D}(\mathbb{R}) \), we have
\[ \langle V_{A,\varepsilon}g \rangle(y) = \int_{|y|>|x|} K_{\varepsilon}(x, y)g(x)A(x)dx. \]
We shall prove that if \( g \geq 0 \) then \( \langle V_{A,\varepsilon}g \rangle \geq 0 \).

For \( s > 0 \) and \( u, v \in \mathbb{R} \), let \( p_s(u, v) := \frac{e^{-|u-v|^2}}{2\sqrt{s\pi}} \) be the Euclidean heat kernel. The key observation is that
\[ \int_{\mathbb{R}} g(x)V_{A,\varepsilon}(p_s(u, \cdot))A(x)dx = \int_{\mathbb{R}} \langle V_{A,\varepsilon}g \rangle(x; p_s(u, \cdot))dx \]
where \( q_s(r) := p_s(r, 0) \) and * is the Euclidean convolution product. Thus, the positivity of \( \langle V_{A,\varepsilon}g \rangle \), reduces to the positivity of \( V_{A,\varepsilon}(p_s(u, \cdot)) \). Now, by (2.4) and (2.3) we prove that for every \( s > 0 \) and \( u, x \in \mathbb{R} \), we have
\[ V_{A,\varepsilon}(p_s(u, \cdot))(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_{A,\varepsilon}(-\lambda, x)e^{-\lambda|u|}e^{i\lambda x}d\lambda, \]
which allowed us to show that \( V_{A,\varepsilon}(p_s(u, \cdot))(x) \geq 0. \)

3. \( L^p \)-Fourier analysis

For \( f \in L^1(\mathbb{R}, A(x)dx) \) put
\[ \mathcal{F}_{A,\varepsilon}f(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{A,\varepsilon}(\lambda, -x)A(x)dx, \]
which is well defined, by Theorem 1.1. I.1.1

For \( -1 \leq \varepsilon \leq 1 \) and \( 0 < p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \), set \( \partial_{p,\varepsilon} := \frac{2}{p} - 1 - \sqrt{1 - \varepsilon^2} \). Observe that \( 1 \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \leq 2 \). We introduce the tube domain
\[ C_{p,\varepsilon} := \{ \lambda \in \mathbb{C} \mid |\text{Im} \lambda| \leq \partial_{p,\varepsilon} \}. \]

**Theorem 3.1.** Let \( f \in L^p(\mathbb{R}, A(x)dx) \) with \( 1 \leq p \leq \frac{2}{1 + \sqrt{1 - \varepsilon^2}} \). Then the following properties hold.

1) For \( p > 1 \), the Fourier transform \( \mathcal{F}_{A,\varepsilon}(f)(\lambda) \) is well defined for all \( \lambda \) in \( \mathfrak{C}_{p,\varepsilon} \), the interior of \( C_{p,\varepsilon} \). Moreover, for all \( \lambda \in \mathfrak{C}_{p,\varepsilon} \), we have \( |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| \leq c\|f\|_p \). For \( p = 1 \), we may replace above the open domain \( \mathfrak{C}_{p,\varepsilon} \) by \( C_{p,\varepsilon} \).
2) The function $\mathcal{F}_{A,\varepsilon}(f)$ is holomorphic on $\hat{\mathbb{C}}_{p,\varepsilon}$.

3) (Riemann-Lebesgue lemma) We have $\lim_{\lambda \to \infty} |\mathcal{F}_{A,\varepsilon}(f)(\lambda)| = 0$.

4) The Fourier transform $\mathcal{F}_{A,\varepsilon}$ is injective on $L^p(\mathbb{R}, A(x)dx)$ for $1 \leq p \leq \frac{2}{1 + \sqrt{1 - p}}$.

**Sketch of Proof.** The first two statements follow from the estimate of $\Psi_{A,\varepsilon}(\lambda, x)$ given in Theorem 1.1.II.4 (with $N = 0$), the fact that $A(x) \leq c|x|^p e^{2|\varepsilon|}$ (a consequence of the hypothesis (C3) on the function $A$), the fact that $\Psi_{A,\varepsilon}(\lambda, \cdot)$ is holomorphic in $\lambda$, and Morera’s theorem. To extend the first statement from $\hat{\mathbb{C}}_{p,\varepsilon}$ to $\mathbb{C}_{p,\varepsilon}$ when $p = 1$, in addition, we show that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq 2$ for all $\lambda \in \mathbb{C}_{1,\varepsilon}$ and for all $x \in \mathbb{R}$. The proof uses the maximum modulus principle and the fact that $|\Psi_{A,\varepsilon}(\lambda, x)| \leq \Psi_{A,\varepsilon}(i \Im \lambda, x)$. For the Riemann-Lebesgue lemma, a classical proof for the Euclidean Fourier transform carries over. The forth statement is based on the following steps:

