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Abstract. This paper deals with the Schrödinger equation $i\partial_s u(\mathbf{z}, t; s) - \mathscr{L}u(\mathbf{z}, t; s) = 0$, where $\mathscr{L}$ is the sub-Laplacian on the Heisenberg group. Assume that the initial data f satisfies $|f(\mathbf{z}, t)| \lesssim q_\alpha(\mathbf{z}, t)$, where q_s is the heat kernel associated to $\mathscr{L}$. If in addition $|u(\mathbf{z}, t; s_0)| \lesssim q_\beta(\mathbf{z}, t)$, for some $s_0 \in \mathbb{R} \setminus \{0\}$, then we prove that $u(\mathbf{z}, t; s) = 0$ for all $s \in \mathbb{R}$ whenever $\alpha\beta < s_0^2$. This result holds true in the more general context of H-type groups. We also prove an analogous result for the Grushin operator on $\mathbb{R}^{n+1}$.

1. Introduction

Consider the solution $u(x, t)$ of the Schrödinger equation

$$i\partial_s u(x, s) = \Delta u(x, s), \quad u(x, 0) = f(x)$$

on $\mathbb{R}^n$. In [2] Chanillo has shown that if the initial condition f has certain Gaussian decay then the solution $u(x, s)$ at a later time cannot have an arbitrary Gaussian decay. This is reminiscent of Hardy's theorem which states that a function f and its Euclidean Fourier transform $\hat{f}$ cannot have arbitrary Gaussian decay. To be more precise, if the definition of the Fourier transform is taken as

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx,$$

then the conditions

$$|f(x)| \lesssim e^{-\alpha|x|^2}, \quad |\hat{f}(\xi)| \lesssim e^{-\beta|\xi|^2}$$

can be satisfied for a nontrivial f only if $\alpha\beta \leq 1/4$. This can be viewed as an uncertainty principle for the Fourier transform. The notation $X \lesssim Y$ is (and will be) used to indicate that $X \leq CY$ with a positive constant C independent of significant quantities.

Since the solution of the aforementioned Schrödinger equation can be expressed in terms of the Fourier transform of f, by a straightforward application of Hardy's theorem Chanillo obtained the following uniqueness theorem for solutions of the Schrödinger equation.

Theorem 1.1. *(cf. [2]) Let $u(x, s)$ be the solution of the equation*

$$i\partial_s u(x, s) = \Delta u(x, s), \quad u(x, 0) = f(x)$$

where f is assumed to satisfy the estimate $|f(x)| \lesssim e^{-\alpha|x|^2}$, $x \in \mathbb{R}^n$, for some positive constant α. If at a later time $s = s_0$ the solution satisfies the estimate $|u(x, s_0)| \lesssim e^{-\beta|x|^2}$, $x \in \mathbb{R}^n$, then $f = 0$ whenever $\alpha\beta < s_0^2$.

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Hardy's theorem as stated above goes back to the work of Hardy in 1933 and later similar results for the Fourier transforms on other Lie groups have been established, see [15]. However, until the work of Chanillo, Hardy's theorem was considered only in the context of heat equation and his work triggered a lot of attention on the Schrödinger equation. Chanillo himself treated the Schrödinger equation on complex Lie groups where the initial condition was assumed to be K-biinvariant. However, if we use Radon transform the problem can be reduced to the Euclidean case and his result holds without any restriction either on the group or on the initial condition. Recently, somewhat more precise results of this kind have been proved by Pasquale and Sundari [11] in the context of symmetric spaces.

Similar uniqueness results for other Schrödinger evolutions and for the Korteweg-de Vries equation have received a good deal of attention in recent years (see for instance [4, 5, 8, 10, 14, 18]). These authors have developed powerful PDE techniques to deal with uniqueness results. Completing a full circle, in a recent work Cowling et al. [3] have used a uniqueness theorem for the Schrödinger equation to give a 'real variable proof' of Hardy's theorem. See also the works [6, 7] where the authors deal with equations with non-constant lower order terms and/or nonlinear equations.

In this paper we are interested in proving an analogue of Chanillo's theorem for H-type groups. Let G be such a group and denote by $\mathscr{L}$ the sub-Laplacian on G. We consider the following initial value problem for the Schrödinger equation associated to $\mathscr{L}$:

$$\begin{aligned} &i\partial_s u(g,s) - \mathscr{L}u(g,s) = 0, \qquad g \in G,\ s \in \mathbb{R}, \\ &u(g,0) = f(g) \end{aligned}$$

where f is assumed to be in $L^2(G)$. Our goal is to find sufficient conditions on the behavior of the solution u at two different times $s = 0$ and $s = s_0$ which guarantee that $u \equiv 0$ is the unique solution to the above initial value problem.

We write the elements of G as $g = (\mathbf{v}, \mathbf{t})$ where $\mathbf{t}$ comes from the center of G. We denote by $h_a(\mathbf{v}, \mathbf{t})$ the heat kernel associated to the sub-Laplacian. We prove:

Theorem 1.2. *Let $u(\mathbf{v}, \mathbf{t}; s)$ be the solution of the Schrödinger equation on $G \times \mathbb{R}$, with initial data f. Assume that $|f(\mathbf{v}, \mathbf{t})| \lesssim h_\alpha(\mathbf{v}, \mathbf{t})$ for some $\alpha > 0$. Further, suppose that there exists $s_0 \in \mathbb{R} \setminus \{0\}$ such that $|u(\mathbf{v}, \mathbf{t}; s_0)| \lesssim h_\beta(\mathbf{v}, \mathbf{t})$ for some $\beta > 0$. If $\alpha\beta < s_0^2$, then $u(\mathbf{v}, \mathbf{t}; s) = 0$ for all $(\mathbf{v}, \mathbf{t}) \in G$ and for all $s \in \mathbb{R}$.*

Our approach uses Hardy's theorem for the Hankel transform obtained in [17], which says that a function and its Hankel transform both cannot have arbitrary Gaussian decay at infinity unless, of course, the function is identically zero. It is interesting to note that we do not need to use Hardy's theorem for the Heisenberg group proved in [15].

The $(2n+1)$-dimensional Heisenberg group, denoted by $\mathbb{H}^n$, is the most well known example of a H-type group. In Section 3 we prove the above theorem for $\mathbb{H}^n$. Once the theorem is proved for $\mathbb{H}^n$ it is not difficult to extend the proof for all H-type groups. This class of groups was introduced in [9] and the list of H-type groups includes the Heisenberg groups and their analogues built up with quaternions or octonions in place of complex numbers, as well as many other groups.

We also prove an analogue of the above theorem for the Grushin operator $\mathscr{L} = -\Delta - |x|^2\partial_t^2$ on $\mathbb{R}^{n+1}$. The behavior of this operator is very similar to that of sub-Laplacian as can be easily seen by comparing the explicit expression for the latter with the above when n is even. The spectral decomposition of $\mathscr{L}$ is explicitly known and we also have a good knowledge of the associated heat kernel. Let $k_s(x, y, t - t')$ stand for the heat kernel so that

$$u((x,t), s) = \int_{\mathbb{R}^{n+1}} k_s(x, y, t - t') f(y, t') dy dt', \qquad (x, t) \in \mathbb{R}^{n+1}, s \in \mathbb{R}$$

solves the heat equation associated to the Grushin operator. The following is the analogue of Theorem 1.2 for the Grushin operator.

