Domains of holomorphy for irreducible admissible uniformly bounded representations of simple Lie groups

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DOMAINS OF HOLOMORPHY FOR IRREDUCIBLE ADMISSIBLE UNIFORMLY BOUNDED REPRESENTATIONS OF SIMPLE LIE GROUPS

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Abstract. In this note, we address a question asked in [Krö08] on the classification of domains of holomorphy of irreducible admissible Banach representations for simple real Lie groups. When G is not of Hermitian type, and the representation is either irreducible uniformly bounded Hilbert or irreducible admissible isometric on a certain class of Banach spaces, we give a full answer. When the group G is Hermitian, our results are only partial.

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1. Introduction

Let G be a simple, non-compact, and connected real algebraic Lie group with Lie algebra $\mathfrak{g}$, and let K be a maximal compact subgroup of G. Further, let $G_{\mathbb{C}}$ be the universal complexification of G, and $K_{\mathbb{C}}$, that of K. We can assume without loss of generality that $G \subseteq G_{\mathbb{C}}$ and that $G_{\mathbb{C}}$ is simply connected (see [Krö08, Remark 5.2]). Let (π, V) be a Banach representation of G i.e. there is a continuous action

$$G \times V \longrightarrow V, \qquad (g, v) \mapsto g \cdot v, \qquad g \in G, v \in V$$

of G on a Banach space V which gives rise to a group homomorphism $g \mapsto \pi(g)$ with $\pi(g)v := g \cdot v$. For much of this introductory material [Wal88, Chapters 1, 3]) is a good reference. We call a vector $v \in V$ an *analytic vector* if the *orbit map* $\gamma_v : G \longrightarrow V$ of v, given by $g \longmapsto \pi(g)v$ and a priori continuous, extends to a holomorphic (V-valued) function on an open neighbourhood of G in $G_{\mathbb{C}}$ or equivalently, if γ_v is a real analytic (V-valued) map. Note that the space V^ω of

Date: November 13, 2015.

The authors would like to thank Bernhard Krötz for introducing the subject to them, and for helpful correspondence.

analytic vectors for the representation (π, V) is a G-invariant subspace which is dense in V. Recall that a Banach representation (π, V) is called *admissible* if for any finite-dimensional K-module W, we have that $\dim \mathrm{Hom_K(W, V_K)} < \infty$. If π is irreducible and admissible, then we know that the space V_K of K-finite vectors of (π, V) is contained in V^ω. So given a non-zero vector $v \in V_K$, one might ask to which natural domain in $G_\mathbb{C}$ does its orbit map γ_v extend holomorphically. A first remark is that it is not unreasonable to expect that such a domain would be independent of the vector $v \in V_K$ because $\mathcal{U}(\mathfrak{g}_\mathbb{C}) \cdot v = V_K$ as $\mathcal{U}(\mathfrak{g}_\mathbb{C})$-modules. Here $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ is the universal enveloping algebra of $\mathfrak{g}$. In fact, in [Krö08] Krötz proved a classification of such domains when G is as above and π is a unitary irreducible representation, and further proposed the following generalisation.

Conjecture: Let (π, V) be an irreducible admissible Banach representation of G. Given $0 \neq v \in V_K$, there exists a unique maximal $G \times K_\mathbb{C}$-invariant domain $D_\pi \subset G_\mathbb{C}$ such that the orbit map $\gamma_v : g \mapsto \pi(g)v$ extends to a holomorphic map $\widetilde{\gamma}_v : D_\pi \longrightarrow V$. In more detail, we have

i) $D_\pi = G_\mathbb{C}$ if π is finite-dimensional.
ii) $D_\pi = \widetilde{\widetilde{\Xi}}^+$ if G is Hermitian, and π is a highest weight representation.
iii) $D_\pi = \widetilde{\widetilde{\Xi}}^-$ if G is Hermitian, and π is a lowest weight representation.
iv) $D_\pi = \widetilde{\widetilde{\Xi}}$ in all other cases.