**Step 1)** For $f \in L^p(\mathbb{R}, A(x)dx)$ et $g \in \mathcal{D}(\mathbb{R})$ we show, by means of Hölder’s inequality and the first statement, that the mapping $f \mapsto (f, g)_A := \int_{\mathbb{R}} f(x)g(-x)A(x)dx$ and $f \mapsto (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_{\varepsilon}} := \int_{\mathbb{R}} \mathcal{F}_{A,\varepsilon}(f)(\lambda)\mathcal{F}_{A,\varepsilon}(g)(\lambda)(1 - \frac{\lambda^2}{\varepsilon^2})\pi_{\varepsilon}(d\lambda)$ are continuous functionals on $L^p(\mathbb{R}, A(x)dx)$. Here $\pi_{\varepsilon}$ is a positive measure with support $\mathbb{R} \setminus \sqrt{1 - \varepsilon^2}, \sqrt{1 - \varepsilon^2}$. 

**Step 2)** We show that $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_{\varepsilon}}$ for all $f, g \in \mathcal{D}(\mathbb{R})$. Thus, by Step 1), $(f, g)_A = (\mathcal{F}_{A,\varepsilon}(f), \mathcal{F}_{A,\varepsilon}(g))_{\pi_{\varepsilon}}$ for all $f \in L^p(\mathbb{R}, A(x)dx)$.

Hence, if we assume that $f \in L^p(\mathbb{R}, A(x)dx)$ and that $\mathcal{F}_{A,\varepsilon}(f) = 0$, then for all $g \in \mathcal{D}(\mathbb{R})$ we have $(f, g)_A = 0$ and therefore $f = 0$.

For $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1 + \sqrt{1 - p}}$, denote by $\mathcal{S}_p(\mathbb{R})$ the space consisting of all functions $f \in C^\infty(\mathbb{R})$ such that

$$a_{s,k}^{(p)}(f) := \sup_{x \in \mathbb{R}}((|x| + 1)^s e^{\frac{2}{\varepsilon}(|x|)} |f^{(k)}(x)|) < \infty,$$

for any $s, k \in \mathbb{N}$. The topology of $\mathcal{S}_p(\mathbb{R})$ is defined by the seminorms $a_{s,k}^{(p)}$. The space $\mathcal{S}(\mathbb{R})$ of smooth functions with compact support on $\mathbb{R}$ is a dense subspace of $\mathcal{S}_p(\mathbb{R})$; see for instance [3], Appendix A).

Let $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ be the Schwartz space consists of all complex valued functions $h$ that are analytic in the interior of $\mathbb{C}_{p,\varepsilon}$, and such that $h$ together with all its derivatives extend continuously to $\mathbb{C}_{p,\varepsilon}$ and satisfy

$$T_{t,\ell}^{(p,\varepsilon)}(h) := \sup_{A \in \mathbb{C}_{p,\varepsilon}} (|A| + 1)^t |h^{(\ell)}(A)| < \infty,$$

for any $t, \ell \in \mathbb{N}$. The topology of $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ is defined by the seminorms $T_{t,\ell}^{(p,\varepsilon)}$.

Using Anker’s approach [1] we prove the following result:

**Theorem 3.2.** Let $-1 \leq \varepsilon \leq 1$ and $0 < p \leq \frac{2}{1 + \sqrt{1 - p}}$. Then the Fourier transform $\mathcal{F}_{A,\varepsilon}$ is a topological isomorphism between $\mathcal{S}_p(\mathbb{R})$ and $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$.

**Sketch of Proof.** The proof is based on the following steps:

**Step 1)** The transform $\mathcal{F}_{A,\varepsilon}$ maps $\mathcal{S}_p(\mathbb{R})$ continuously into $\mathcal{S}(\mathbb{C}_{p,\varepsilon})$ and is injective.
Step 2) The inverse Fourier transform $\mathcal{F}^{-1}_{A\epsilon}: PW(\mathbb{C}) \rightarrow \mathcal{D}(\mathbb{R})$ given by

$$\mathcal{F}^{-1}_{A\epsilon} h(x) = c \int_{\mathbb{R}} h(\lambda) \Psi_{A\epsilon}(\lambda, x) \left(1 - \frac{\epsilon^2}{\lambda^2}\right) \pi_{\epsilon}(d\lambda)$$

is continuous for the topologies induced by $\mathcal{S}(\mathbb{C}_{p,\epsilon})$ and $\mathcal{S}(\mathbb{R})$. Here $PW(\mathbb{C})$ is the space of entire functions on $\mathbb{C}$ which are of exponential type and rapidly decreasing, and $\pi_{\epsilon}$ is a positive measure with support $\mathbb{R} \setminus [-1/\epsilon^2, 1/\epsilon^2]$.

We pin down that $PW(\mathbb{C})$ is dense in $\mathcal{S}(\mathbb{C}_{p,\epsilon})$.