Theorem 1.3. *Let $u((x,t), s)$ be the solution of the Schrödinger equation associated to the Grushin operator with initial condition $f \in L^2(\mathbb{R}^{n+1})$. Suppose that*

$$|f(x,t)| \lesssim k_\alpha(x, 0, t) \tag{1.1}$$

$$|u((x,t), s_0)| \lesssim k_\beta(x, 0, t) \tag{1.2}$$

for some $\alpha, \beta > 0$ and for a fixed $s_0 \in \mathbb{R}^*$. Then $u((x,t), s) = 0$ on $\mathbb{R}^{n+1} \times \mathbb{R}$ whenever $\alpha\beta < s_0^2$.

We indicate a proof of this theorem in Section 5. As we mentioned above, the proof of the main theorem for the Heisenberg group uses an analogue of Hardy's theorem for Hankel transforms. An important role is played by Hecke-Bochner type formula for the special Hermite projections in reducing the problem to the Euclidean setup. The proof can also be carried out for Grushin operators, which are very similar to the sub-Laplacians, thanks to an analogue of Hecke-Bochner formula for Hermite projection operators. In the last section we briefly indicate how other versions of our main result can be proved for the Heisenberg group.

2. Background

The $(2n + 1)$-dimensional Heisenberg group, denoted by $\mathbb{H}^n$, is $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law

$$(\mathbf{z}, t)(\mathbf{w}, s) = (\mathbf{z} + \mathbf{w}, t + s + \frac{1}{2}\mathrm{Im}(\mathbf{z} \cdot \bar{\mathbf{w}})).$$

Under this multiplication $\mathbb{H}^n$ becomes a nilpotent unimodular Lie group, the Haar measure being the Lebesgue measure $d\mathbf{z}dt$ on $\mathbb{C}^n \times \mathbb{R}$. The corresponding Lie algebra is generated by the vector fields

$$X_j := \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}, \qquad j = 1, 2, \ldots, n,$$

$$Y_j := \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}, \qquad j = 1, 2, \ldots, n,$$

and $T := \frac{\partial}{\partial t}$. The sub-Laplacian

$$\mathscr{L} := -\sum_{j=1}^{n} X_j^2 + Y_j^2$$

can be written as

$$\mathscr{L} = -\Delta_{\mathbb{R}^{2n}} - \frac{1}{4}|\mathbf{z}|^2 \partial_t^2 + N\partial_t,$$

where

$$N=\sum_{j=1}^{n} x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}.$$

This second order differential operator $\mathscr{L}$ is hypoelliptic, self-adjoint and nonnegative. It generates a semigroup with kernel $q_s(\mathbf{z},t)$, called the heat kernel. In particular, $q_s(\mathbf{z},t)$ is nonnegative and has the property

$$q_{r^2s}(\mathbf{z},t)=r^{-2(n+1)}q_s(r^{-1}\mathbf{z},r^{-2}t),\qquad r\neq 0.$$

Moreover,

$$\int_{\mathbb{R}} e^{i\lambda t}q_s(\mathbf{z},t)dt=(4\pi)^{-n}\left(\frac{\lambda}{\sinh\lambda s}\right)^n e^{-\frac{1}{4}\lambda(\coth s\lambda)|\mathbf{z}|^2}$$

(see [15]). Henceforth, for $f\in L^1(\mathbb{H}^n)$ and $\lambda\in\mathbb{R}$, we will write

$$f^\lambda(\mathbf{z}):=\int_{\mathbb{R}} e^{i\lambda t}f(\mathbf{z},t)dt.$$

We now collect some properties of the heat kernel $q_s(\mathbf{z},t)$.

Fact 2.1. *The heat kernel satisfies the semigroup property* $q_\alpha * q_\beta(\mathbf{z},t)=q_{\alpha+\beta}(\mathbf{z},t)$.

The following is a slight modification of [15, Proposition 2.8.2].

Fact 2.2. *The heat kernel* $q_s(\mathbf{z},t)$ *satisfies the following estimate*

$$q_s(\mathbf{z},t)\lesssim s^{-n-1}e^{-\frac{\pi}{2}\frac{|t|}{s}}e^{-\frac{1}{4}\frac{|\mathbf{z}|^2}{s}},\qquad s>0. \tag{2.1}$$

Indeed, for $s=1$ by [15, (2.8.9-2.8.10)], we have

$$q_1(\mathbf{z},t)\lesssim e^{-\frac{\pi}{2}|t|}e^{-\frac{1}{4}|\mathbf{z}|^2}.$$

Now Fact 2.2 follows from the fact that $q_s(\mathbf{z},t)=s^{-n-1}q_1(s^{-1/2}\mathbf{z},s^{-1}t)$ for all $s>0$.

Let f and g be two functions on $\mathbb{H}^n$. The convolution of f with g is defined by

$$(f*g)(\mathbf{z},t)=\int_{\mathbb{H}^n} f((\mathbf{z},t)(-\mathbf{w},s))g(\mathbf{w},s)d\mathbf{w}ds.$$

An easy calculation shows that

$$(f*g)^\lambda(\mathbf{z})=\int_{\mathbb{C}^n} f^\lambda(\mathbf{z}-\mathbf{w})g^\lambda(\mathbf{w})e^{i\frac{\lambda}{2}\operatorname{Im}(\mathbf{z}\cdot\bar{\mathbf{w}})}d\mathbf{w}.$$

The right hand side is called the λ-twisted convolution of f^λ with g^λ, and will be denoted by $f^\lambda *_\lambda g^\lambda$.

Let $\mathscr{P}$ be the set of all polynomials of the form $P(\mathbf{z})=\sum_{|\alpha|+|\beta|\le m}a_{\alpha,\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta$. For each pair of nonnegative integers (p,q), we define $\mathscr{P}_{p,q}$ to be the subspace of $\mathscr{P}$ consisting of all polynomials of the form $P(\mathbf{z})=\sum_{|\alpha|=p}\sum_{|\beta|=q}a_{\alpha,\beta}\mathbf{z}^\alpha\overline{\mathbf{z}}^\beta$.

Let $\mathscr{H}_{p,q}:=\{P\in\mathscr{P}_{p,q}\mid \Delta P=0\}$, where Δ denotes the Laplacian on $\mathbb{C}^n$. The elements of $\mathscr{H}_{p,q}$ are called bigraded solid harmonics of degree (p,q). We will denote by $\mathscr{S}_{p,q}$ the space of all restrictions of bigraded solid harmonics of degree (p,q) to the sphere S^{2n-1}. By [15], the space $L^2(S^{2n-1})$ is the orthogonal direct sum of the spaces $\mathscr{S}_{p,q}$, with $p,q\ge 0$. We choose an orthonormal basis $\{Y^j_{p,q} \mid 1 \le j \le d(p,q)\}$ for $\mathscr{S}_{p,q}$. Then by standard arguments it follows that every continuous function f on $\mathbb{C}^n$ can be expanded as

$$f(r\omega) = \sum_{p,q\ge 0} \sum_{j=1}^{d(p,q)} f_{p,q,j}(r) Y^j_{p,q}(\omega), \qquad r > 0,\ \omega \in S^{2n-1},$$

where

$$f_{p,q,j}(r) := \int_{S^{2n-1}} f(r\omega)\overline{Y^j_{p,q}(\omega)} d\sigma(\omega). \tag{2.2}$$

For $k \in \mathbb{N}$, we write L^{n-1}_k for the Laguerre polynomial defined by

$$L^{n-1}_k(t) = \sum_{j=0}^{k} \frac{(-1)^j \Gamma(n+k)}{(k-j)!\Gamma(n+j)} t^j.$$

For $\lambda \in \mathbb{R}^*$, define the Laguerre functions $\varphi^{n-1}_{k,\lambda}$ by

$$\varphi^{n-1}_{k,\lambda}(\mathbf{z}) = L^{n-1}_k \left(\frac{|\lambda|}{2} |\mathbf{z}|^2 \right) e^{-\frac{|\lambda|}{4}|\mathbf{z}|^2}, \tag{2.3}$$

for $\mathbf{z} \in \mathbb{C}^n$. Suppose that f if a radial function in $L^1(\mathbb{H}^n)$. Then $f(r)$ is in $L^1(\mathbb{R}^+, r^{2n-1}dr)$, where $f(r)$ stands for $f(\mathbf{w})$ with $|\mathbf{w}| = r$. For the following Hecke-Bochner formula we refer to [15, Theorem 2.6.1].