Here we wrote $\widetilde{\widetilde{\Xi}} = q^{-1}(\Xi)$, $\widetilde{\widetilde{\Xi}}^\pm = q^{-1}(\Xi^\pm)$, where $q : G_\mathbb{C} \longrightarrow G_\mathbb{C}/K_\mathbb{C}$ is the canonical projection, and where Ξ is the so-called *crown domain*, Ξ^+, Ξ^- are related domains in $\mathbb{X}_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$ containing the Riemannian symmetric space $\mathbb{X} = G/K$. Notice that for a finite-dimensional representation π, the fact that $D_\pi = G_\mathbb{C}$ follows directly from definitions. Henceforth all the representations that we consider will be infinite-dimensional. We also remark here that the conjecture is to be viewed as a complex-geometric description of the admissible dual of G.
Recall that a Banach representation (π, V) of G is called *uniformly bounded* if there exists a constant $C > 0$ such that

$$\|\pi(g)\| \leq C \qquad \forall\, g \in G. \tag{1}$$

Here $\|\cdot\|$ denotes the operator norm on the space of bounded linear operators on V. A Banach representation π is called *isometric* if $\|\pi(g)v\| = \|v\|$, $\forall\, g \in G$, $\forall\, v \in V$. Further, a Banach space is called *uniformly convex and uniformly smooth*, abbreviated henceforth as *ucus*, if both V and its continuous dual V^* are uniformly convex (See [BFGM07, Section 2.a], for instance). We remark that Hilbert spaces as well as L^p-spaces, $1 < p < \infty$, belong to this class.
In this note, we give a proof of the conjecture in the following interesting cases:

- *uniformly bounded* irreducible Hilbert representations (except when G is Hermitian and π is not a highest/lowest weight representation of G).

- *isometric* irreducible admissible representations on ucus Banach spaces (except when G is Hermitian and π is not a highest/lowest weight representation of G).

Since the class of groups which are not Hermitian is a very large one, our results, while not complete, are, in our opinion, of interest.

Remark 1. Given a uniformly bounded Banach representation (π, V), we can endow V with an equivalent norm $\|v\|_{\text{isom}} := \sup_{g\in G} \|\pi(g)v\|$, $v \in V$, so that π is isometric with respect to the new norm $\|\cdot\|_{\text{isom}}$.

It is worth mentioning here that uniformly bounded representations on Hilbert spaces have been well-studied in the context of harmonic analysis on semisimple Lie groups (Kunze-Stein phenomenon, $(\mathfrak{g}, K)$-module cohomology etc.). This is true also for representations on L^p-spaces while on general ucus Banach spaces, to our knowledge, such representations have been studied in the context of Property (T) and rigidity.

One underlying idea in the proof is that the holomorphic extension of the orbit map γ_v of a non-zero K-finite vector v depends essentially on the smooth Fréchet structure of the Casselman-Wallach globalisation of the underlying $(\mathfrak{g}, K)$-module. Another key observation is the vanishing at infinity of the matrix coefficients for irreducible uniformly bounded representations, from which one derives the appropriate properness of the G-action.

2. Complex geometric setting

We begin by briefly describing the complex geometric setting, and refer to [KS05], [KO08] for comprehensive accounts. With G and K as before, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition such that K is the analytic subgroup corresponding to $\mathfrak{k}$, and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Set $\widehat{\Omega} := \{Y \in \mathfrak{p} \mid \operatorname{spec}(\operatorname{ad}(Y)) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})\}$, and $\Omega = \widehat{\Omega} \cap \mathfrak{a}$ which is invariant under the Weyl group $W = W(\mathfrak{g}, \mathfrak{a})$ associated to the set of restricted roots $\Sigma(\mathfrak{g}, \mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{a})$. Then we define the domain $\widetilde{\Xi} := G\exp(i\Omega)K_{\mathbb{C}} \subset G_{\mathbb{C}}$, and thereby the so-called *elliptic model* of the crown domain by $\Xi = \widetilde{\Xi}/K_{\mathbb{C}}$. It is known that Ξ is a Stein domain admitting a proper G-action. Notice that $\mathbb{X} \subset \Xi \subset \mathbb{X}_{\mathbb{C}}$. Now, if we define the set of elliptic elements in $\mathbb{X}_{\mathbb{C}}$ by $\mathbb{X}_{\mathbb{C},\,\text{ell}} := G\exp(i\mathfrak{a})K_{\mathbb{C}}/K_{\mathbb{C}}$, then it is known that the crown domain Ξ is the maximal domain contained in $\mathbb{X}_{\mathbb{C},\,\text{ell}}$ which admits a proper G-action. However, Ξ is *not* a maximal domain in all of $\mathbb{X}_{\mathbb{C}}$ which admits a proper G-action. This is related to an alternative description of the crown domain - the so-called *unipotent model*, and thence to the domains $\Xi^{\pm}$. Fixing an order on the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition, and set Λ to be the connected component of $\{Y \in \mathfrak{n} \mid \exp(iY)K_{\mathbb{C}}/K_{\mathbb{C}} \in \Xi\}$ containing 0. Then, as described in [KO08, Section 8], we have the unipotent model $\Xi = G\exp(i\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$ of the crown domain. This model enables one to have a precise understanding of the boundary of