For Step 1), we prove that $\mathcal{F}_{A\epsilon}(f)$ is well defined for all $f \in \mathcal{S}(\mathbb{R})$. This is due to the growth estimates for $\Psi_{A\epsilon}(\lambda, x)$ stated in Theorem I.II.4. Moreover, since the map $\lambda \mapsto \Psi_{A\epsilon}(\lambda, x)$ is holomorphic on $\mathbb{C}$, it follows that for all $f \in \mathcal{S}(\mathbb{R})$, the function $\mathcal{F}_{A\epsilon}(f)$ is analytic in the interior of $\mathbb{C}_{p,\epsilon}$, and continuous on $\mathbb{C}_{p,\epsilon}$. Finally, we prove that given a continuous seminorm $\tau$ on $\mathcal{S}(\mathbb{R})$, there exists a continuous seminorm $\sigma$ on $\mathcal{S}(\mathbb{R})$ such that $\tau(\mathcal{F}_{A\epsilon}(f)) \leq c \sigma(f)$ for all $f \in \mathcal{S}(\mathbb{R})$. Indeed, by means of the growth estimates for $\partial_{\lambda}^j \Psi_{A\epsilon}(\lambda, x)$ stated in Theorem I.II.4, we show first that

$$\left| (i\lambda)^j \mathcal{F}_{A\epsilon}(f)(\lambda) \right| \leq c \int_{\mathbb{R}} \left| A_{\epsilon, A \tau}(f, x) \right| \left(1 + \frac{\epsilon^2}{\lambda^2}\right)^{\frac{j+1}{2}} A(x) dx,$$

and then we prove that $|A_{\epsilon, A \tau}(f, x)|$ is bounded by finite sums of the derivatives of $f$. Thus $\tau(\mathcal{F}_{A\epsilon}(f)) \leq c \sum_{j=0}^{\infty} \sigma(f)$ for all $f \in \mathcal{S}(\mathbb{R})$. The injectivity of $\mathcal{F}_{A\epsilon}$ on $\mathcal{S}(\mathbb{R})$ follows from Theorem I.II.4 and the fact that $\mathcal{S}(\mathbb{R}) \subseteq L^1(\mathbb{R}, A(x) dx)$ for all $q < \infty$ so that $p \leq q$.

For Step 2), we start by proving a Paley-Wiener theorem for $\mathcal{F}_{A\epsilon}$, i.e. we prove that $\mathcal{F}_{A\epsilon}$ is a linear isomorphism between the space $\mathcal{D}(\mathbb{R})$ of smooth functions with support inside $[-R, R]$ and the space $PW_R(\mathbb{C})$ of entire functions which are of $R$-exponential type and rapidly decreasing. We note that $PW(\mathbb{C}) = \bigcup_{R>0} PW_R(\mathbb{C})$.

Next, we take $f \in \mathcal{D}(\mathbb{R})$ and $h \in PW(\mathbb{C})$ so that $f = \mathcal{F}^{-1}_{A\epsilon}(h)$. Denote by $g$ the image of $h$ by the inverse Euclidean Fourier transform $\mathcal{F}^{-1}_{euc}$. Making use of the Paley-Wiener theorem for $\mathcal{F}_{A\epsilon}$ and the classical Paley-Wiener theorem for $\mathcal{F}_{euc}$, we have the following support conservation property: $\text{supp}(f) \subset I_R := [-R, R] \Leftrightarrow \text{supp}(g) \subset I_R$.

For $j \in \mathbb{N}_{\geq 1}$, let $\omega_j \in C^{\infty}(\mathbb{R})$ with $\omega_j = 0$ on $I_{j-1}$ and $\omega_j = 1$ outside of $I_j$. Assume that $\omega_j$ and all its derivatives are bounded, uniformly in $j$. Write $g_j := \omega_j g$, and define $h_j := \mathcal{F}_{euc}(g_j)$ and $f_j := \mathcal{F}^{-1}_{A\epsilon}(h_j)$. Note that $g_j = g$ outside $I_j$. Hence, by the above support property, $f_j = f$ outside $I_j$.

In view of the growth estimate for $\partial_{\epsilon}^k \Psi_{A\epsilon}(\lambda, x)$ stated in Theorem I.II.3, we prove that for $j \in \mathbb{N}_{\geq 1}$,

$$\sup_{x \in I_{j+1}\setminus I_j} \left(1 + \frac{\epsilon^2}{\lambda^2}\right)^{\frac{j+1}{2}} e^{\frac{\epsilon^2}{\lambda^2} |x|} |f_j^{(k)}(x)| \leq c \sum_{r=0}^{j+3} r^3 \tau_{\epsilon, A}(h),$$

for some integer $r > 0$. For $I_1$, we show first that there exists an integer $m_k \geq 1$ such that

$$|\partial_{\epsilon}^k \Psi_{A\epsilon}(\lambda, x)| \leq c(|\lambda| + 1)^{m_k} (|\lambda| + 1)e^{-\epsilon^2 |x|}$$

for some integer $k > 0$. □
Details of this paper will be given in another article [3).

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