Theorem 2.3. *Let $f(\mathbf{z}) = P(\mathbf{z})g(|\mathbf{z}|)$, where $P \in \mathscr{H}_{p,q}$ and $g \in L^1(\mathbb{R}^+, r^{2n-1}dr)$. Then for $\lambda \in \mathbb{R}^*$, we have*

$$f *_\lambda \varphi^{n-1}_{k,\lambda}(\mathbf{z}) = (2\pi)^{-n} |\lambda|^{p+q} P(\mathbf{z}) g *_\lambda \varphi^{n+p+q-1}_{k-p,\lambda}(\mathbf{z}),$$

where the convolution on the right hand side is taken on $\mathbb{C}^{n+p+q}$ treating the radial functions g and $\varphi^{n+p+q-1}_{k-p,\lambda}$ as functions on $\mathbb{C}^{n+p+q}$. More explicitly we have

$$\begin{aligned} g *_\lambda \varphi^{n+p+q-1}_{k-p,\lambda}(\mathbf{z}) \quad &= \quad \frac{(2\pi)^{n+p+q} |\lambda|^{\frac{n+p+q}{2}} 2^{-(n+p+q)+1} \Gamma(k-p+1)}{\Gamma(k+n+q)} \\ &\quad \Big(\int_0^\infty g(s) L^{n+p+q-1}_{k-p}\Big(\frac{|\lambda|}{2} s^2\Big) e^{-\frac{|\lambda|}{4}s^2} s^{2(n+p+q)-1} ds \Big) L^{n+p+q-1}_{k-p} \left(\frac{|\lambda|}{2} |\mathbf{z}|^2 \right) e^{-\frac{|\lambda|}{4}|\mathbf{z}|^2}. \end{aligned} \tag{2.4}$$

To end this section, let us recall Hardy's uncertainty principle for the Hankel transform. For $\nu > -\frac{1}{2}$ and $F \in S(\mathbb{R}^+)$, the Hankel transform of order ν is defined by

$$\mathscr{H}_\nu F(s) = \int_0^\infty F(r) \frac{J_\nu(rs)}{(rs)^\nu} r^{2\nu+1} dr, \tag{2.5}$$

where $J_\nu(w)$ is the Bessel function of order ν defined by

$$J_\nu(w) = \Big(\frac{w}{2}\Big)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{w}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)}.$$

Theorem 2.4. *(Hardy's theorem,* [17]*) Let F be a measurable function on $\mathbb{R}^+$ such that*

$$F(r) = O(e^{-\alpha r^2}), \quad \mathscr{H}_\nu F(s) = O(e^{-\beta s^2})$$

for some positive α and β. Then $F = 0$ whenever $\alpha\beta > \frac{1}{4}$ and $F(r) = Ce^{-\alpha r^2}$ whenever $\alpha\beta = \frac{1}{4}$.

3. Schrödinger equation on $\mathbb{H}^n \times \mathbb{R}$

Let us consider the Schrödinger equation on $\mathbb{H}^n \times \mathbb{R}$

$$i\partial_s u(\mathbf{z}, t; s) = \mathscr{L} u(\mathbf{z}, t; s),$$

with the initial condition $u(\mathbf{z}, t; 0) = f(\mathbf{z}, t)$. As the closure of $\mathscr{L}$ on $C_c^\infty(\mathbb{H}^n)$ is a self-adjoint operator, $-i\mathscr{L}$ generates a unitary semi-group $e^{-is\mathscr{L}}$ on $L^2(\mathbb{H}^n)$, and the solution of the above Schrödinger equation is given by

$$u(\mathbf{z}, t; s) = e^{-is\mathscr{L}} f(\mathbf{z}, t).$$

Theorem 3.1. *Let $u(\mathbf{z}, t; s)$ be the solution to the Schrödinger equation for the sub-Laplacian $\mathscr{L}$ with initial condition f. Suppose that*

$$|f(\mathbf{z}, t)| \lesssim q_\alpha(\mathbf{z}, t), \tag{3.1 a}$$

$$|u(\mathbf{z}, t; s_0)| \lesssim q_\beta(\mathbf{z}, t), \tag{3.1 b}$$

for some $\alpha, \beta > 0$ and for a fixed $s_0 \in \mathbb{R}^*$. Then $u(\mathbf{z}, t; s) = 0$ on $\mathbb{H}^n \times \mathbb{R}$ whenever $\alpha\beta < s_0^2$.

The remaining part of this section is devoted to the proof of the above statement.

The heat kernel $q_s(\mathbf{z}, t)$ has an analytic continuation in s as long as real part of s is positive. However, due to the zeros of the sine function, the kernel $q_{is}(\mathbf{z}, t)$ does not exist as can be seen from the formula for $q_s^\lambda(\mathbf{z})$. Hence the solution $u(\mathbf{z}, t; s)$ does not have an integral representation. We will therefore consider the following regularized problem on $\mathbb{H}^n \times \mathbb{R}$:

$$\begin{aligned} &i\partial_s u_\epsilon(\mathbf{z}, t; s) = \mathscr{L} u_\epsilon(\mathbf{z}, t; s), \qquad \epsilon > 0, \\ &u_\epsilon(\mathbf{z}, t; 0) = f_\epsilon(\mathbf{z}, t), \end{aligned}$$

where $f_\epsilon(\mathbf{z}, t) := e^{-\epsilon\mathscr{L}} f(\mathbf{z}, t)$. The solution u_ϵ on $\mathbb{H}^n \times \mathbb{R}$ is given by

$$u_\epsilon(\mathbf{z}, t; s) = e^{-is\mathscr{L}} f_\epsilon(\mathbf{z}, t) = f * q_\zeta(\mathbf{z}, t),$$

where $\zeta = \epsilon + is$ and

$$q_\zeta(\mathbf{z}, t) := \frac{1}{(8\pi^2)^n} \int_{\mathbb{R}} e^{-i\lambda t} \left(\frac{\lambda}{\sinh \lambda\zeta}\right)^n e^{-\frac{1}{4}\lambda(\coth \zeta\lambda)|\mathbf{z}|^2} d\lambda.$$

Observe that the kernel $q_\zeta(\mathbf{z}, t)$ is well defined.