Ξ which, in turn, then allows for a description of the directions in which the crown domain Ξ can be extended to obtain a domain D in such a way that it still admits a proper G-action. Denote by $\partial\Xi$ the topological boundary of the crown domain Ξ. The *elliptic* part $\partial_{\text{ell}}\Xi$ of the boundary is then given by $\partial_{\text{ell}}\Xi = G\exp(i\partial\,\Omega)K_{\mathbb{C}}$, and we define the *unipotent* part of the boundary to be $\partial_u\Xi := \partial\,\Xi \setminus \partial_{\text{ell}}\Xi$. Indeed it can be seen that $\partial_u\Xi = G\exp(i\partial\Lambda)K_{\mathbb{C}}/K_{\mathbb{C}}$. Since the G-stabiliser of any point in $\partial_{\text{ell}}\Xi$ is a non-compact subgroup of G, it follows that for any G-invariant domain D with $\mathbb{X} \subseteq D \subseteq \mathbb{X}_{\mathbb{C}}$ on which G acts properly, we have that $D \cap \partial_{\text{ell}}\Xi = \emptyset$. Further, $\partial_u\Xi \not\subset D$, and so if $D \not\subset \Xi$, then $D \cap \partial_u\Xi \neq \emptyset$ (in fact, D intersects the so-called *regular* part of $\partial_u\Xi$).

For later use, we also mention that a simple real Lie group is called *Hermitian* if the corresponding symmetric space G/K admits the structure of a complex manifold. In this case, as a $\mathfrak{k}_{\mathbb{C}}$-module, $\mathfrak{p}_{\mathbb{C}}$ splits into irreducible components $\mathfrak{p}_{\mathbb{C}}^{+}$, and $\mathfrak{p}_{\mathbb{C}}^{-}$, and let $\mathsf{P}^{\pm}$ denote the corresponding analytic subgroups of $G_{\mathbb{C}}$. Then it can be seen that, $\widetilde{\Xi}^{\pm} = GK_{\mathbb{C}}\mathsf{P}^{\pm}$.

3. Holomorphic extensions of irreducible admissible representations

In this section, we relate the complex geometric setting discussed above to irreducible admissible G-representations. The first important observation in this direction is the following result on the holomorphic extension of the orbit map of a non-zero K-finite vector to the $G \times K_{\mathbb{C}}$-invariant domain $\widetilde{\Xi}$. This is essentially [KS04, Theorem 3.1], and we briefly sketch the idea of the proof for the convenience of the reader and because of its importance. Note that admissibility is a crucial assumption in what follows.

Theorem 1. *Let G be a connected non-compact simple Lie group, and (π, V) an irreducible admissible Banach representation of G. If $0 \neq v \in V_K$ is a K-finite vector, then the orbit map $\gamma_v : G \longrightarrow V$ extends to a G-equivariant holomorphic map on $\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}$.*

Proof. Let V^∞ be the subspace of smooth vectors in V. Then $V_K \subset V^\infty$ and it is clear that $\gamma_v(G) \subset V^\infty$. Now, by the admissibility of π, V^∞ equipped with the topology induced by the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ becomes the (smooth) Casselman-Wallach globalisation of the Harish-Chandra module V_K (See [Cas89], [Wal92, Chapter 11] or see [BK14] for a different approach). On the other hand, since the topology on V^∞ as a Casselman-Wallach globalisation is finer than the topology on it induced from V, we only need to prove that the orbit map $\gamma_v : G \longrightarrow V^\infty$ extends to a G-equivariant map on $\widetilde{\Xi} = G\exp(i\Omega)K_{\mathbb{C}}$ which is holomorphic with respect to the topology of the Casselman-Wallach globalisation (π^∞, V^∞).