Lemma 3.2. *Under the assumptions* (3.1 a) *and* (3.1 b)*, we have*

$$|f_\epsilon(\mathbf{z}, t)| \lesssim q_{\alpha+\epsilon}(\mathbf{z}, t), \tag{3.2 a}$$

$$|u_\epsilon(\mathbf{z}, t; s_0)| \lesssim q_{\beta+\epsilon}(\mathbf{z}, t). \tag{3.2 b}$$

Proof. For the first estimate, we have

$$\begin{aligned}|f_\epsilon(\mathbf{z},t)| = |e^{-\epsilon\mathscr{L}}f(\mathbf{z},t)| &= |f * q_\epsilon(\mathbf{z},t)|\\ &\lesssim q_{\alpha+\epsilon}(\mathbf{z},t).\end{aligned}$$

Above we have used the fact that q_s is nonnegative and Fact 2.1. Similarly we have

$$\begin{aligned}|u_\epsilon(\mathbf{z},t;s_0)| &= |u(\cdot,\cdot\,;s_0) * q_\epsilon(\mathbf{z},t)|\\ &\lesssim q_{\beta+\epsilon}(\mathbf{z},t).\end{aligned}$$

□

Recall that for $\lambda \in \mathbb{R}$, the notation $f^\lambda(\mathbf{z})$ stands for the inverse Fourier transform of $f(\mathbf{z},t)$ in the t-variable. In view of the hypothesis (3.1 a) on f and the estimate (2.1) on the heat kernel, one can see that the function $\lambda \mapsto f^\lambda(\mathbf{z})$ extends to a holomorphic function of λ on the strip $|\mathrm{Im}(\lambda)| < \frac{\pi}{2a}$. Thus the following statement is true.

Lemma 3.3. *Under the hypothesis* (3.1 a) *on* f, *the inverse Fourier transform* $f^\lambda(\mathbf{z})$ *of* $f(\mathbf{z},t)$ *in the t-variable extends to a holomorphic function of λ in a tubular neighborhood in $\mathbb{C}$ of the real line.*

We point out that the above lemma also holds for the function $\lambda \mapsto f_\epsilon^\lambda$.

Strategy. *To prove Theorem 3.1, our strategy is to show that $f = 0$ on $\mathbb{H}^n$ whenever $\alpha\beta < s_0^2$. However, by the above lemma, showing that $f^\lambda = 0$ on $\mathbb{C}^n$ for $0 < \lambda < \delta$, for some $\delta > 0$, will force $f^\lambda = 0$ on $\mathbb{C}^n$ for all $\lambda \in \mathbb{R}$ and hence $f = 0$ on $\mathbb{H}^n$. Furthermore, since $f_\epsilon^\lambda = f^\lambda *_\lambda q_\epsilon^\lambda$, then proving that $f^\lambda = 0$ on $\mathbb{C}^n$ for $0 < \lambda < \delta$ is equivalent to show the same statement for f_ϵ^λ. On the other hand, in order to prove that $f_\epsilon^\lambda(\mathbf{z}) = 0$ for $0 < \lambda < \delta$, for some $\delta > 0$, it is enough to prove that the spherical harmonic coefficients*

$$(f_\epsilon^\lambda)_{p,q,j}(r) = \int_{S^{2n-1}} f_\epsilon^\lambda(r\omega)\overline{Y_{p,q}^j(\omega)}d\sigma(\omega)$$

vanish for $0 < \lambda < \delta$, for all $p,q \geq 0$ and $1 \leq j \leq d(p,q)$. In conclusion, the proof of Theorem 3.1 reduces to prove that if $\alpha\beta < s_0^2$, then $(f_\epsilon^\lambda)_{p,q,j} = 0$ on $\mathbb{R}^+$ for $0 < \lambda < \delta$, for all $p,q \geq 0$ and $1 \leq j \leq d(p,q)$.

The following theorem will be of crucial importance to us.

Theorem 3.4. *Let us fix $p_0, q_0 \geq 0$ and $1 \leq j_0 \leq d(p_0,q_0)$. For all $r > 0$, there exists a constant c_λ which depends only on λ such that*

$$\int_{S^{2n-1}} u_\epsilon^\lambda(r\omega;s_0)\overline{Y_{p_0,q_0}^{j_0}(\omega)}d\sigma(\omega) = c_\lambda r^{p_0+q_0} e^{i\frac{\lambda}{4}r^2\operatorname{cotg}(\lambda s_0)}$$
$$\mathscr{H}_{n+p_0+q_0-1}\left(e^{i\frac{\lambda}{4}(\cdot)^2\operatorname{cotg}(\lambda s_0)}(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}\right)\left(\frac{\lambda r}{2\sin(\lambda s_0)}\right),$$

where $u_\epsilon^\lambda(\mathbf{z};s_0)$ denotes the inverse Fourier transform of $u_\epsilon(\mathbf{z},t;s_0)$ in the t-variable, $\mathscr{H}_\nu$ denotes the Hankel transform of order ν (see (2.5)*), and $(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}(t) := t^{-(p_0+q_0)}(f_\epsilon^\lambda)_{p_0,q_0,j_0}(t)$.*

Proof. In what follows c_λ will stand for constants depending only on λ which will vary from one line to another. Using Fact 2.2 we can rewrite $u_\epsilon^\lambda(\mathbf{z}; s_0)$ as

$$u_\epsilon^\lambda(\mathbf{z}; s_0) = f_\epsilon^\lambda *_\lambda q_{is_0}^\lambda(\mathbf{z}),$$

where

$$q_{is_0}^\lambda(\mathbf{z}) = (4\pi)^{-n}\left(\frac{\lambda}{i\sin\lambda s_0}\right)^n e^{\frac{i}{4}\lambda(\operatorname{cotg}\lambda s_0)|\mathbf{z}|^2}$$

which exits for all but a countably many values of λ. Thus

$$\begin{aligned}
&\int_{S^{2n-1}} u_\epsilon^\lambda(r\omega; s_0)\overline{Y_{p_0,q_0}^{j_0}(\omega)}d\sigma(\omega)\\
&= \int_{S^{2n-1}}\Big[\int_{\mathbb{C}^n} f_\epsilon^\lambda(r\omega-\mathbf{w})q_{is_0}^\lambda(\mathbf{w})e^{i\frac{\lambda}{2}\mathrm{Im}(r\omega\cdot\bar{\mathbf{w}})}d\mathbf{w}\Big]\overline{Y_{p_0,q_0}^{j_0}(\omega)}d\sigma(\omega)\\
&= \int_{S^{2n-1}}\Big[\int_{\mathbb{C}^n} f_\epsilon^\lambda(\mathbf{w})q_{is_0}^\lambda(r\omega-\mathbf{w})e^{-i\frac{\lambda}{2}\mathrm{Im}(r\omega\cdot\bar{\mathbf{w}})}d\mathbf{w}\Big]\overline{Y_{p_0,q_0}^{j_0}(\omega)}d\sigma(\omega).
\end{aligned}$$

We now expand f_ϵ^λ in terms of bigraded spherical harmonics as

$$f_\epsilon^\lambda(t\eta) = \sum_{p,q\geq 0}\sum_{j=1}^{d(p,q)}(f_\epsilon^\lambda)_{p,q,j}(t)Y_{p,q}^j(\eta),$$

where $(f_\epsilon^\lambda)_{p,q,j}$ is as in (2.2). Further, by [15, (2.8.7)] we have

$$q_{is_0}^\lambda(r\omega-t\eta) = (2\pi)^{-n}|\lambda|^n\sum_{k=0}^\infty e^{-i(2k+n)|\lambda|s_0}\varphi_{k,\lambda}^{n-1}(r\omega-t\eta),$$

where $\varphi_{k,\lambda}^{n-1}$ is given by (2.3). Now the Hecke-Bochner formula for the λ-twisted convolution (see Theorem 2.3) gives us

$$\begin{aligned}
&\int_0^\infty\int_{S^{2n-1}}(f_\epsilon^\lambda)_{p,q,j}(t)Y_{p,q}^j(\eta)\varphi_{k,\lambda}^{n-1}(r\omega-t\eta)e^{-i\frac{\lambda}{2}rt\mathrm{Im}(\omega\cdot\bar\eta)}t^{2n-1}dtd\sigma(\eta)\\
&= \int_0^\infty\int_{S^{2n-1}}(f_\epsilon^\lambda)^\sim_{p,q,j}(t)P_{p,q}^j(t\eta)\varphi_{k,\lambda}^{n-1}(r\omega-t\eta)e^{-i\frac{\lambda}{2}rt\mathrm{Im}(\omega\cdot\bar\eta)}t^{2n-1}dtd\sigma(\eta)\\
&= \Big[(f_\epsilon^\lambda)^\sim_{p,q,j}P_{p,q}^j\Big]*_{-\lambda}\varphi_{k,\lambda}^{n-1}(r\omega)\\
&= (2\pi)^{-n}|\lambda|^{p+q}P_{p,q}^j(r\omega)\Big[(f_\epsilon^\lambda)^\sim_{p,q,j}*_{-\lambda}\varphi_{k-p,\lambda}^{n+p+q-1}\Big](r\omega),
\end{aligned}$$

where the convolution on the right hand side is on $\mathbb{C}^{n+p+q}$. Here $P_{p,q}^j(r\omega) := r^{p+q}Y_{p,q}^j(\omega)$ and $F^\sim(t) = t^{-(p+q)}F(t)$. Above we have used the fact that $\varphi_{k,\lambda}^{n-1} = \varphi_{k,-\lambda}^{n-1}$. Using the orthogonality of the basis $\{Y^j_{p,q} \ : \ 1 \le j \le d(p,q)\}$ we obtain:

$$\begin{aligned}
&\int_{S^{2n-1}} \Big[\int_{\mathbb{C}^n} f^\lambda_\epsilon(\mathbf{w}) q^\lambda_{is_0}(r\omega - \mathbf{w}) e^{-i\frac{\lambda}{2}\mathrm{Im}(r\omega\cdot\bar{\mathbf{w}})} d\mathbf{w} \Big] \overline{Y^{j_0}_{p_0,q_0}(\omega)} d\sigma(\omega) \\
&= c_\lambda r^{p_0+q_0} \sum_{k\ge p_0} e^{-i(2k+n)|\lambda|s_0} (f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0} *_{-\lambda} \varphi^{n+p_0+q_0-1}_{k-p_0,\lambda}(r\omega) \\
&= c_\lambda r^{p_0+q_0} \sum_{k=0}^\infty e^{-i(2k+n+2p_0)|\lambda|s_0} (f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0} *_{-\lambda} \varphi^{n+p_0+q_0-1}_{k,\lambda}(r\omega).
\end{aligned}$$

On the other hand, by (2.4) we have

$$\begin{aligned}
(f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0} *_{-\lambda} \varphi^{n+p_0+q_0-1}_{k,\lambda}(r\omega) &= c_\lambda \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)} \\
&\quad \left(\int_0^\infty (f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0}(t) \varphi^{n+p_0+q_0-1}_{k,\lambda}(t) t^{2(n+p_0+q_0)-1} dt \right) \varphi^{n+p_0+q_0-1}_{k,\lambda}(r\omega).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\int_{S^{2n-1}} u^\lambda_\epsilon(r\omega; s_0) \overline{Y^{j_0}_{p_0,q_0}(\omega)} d\sigma(\omega) \\
&= c_\lambda r^{p_0+q_0} \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)} e^{-i(2k+n+2p_0)|\lambda|s_0} L^{n+p_0+q_0-1}_k \Big(\frac{|\lambda|}{2} r^2\Big) e^{-\frac{|\lambda|}{4} r^2} \\
&\qquad \Big(\int_0^\infty (f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0}(t) \varphi^{n+p_0+q_0-1}_{k,\lambda}(t) t^{2(n+p_0+q_0)-1} dt \Big) \\
&= c_\lambda r^{p_0+q_0} \int_0^\infty (f^\lambda_\epsilon)^\sim_{p_0,q_0,j_0}(t) K_\lambda(r,t;s_0) t^{2(n+p_0+q_0)-1} dt,
\end{aligned}$$

where

$$K_\lambda(r,t;s_0) := \sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+n+p_0+q_0)} e^{-i(2k+n+2p_0)|\lambda|s_0} e^{-\frac{|\lambda|}{4}(r^2+t^2)} L^{n+p_0+q_0-1}_k \left(\frac{|\lambda|}{2} r^2\right) L^{n+p_0+q_0-1}_k \left(\frac{|\lambda|}{2} t^2\right).$$

Now we can use the following Hille-Hardy identity (see for instance [16])

$$\sum_{k=0}^\infty \frac{\Gamma(k+1)}{\Gamma(k+\nu+1)} L^\nu_k(x) L^\nu_k(y) w^k = (1-w)^{-(\nu+1)} e^{-\frac{w}{1-w}(x+y)} \widetilde{J_\nu}\left(\frac{2(-xyw)^{1/2}}{1-w} \right),$$

where $\widetilde{J_\nu}(w) := \left(\frac{w}{2}\right)^{-\nu} J_\nu(w)$ and J_ν is the Bessel function of order ν. Thus we may rewrite the kernel K_λ as

$$K_\lambda(r,t;s_0) = e^{i|\lambda|s_0(q_0-p_0)} (2i\sin(|\lambda|s_0))^{-(n+p_0+q_0)} e^{i\frac{\lambda}{4}(r^2+t^2)\operatorname{cotg}(\lambda s_0)} \widetilde{J}_{n+p_0+q_0-1} \left(\frac{\lambda}{2} \frac{rt}{\sin(\lambda s_0)} \right).$$

Thus we arrive at

$$\begin{aligned}
&\int_{S^{2n-1}} u_\epsilon^\lambda(r\omega; s_0)\overline{Y^{j_0}_{p_0,q_0}(\omega)}d\sigma(\omega)\\
&= c_\lambda r^{p_0+q_0}\int_0^\infty e^{i\frac{\lambda}{4}(r^2+t^2)\operatorname{cotg}(\lambda s_0)}(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}(t)\widetilde{J}_{n+p_0+q_0-1}\left(\frac{\lambda}{2}\frac{rt}{\sin(\lambda s_0)}\right)t^{2(n+p_0+q_0)-1}dt\\
&= c_\lambda r^{p_0+q_0}e^{i\frac{\lambda}{4}r^2\operatorname{cotg}(\lambda s_0)}\mathscr{H}_{n+p_0+q_0-1}\left(e^{i\frac{\lambda}{4}(\cdot)^2\operatorname{cotg}(\lambda s_0)}(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}\right)\left(\frac{\lambda r}{2\sin(\lambda s_0)}\right).
\end{aligned}$$

Hence Theorem 3.4 has been proved. □

Now we are ready to complete the proof of Theorem 3.1.