Now by Casselman's submodule theorem, (π^∞, V^∞) is embedded (as a closed G-submodule) into a smooth principal series representation $(\pi^\infty_{\tau,\lambda}, \mathcal{H}^\infty_{\tau,\lambda})$ arising from a minimal parabolic subgroup P of G. In this way, we can assume that $v \in V^\infty \subseteq H^\infty_{\tau,\lambda}$.

Then we can follow the argument in [KS04, Theorem 3.1] in order to conclude that $\gamma_v : G \longrightarrow V^\infty$ extends to a G-equivariant holomorphic map on $\widetilde{\Xi}$. □

The above theorem tells us that the domains of holomorphy that we seek necessarily contain the domain $\widetilde{\Xi}$. The question then is whether such domains can be larger than $\widetilde{\Xi}$, and if so, to understand the connection between the geometry of the domains and representation theory. This crucial link is established by using the vanishing property of matrix coefficients at infinity of the irreducible admissible representations under consideration, and relating this to properness of the G-action.

We can then use the finer group theoretic structure according to whether G is Hermitian or non-Hermitian to establish precisely what the sought-after domains of holomorphy are. We first remark that a non-trivial irreducible uniformly bounded Hilbert representation is necessarily admissible (see [BW80, Theorem 5.2, Chapter IV]). Then we have

Proposition 1. *Let (π, V) be either an infinite-dimensional uniformly bounded irreducible Hilbert representation or an infinite irreducible isometric representation of G on a ucus Banach space. Then all the matrix coefficients of π vanish at infinity.*

Proof. For the uniformly bounded Hilbert case see [BW80, Theorem 5.4]). The result for ucus Banach spaces is due to Shalom and can be found, for instance, in [BFGM07, Theorem 9.1] □

This proposition then enables us to adapt the argument in [KO08, Theorem 4.3], used there in the context of unitary irreducible representations, to conclude the following theorem.

Theorem 2. *Let (π, V) be either an infinite-dimensional uniformly bounded irreducible Hilbert representation or an infinite-dimensional irreducible isometric representation on a ucus Banach space of a noncompact simple real Lie group G. Then for any maximal $G \times K_{\mathbb{C}}$-invariant domain $\widetilde{D} \subset G_{\mathbb{C}}$ to which the orbit map γ_v of a non-zero K-finite vector v extends holomorphically, G acts properly on $\widetilde{D}/K_{\mathbb{C}}$.*

The following lemma is what allows us to prove the above theorem not just for uniformly bounded Hilbert representations but also for irreducible isometric representations on ucus Banach spaces.

Lemma 1. *Let (π, V) be a infinite-dimensional irreducible uniformly bounded representation of G on a Hilbert space or an infinite dimensional irreducible isometric representation on a ucus Banach space. Then G acts properly on $V \setminus \{0\}$.*

Proof. The G-action on $V \setminus \{0\}$ is proper if for every compact subset W of $V \setminus \{0\}$, the set $W_G := \{g \in G \,|\, \pi(g)W \cap W \neq \emptyset\}$ is also compact. Suppose not. Then there exist sequences $(g_n)_{n\in\mathbb{N}}$ in W_G and $(v_n)_{n\in\mathbb{N}}$ in $V \setminus \{0\}$ such that $\lim_{n\to\infty} g_n = \infty$ but $\pi(g_n)v_n \in W$ for all $n \in \mathbb{N}$. By the compactness of W we have, by going to subsequences if necessary, that there exist $v, v' \in W$ such that $\lim_{n\to\infty} v_n = v$ and $\lim_{n\to\infty} \pi(g_n)v_n = v'$. Now, since π is uniformly bounded, we have that $\|\pi(g_n)v_n - \pi(g_n)v\| \leq C\,\|v_n - v\|$, and so it follows that $\lim_{n\to\infty} \pi(g_n)v = v'$. Since $v' \neq 0$, there is an $f \in V^*$ such that $\langle v', f\rangle \neq 0$. But then $\lim_{n\to\infty} \langle \pi(g_n)v, f\rangle = \langle v', f\rangle \neq 0$ - a contradiction to the vanishing of all the matrix coefficents at infinity given by Proposition 1. This concludes the proof. □