The estimate (3.2 a) on $f_\epsilon(\mathbf{z},t)$ together with Fact 2.1 lead us to

$$|f_\epsilon^\lambda(\mathbf{z})| \lesssim e^{-\frac{1}{4}\frac{|\mathbf{z}|^2}{\alpha+\epsilon}}.$$

Thus, the spherical harmonic coefficient $(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}$ satisfies

$$|(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}(t)| \lesssim t^{-(p_0+q_0)}e^{-\frac{1}{4}\frac{t^2}{\alpha+\epsilon}}.$$

On the other hand, by means of Theorem 3.4 and the estimate (3.2 b) on $u_\epsilon(\mathbf{z},t;s_0)$, we deduce that

$$\left|\mathscr{H}_{n+p_0+q_0-1}\left(e^{i\frac{\lambda}{4}(\cdot)^2\operatorname{cotg}(\lambda s_0)}(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}\right)\left(\frac{\lambda r}{2\sin(\lambda s_0)}\right)\right| \le c_\lambda r^{-(p_0+q_0)}e^{-\frac{1}{4}\frac{r^2}{\beta+\epsilon}}.$$

That is

$$\left|\mathscr{H}_{n+p_0+q_0-1}\left(e^{i\frac{\lambda}{4}(\cdot)^2\operatorname{cotg}(\lambda s_0)}(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}\right)(r)\right| \le c_\lambda r^{-(p_0+q_0)}e^{-\frac{1}{4}\left(\frac{2\sin(\lambda s_0)}{\lambda s_0}\right)^2\frac{r^2 s_0^2}{\beta+\epsilon}}.$$

Given $\alpha,\beta>0$ such that $\alpha\beta<s_0^2$ we can choose $\epsilon>0$ such that $(\alpha+\epsilon)(\beta+\epsilon)<s_0^2$. We can also choose $\delta>0$ small enough in such a way that for $0<\lambda<\delta$ we have $(\alpha+\epsilon)(\beta+\epsilon)<s_0^2\left(\frac{\sin(\lambda s_0)}{\lambda s_0}\right)^2$. This inequality can be written as

$$\frac{1}{4(\alpha+\epsilon)}\frac{s_0^2}{4(\beta+\epsilon)}\left(\frac{2\sin(\lambda s_0)}{\lambda s_0}\right)^2 > \frac{1}{4}.$$

Therefore, by Hardy's theorem for the Hankel transform (see Theorem 2.4), we deduce that for $0<\lambda<\delta$ we have $(f_\epsilon^\lambda)^\sim_{p_0,q_0,j_0}=0$, for all $p_0,q_0\ge 0$ and $1\le j_0\le d(p_0,q_0)$. That is $f_\epsilon^\lambda=0$ on $\mathbb{C}^n$ for $0<\lambda<\delta$, which forces $f_\epsilon^\lambda=0$ for all λ and hence $f_\epsilon=0$ on $\mathbb{H}^n$. That is $f=0$ on $\mathbb{H}^n$. This finishes the proof Theorem 3.1.

4. Proof of Theorem 1.2

Let $\mathfrak{g}$ be a two step nilpotent Lie algebra over $\mathbb{R}$ with an inner product $\langle\cdot,\cdot\rangle$. The corresponding simply connected Lie group is denoted by G. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called an H-type algebra if for every $\mathbf{v}\in\mathfrak{v}$, the map $\mathrm{ad}_{\mathbf{v}}:\mathfrak{v}\to\mathfrak{z}$ is a surjective isometry when restricted to the orthogonal complement of its kernel.

For the H-type algebra $\mathfrak{g}=\mathfrak{v}\oplus\mathfrak{z}$, let $\dim(\mathfrak{v})=2n$ and $\dim(\mathfrak{z})=k$. The class of groups of H-type includes the Heisenberg group $\mathbb{H}^n$ when $k=1$. Let η be a unit element in $\mathfrak{z}$ and denote its orthogonal complement in $\mathfrak{z}$ by $\eta^\perp$. The quotient algebra $\mathfrak{g}/\eta^\perp$ is a Lie algebra with Lie bracket $[X,Y]_\eta = \langle [X,Y], \eta\rangle$.

The quotient $\mathfrak{g}/\eta^\perp$ is an H-type algebra with inner product $\langle\cdot,\cdot\rangle_\eta$ given by $\langle(\mathbf{v}_1,t_1),(\mathbf{v}_2,t_2)\rangle_\eta = \langle \mathbf{v}_1,\mathbf{v}_2\rangle + t_1t_2$, where $\mathbf{v}_1,\mathbf{v}_2 \in \mathfrak{v}$, $t_1,t_2\in\mathbb{R}$, and $\langle \mathbf{v}_1,\mathbf{v}_2\rangle$ is the inner product in $\mathfrak{g}$. Here $(\mathbf{v},t)$ stands for the coset of $\mathbf{v}+t\eta$ in $\mathfrak{g}/\eta^\perp$. Moreover, if we denote by G_η the simply connected Lie group with Lie algebra $\mathfrak{g}/\eta^\perp$, then by [13], the Lie group G_η is isomorphic to the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n\times\mathbb{R}$. We refer to [1] for more details on the theory of H-type groups.

We fix an orthonormal basis $X_1,\dots,X_{2n}$ for $\mathfrak{v}$, and define the sub-Laplacian by

$$\mathscr{L} = -\sum_{j=1}^{2n} X_j^2.$$

It is known that $\mathscr{L}$ generates a semigroup which is given by convolution with the heat kernel for G. As in the case of the Heisenberg group, the kernel is explicitly known and is given by

$$h_s(\mathbf{v},\mathbf{t}) = \frac{1}{2^n(2\pi)^{n+k/2}}\int_0^\infty \frac{\lambda^{k/2}}{|\mathbf{t}|^{k/2-1}} J_{k/2-1}(\lambda|\mathbf{t}|)\left(\frac{\lambda}{\sinh(s\lambda)}\right)^n e^{-\frac{1}{4}\lambda(\coth s\lambda)|\mathbf{v}|}d\lambda,$$

for $(\mathbf{v},\mathbf{t})\in G$ and $s>0$. Here J_ν denotes the Bessel function of order ν. This formula has been proved in [12], where the author also obtains the integral expression for the analytic continuation, h_ζ, of the heat kernel h_s as long as $\mathrm{Rel}(\zeta)>0$.

We now consider the solution of the Schrödinger equation on $G\times\mathbb{R}$

$$\begin{aligned} i\partial_s u(\mathbf{v},\mathbf{t};s) &= \mathscr{L}u(\mathbf{v},\mathbf{t};s),\\ u(\mathbf{v},\mathbf{t};0) &= f(\mathbf{v},\mathbf{t}),\end{aligned}$$

which is given by $u(\mathbf{v},\mathbf{t};s) = e^{-is\mathscr{L}}f(\mathbf{v},\mathbf{t})$. When we replace the initial condition f by $e^{-\epsilon\mathscr{L}}f$, for some $\epsilon>0$, then the solution is given by

$$u_\epsilon(\mathbf{v},\mathbf{t};s) = f * h_\zeta(\mathbf{v},\mathbf{t}), \qquad \zeta = \epsilon + is.$$

We claim that the uniqueness Theorem 3.1 for the Schrödinger equation on $\mathbb{H}^n\times\mathbb{R}$ is true in the more general setting $G\times\mathbb{R}$. The rest of this section is devoted to the proof of Theorem 1.2.

For a suitable function f on G we define its partial Radon transform $\mathscr{R}_\eta f(\mathbf{v},t)$ on G_η by

$$\mathscr{R}_\eta f(\mathbf{v},t) = \int_{\eta^\perp} f(\mathbf{v}, t\eta+\nu)d\nu$$

where $d\nu$ is the Lebesgue measure on $\eta^\perp$. Since G_η can be identified with the Heisenberg group $\mathbb{H}^n$, we can think of $\mathscr{R}_\eta f$ as a function on $\mathbb{H}^n$. With this identification, it has been proved in [12] that $\mathscr{R}_\eta h_s(\mathbf{v},t) = q_s(\mathbf{v},t)$, for $s>0$, where $q_s(\mathbf{v},t)$ is the heat kernel from section 2. The latter identity between the heat kernels holds true even when s is complex with $\mathrm{Rel}(s)>0$.

In view of the assumptions on $f(\mathbf{v},\mathbf{t})$ and $u(\mathbf{v},\mathbf{t};s_0)$ it follows that $\mathscr{R}_\eta f(\mathbf{v},t)$ and $\mathscr{R}_\eta u(\mathbf{v},t;s_0)$ satisfy

$$\begin{aligned} |\mathscr{R}_\eta f(\mathbf{v},t)| &\lesssim q_\alpha(\mathbf{v},t),\\ |\mathscr{R}_\eta u(\mathbf{v},t;s_0)| &\lesssim q_\beta(\mathbf{v},t).\end{aligned}$$

Moreover, using the fact that under the Radon transform $\mathscr{R}_\eta$, the sub-Laplacian $\mathscr{L}$ on G goes into the sub-Laplacian $\mathscr{L}$ on $\mathbb{H}^n$ (see [13]), it follows that $\mathscr{R}_\eta u$ solves the Schrödinger equation on $\mathbb{H}^n \times \mathbb{R}$ with initial data $\mathscr{R}_\eta f(\mathbf{v}, t)$. Hence we can appeal to Theorem 3.1 to conclude that $\mathscr{R}_\eta u(\mathbf{v}, t; s) = 0$ for all $s \in \mathbb{R}$ and for all $\eta \in \mathfrak{z}$ whenever $\alpha\beta < s_0^2$. Now the injectivity of the Radon transform implies that if $\alpha\beta < s_0^2$, then $u(\mathbf{v}, \mathbf{t}; s) = 0$ for all $(\mathbf{v}, \mathbf{t}) \in G$ and $s \in \mathbb{R}$. This establishes Theorem 1.2.

5. The Grushin operator

The spectral decomposition of the Grushin operator $\mathscr{L} = -\Delta_{\mathbb{R}^n} - |x|^2\partial_t^2$ is given by

$$\mathscr{L} f(x,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\lambda t} \left(\sum_{k=0}^{\infty} (2k+n)|\lambda| P_k(\lambda) f^\lambda(x) \right) d\lambda$$

where $P_k(\lambda)$ are the spectral projections of the Hermite operator $H(\lambda) = -\Delta_{\mathbb{R}^n} + \lambda^2|x|^2$. The heat kernel associated to $\mathscr{L}$ is given by

$$k_s(x,y,t) = (2\pi)^{-1} \int_{-\infty}^{\infty} K_s^\lambda(x,y) e^{-it\lambda} d\lambda$$

where $K_s^\lambda(x,y)$ is the heat kernel associated to the Hermite operator which is known explicitly (see [16]):

$$K_s^\lambda(x,y) = c_n \left(\frac{\lambda}{\sinh 2\lambda s} \right)^{\frac{n}{2}} e^{-\frac{\lambda}{2}(|x|^2+|y|^2)\coth 2\lambda s} e^{\frac{2\lambda x\cdot y}{\sinh 2\lambda s}}.$$

The solution of the heat equation associated to $\mathscr{L}$ is given by

$$e^{-s\mathscr{L}} f(x,t) = \int_{\mathbb{R}^{n+1}} k_s(x,y,t-t') f(y,t') dy dt'.$$

Note that due to the zeros of the sine function appearing in the expression for $K_s^\lambda(x,y)$ the kernel $k_s(x,y,t)$ cannot be analytically continued for purely imaginary values of s. However, as in the case of the sub-Laplacian, the kernel $k_{\epsilon+is}(x,y,t)$ is well defined for all $\epsilon > 0$ and the function

$$u_\epsilon((x,t),s) = \int_{\mathbb{R}^{n+1}} k_{\epsilon+is}(x,y,t-t') f(y,t') dy dt'$$

solves the Schrödinger equation $i\partial_s u_\epsilon((x,t),s) = \mathscr{L} u_\epsilon((x,t),s)$ with initial condition $u_\epsilon((x,t),0) = e^{-\epsilon\mathscr{L}} f(x,t) = f_\epsilon(x,t)$. The heat kernel $k_s(x,y,t)$ satisfies the semigroup property

$$\int_{\mathbb{R}^{n+1}} k_\alpha(x,z,t-t') k_\beta(z,y,t') dz dt' = k_{\alpha+\beta}(x,y,t).$$

Therefore, when f and u are as in Theorem 1.3 then we have the estimates

$$|f_\epsilon(x,t)| \lesssim k_{\alpha+\epsilon}(x,0,t), \quad |u_\epsilon((x,t),s_0)| \lesssim k_{\beta+\epsilon}(x,0,t).$$

Hence we will be working with these functions instead of f and u.

Following the same strategy used in the proof of Theorem 3.1, we reduce the proof of Theorem 1.3 to Hardy's theorem for the Hankel transform. In order to do that we need the Hecke-Bochner formula for the Hermite projection operators $P_k(\lambda)$. For each integer p, we define $\mathscr{P}_p$ to be the space of all polynomials of the form $\sum_{|\alpha|=p} a_\alpha x^\alpha$. Let $\mathscr{H}_p := \{P \in \mathscr{P}_p : \Delta p = 0\}$, where Δ denotes the Laplacian on $\mathbb{R}^n$. The elements of $\mathscr{H}_p$ are called solid harmonics of degree p. We will denote by $\mathscr{S}_p$ the space of all restrictions of solid harmonics of degree p, to the sphere S^{n-1}. Then it is well known that the space $L^2(S^{n-1})$ is the orthogonal direct sum of the spaces $\mathscr{S}_p$, with $p \geq 0$. We choose an orthonormal basis $\{Y_p^j | 1 \leq j \leq d(p)\}$ for $\mathscr{S}_p$.

Theorem 5.1. *Let $p_0 \geq 0$ and $1 \leq j_0 \leq p_0$ be fixed integers. Then*

$$\int_{S^{n-1}} u_\epsilon^\lambda(r\omega, s_0) Y_{p_0}^{j_0}(\omega) d\sigma(\omega) = C_\lambda \, e^{-i\frac{\lambda}{2}\coth 2\lambda s_0 r^2}$$
$$\mathscr{H}_{\frac{n}{2}+p_0-1}\left(e^{-i\frac{\lambda}{2}\coth 2\lambda(.)^2}(f_\epsilon^\lambda)_{p_0,j_0}(.)\right)\left(\frac{\lambda r}{\sinh 2\lambda s_0}\right),$$

where C_λ is a constant which depends only on λ and $u_\epsilon^\lambda(x, s_0)$ is the inverse Fourier transform of $u_\epsilon((x,t), s_0)$ with respect to t.

In order to prove the above theorem we make use of the Hecke-Bochner formula for the Hermite projection operators which is stated below, see [16].