For a uniformly bounded representation, we note that sub-multiplicativity of the operator norm then gives us also a bound from below

$$\frac{1}{C} \leq \|\pi(g)\| \leq C \qquad \forall g \in G,$$

where C is as in (1). Now using this, together with Lemma 1, and the vanishing of matrix coefficients at infinity in Proposition 1, we can use an argument similar to the one in the proof of [KO08, Theorem 4.3] to prove Theorem 2. We do not give all the details and instead to refer to [KO08, Theorem 4.3] for it.

Now suppose G is non-Hermitian. Then as in [Krö08, Lemma 4.4], using an A_2-reduction, we obtain that for any root $\alpha \in \Sigma$, and $Y \in \mathfrak{g}_\alpha$ there exists an $m \in M = Z_K(\mathfrak{a})$ such that $\mathrm{Ad}\,(m)Y = -Y$. This together with a certain precise description of the boundary of $\widetilde{\Xi}$ in the $SL(2,\mathbb{R})$ case leads to the geometric result, as in [Krö08, Theorem 4.1], that for a G-invariant domain D such that $\mathbb{X} \subset D \subset \mathbb{X}_\mathbb{C}$ on which G acts properly, we have that $D \subset \Xi$. Therefore, we have that $\widetilde{D}/K_\mathbb{C}$ is contained in the crown domain Ξ. Now suppose $\widetilde{D}$ is a maximal $G \times K_\mathbb{C}$-invariant domain to which γ_v extends holomorphically. Observe that by Theorem 1, $\widetilde{D}$ necessarily contains the domain $\widetilde{\Xi}$. Theorem 2 then tells us that G acts properly on $\widetilde{D}/K_\mathbb{C}$, and so it follows that it is equal to Ξ. Thus we have the following

Theorem 3. *Let G be non-Hermitian. If (π, V) is either an infinite-dimensional irreducible uniformly bounded Hilbert representation of G or an infinite-dimensional irreducible isometric representation of G on a ucus Banach space, then the corresponding domain of holomorphy D_π is equal to $\widetilde{\Xi}$.*

Next we handle the case when G is Hermitian. Suppose (π, V) is a representation of highest weight, i.e. P^+ acts finitely on V_K. From this, and the fact that V_K is also naturally a $K_\mathbb{C}$-module, it follows that for $0 \neq v \in V_K$, the orbit map γ_v extends holomorphically to the $G \times K_\mathbb{C}$-invariant domain $GK_\mathbb{C}\mathsf{P}^+ = \widetilde{\Xi}^+$ described in section 2. Note that for Hermitian G, it is in fact true that if a G-invariant domain D is such that $\mathbb{X} \subset D \subset \mathbb{X}_\mathbb{C}$, and admits a proper G-action then either $D \subset \widetilde{\Xi}^+/K_\mathbb{C}$ or $D \subset \widetilde{\Xi}^-/K_\mathbb{C}$ [Krö08]. Using this we can conclude that $\widetilde{D}_\pi = \widetilde{\Xi}^\pm$ depending on whether (π, V) is of highest weight or lowest weight, respectively.

When G is Hermitian, and π is neither a highest nor a lowest weight unitary representation, then in [Krö08] an $SL(2,\mathbb{R})$-reduction is used together with the uniqueness of the direct integral decomposition for unitary representations. This method fails in our setting because we do not have an analogous result on the uniqueness of direct integral decomposition. Perhaps one needs new methods, and we do not see how to do this at the moment. Nevertheless, we wish to reiterate that we can handle all uniformly bounded Hilbert representations as well as some uniformly bounded Banach representations for all noncompact simple non-Hermitian real Lie groups.

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