Theorem 5.2. *Let $f(x) = f_0(|x|)P(x)$, where P is a solid harmonic of degree m. Then for $|w| < 1$,*

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(\lambda) f(x) w^k &= 2i^{-(\frac{n}{2}+m-1)}(1-w^2)^{-1} w^{-(\frac{n}{2}-1)} r^{-(\frac{n}{2}+m-1)} \\ &\quad \left(\int_0^\infty f_0(s) e^{-\frac{|\lambda|}{2}\frac{1+w^2}{1-w^2}s^2} J_{\frac{n}{2}+m-1}\left(\frac{2iw|\lambda|}{1-w^2} rs\right) s^{\frac{n}{2}+m} ds\right) \\ &\quad e^{-\frac{|\lambda|}{2}\frac{1+w^2}{1-w^2}r^2} P(x).\end{aligned}$$

where $r = |x|$.

Theorem 5.1 is proved using the above formula as in the case of the sub-Laplacian. We omit the details. Once Theorem 5.1 is proved, the uniqueness theorem for the Grushin operator follows immediately from the Hardy's theorem for Hankel transforms.

6. Some concluding remarks

It would be interesting to see if Theorem 3.1 is sharp. Though we believe it is sharp we are not able to prove it. The main reason for the difficulty lies in the fact that the heat kernel $q_s(\mathbf{z}, t)$ on $\mathbb{H}^n$ does not have Gaussian decay in the central variable. For the same reason the equality case of Hardy's theorem for the group Fourier transform on the Heisenberg group is still an open problem. However, if we assume conditions on f^λ and u^λ instead of on f and u we can prove the following result.

Theorem 6.1. *Let $u(\mathbf{z}, t; s)$ be the solution to the Schrödinger equation for the sub-Laplacian $\mathscr{L}$ on $\mathbb{H}^n$ with initial condition f. Fix $\lambda \neq 0$ and suppose that*

$$|f^\lambda(\mathbf{z})| \lesssim q_\alpha^\lambda(\mathbf{z}), \qquad |u^\lambda(\mathbf{z}; s_0)| \lesssim q_\beta^\lambda(\mathbf{z})$$

for some $\alpha, \beta > 0$ and for a fixed $s_0 \in \mathbb{R}^*$. Then we have $f^\lambda(\mathbf{z}) = c_\lambda q_\alpha^\lambda(\mathbf{z}) e^{-i\frac{\lambda}{4}|\mathbf{z}|^2 \operatorname{cotg}(\lambda s_0)}$ whenever $\tanh(\alpha\lambda)\tanh(\beta\lambda) = \sin^2(\lambda s_0)$.

To prove this theorem, we can proceed as in the proof of Theorem 3.1. We end up with the estimates

$$\left|\mathscr{H}_{n+p_0+q_0-1}\left(e^{i\frac{\lambda}{4}(\cdot)^2\operatorname{cotg}(\lambda s_0)}(f^\lambda_{p_0,q_0,j_0})^{\sim}\right)(r)\right| \le c_\lambda r^{-(p_0+q_0)}e^{-\frac{\lambda}{4}\coth(\lambda\beta)\left(\frac{2\sin(\lambda s_0)}{\lambda s_0}\right)^2 r^2}$$

and

$$\left|(f^\lambda_{p_0,q_0,j_0})^{\sim}(r)\right| \le c_\lambda e^{-\frac{\lambda}{4}\coth(\lambda\alpha)r^2}.$$

We can now appeal to the equality case of Hardy's theorem for the Hankel transform (Theorem 2.4) to conclude that

$$f^\lambda_{p_0,q_0,j_0}(r) = c_\lambda(p_0,q_0,j_0)r^{p_0+q_0}e^{-\frac{\lambda}{4}\coth(\lambda\alpha)r^2}e^{-i\frac{\lambda}{4}r^2\operatorname{cotg}(\lambda s_0)}.$$

But this is not compatible with the hypothesis on f^λ unless $c_\lambda(p_0,q_0,j_0)=0$ for all $(p_0,q_0)\neq(0,0)$. Hence f^λ is radial and equals $c_\lambda q^\lambda_\alpha(\mathbf{z})e^{-i\frac{\lambda}{4}|\mathbf{z}|^2\operatorname{cotg}(\lambda s_0)}$. This proves Theorem 6.1.

The above result can be viewed as a uniqueness theorem for solutions of the Schrödinger equation associated to the twisted Laplacian $\mathscr{L}_\lambda$ defined by $\mathscr{L}(e^{i\lambda t}f(\mathbf{z})) = e^{i\lambda t}\mathscr{L}_\lambda f(\mathbf{z})$. Indeed, $q^\lambda_a(\mathbf{z})$ is the heat kernel associated to this operator. We refer to [15, (2.3.7)] for the explicit expression of $\mathscr{L}_\lambda$. We can also consider the result as an analogue of Hardy's theorem for fractional powers of the symplectic Fourier transform. In fact, the unitary operator $e^{ins}e^{-is\mathscr{L}_1}$ with $s=\frac{\pi}{2}$ is just the symplectic Fourier transform. Thus the above theorem for $s_0=\frac{\pi}{2}$ follows immediately from Hardy's theorem for the Fourier transform whereas for other values of s_0 we require a long-winding proof.

For the sake of completeness we state another result which can be considered as a theorem for fractional Fourier transform as well as a theorem for solutions of the Schrödinger equation associated to the Hermite operator $H=-\Delta+|x|^2$ on $\mathbb{R}^n$. This elliptic operator generates the Hermite semigroup whose kernel is known explicitly. We also know that $e^{\frac{i}{4}n\pi}e^{-\frac{i}{4}\pi H}$ is the Fourier transform on $\mathbb{R}^n$.

Theorem 6.2. *Let $u(x,s)=e^{-isH}f(x)$ be the solution to the Schrödinger equation*

$$i\partial_s u(x,s) - Hu(x,s) = 0,$$

with initial condition f. Suppose

$$|f(x)| = O(e^{-\alpha|x|^2}), \qquad |u(x,s_0)| = O(e^{-\beta|x|^2})$$

for some $\alpha,\beta>0$. Then $u=0$ on $\mathbb{R}^n\times\mathbb{R}$ whenever $\alpha\beta\sin^2(2s_0)>\frac{1}{4}$.

The theorem follows from Hardy's theorem for $\mathbb{R}^n$ once we realize u as the Fourier transform of a function. But this is easy to check in view of the Mehler's formula (see [15]) for the Hermite functions. In view of this formula, the kernel of e^{-isH} is given by

$$K_r(x,y) = \pi^{-n/2}(1-r^2)^{-n/2}e^{-\frac{1}{2}\frac{1+r^2}{1-r^2}(|x|^2+|y|^2)+\frac{2r}{1-r^2}x\cdot y},$$

where $r=e^{-2is}$. Therefore, the solution $u(x,s)$ can be written as

$$u(x,s) = e^{-isH}f(x) = c_{n,s}e^{\frac{i}{2}|x|^2\cot(2s)}\hat{g}_s\left(\frac{1}{\sin(2s)}y\right)$$

where $g_s(x) := e^{\frac{i}{2}|x|^2\cot(2s)}f(x)$. The assumptions on f and u translate into

$$|g_{s_0}(x)| \lesssim e^{-\alpha|x|^2}, \quad |\hat{g_{s_0}}(y)| \lesssim e^{-\beta|y|^2\sin^2(2s_0)},$$

and hence the theorem follows from Hardy's theorem.

We conclude this paper with one more remark. If we use other uncertainty principles such as Beurling's theorem or Benedick's theorem in place of Hardy's theorem we can obtain different versions of uniqueness theorems for the solutions of Schrödinger equations. For example, in [11] the authors have proved a uniqueness theorem (for symmetric spaces) under a condition of Beurling type on f and u. We restrain from stating such results as the proofs do not involve any new ideas.

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