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► To cite this version:

Salem Ben Saïd, Khalid Koufany, Genkai Zhang. Invariant trilinear forms on spherical principal series of real rank one semisimple Lie groups. *International Journal of Mathematics*, 2014, 25 (3), pp.1450017. 10.1142/S0129167X14500177 . hal-01282454

HAL Id: hal-01282454

<https://hal.science/hal-01282454>

Submitted on 3 Mar 2016

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INVARIANT TRILINEAR FORMS ON SPHERICAL PRINCIPAL SERIES OF REAL-RANK ONE SEMISIMPLE LIE GROUPS

SALEM BEN SAÏD, KHALID KOUFANY, AND GENKAI ZHANG

ABSTRACT. Let G be a connected semisimple real-rank one Lie group with finite center. We consider intertwining operators on tensor products of spherical principal series representations of G . This allows us to construct an invariant trilinear form $\mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3)$, indexed by a complex multiparameter $\underline{\nu} = (\nu_1, \nu_2, \nu_3)$ and defined on the space of smooth functions on the product of three spheres in $\mathbb{F}^n$, where $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. We then study the analytic continuation of the trilinear form with respect to (ν_1, ν_2, ν_3) , where we locate the hyperplanes containing the poles. Using a result due to Johnson and Wallach on the so-called “partial intertwining operator”, we obtain an expression for the generalized Bernstein-Reznikov integral $\mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1)$ in terms of hypergeometric functions.

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2000 Mathematics Subject Classification. Primary 43A85, 22E45; Secondary 33A75, 22E25.

Key words and phrases. Spherical principal series, intertwining operators, invariant trilinear forms, H -type groups, meromorphic continuation, generalized Bernstein-Reznikov integrals.

Research by G. Zhang partially supported by Swedish Research Council. This work started during the visit of G. Zhang to the Institut Élie Cartan at Université de Lorraine-Nancy as an invited professor. He would like to express his gratitude to the University for the support.

1. INTRODUCTION

The study of invariant trilinear forms on spherical principal series representations of $SL(2, \mathbb{R})$ have received much interests due to its applications in automorphic representation theory [4]; see also [26, 3, 31, 11, 25, 32, 8, 9] and references therein. It is closely related to the decomposition of tensor products of principals series representations. We give a brief account of some general considerations and motivate our problem.

Let π_1 and π_2 be unitary representations of a semisimple Lie group G . The appearance of a unitary representation π_3 in the tensor product $\pi_1 \otimes \pi_2$ is equivalent to the existence of *bounded* trilinear forms on the tensor product $\pi_1 \otimes \pi_2 \otimes \pi_3^*$, where π_3^* is the dual representation of π_3 . Tensor product decompositions of unitary representations have been extensively studied; for the group $SL(2, \mathbb{R})$ all decompositions of $\pi_1 \otimes \pi_2$ are known [27] and one can then construct invariant trilinear forms by following the decomposition and by using the concrete models of the representations. In particular for some triples (π_1, π_2, π_3) of unitary representations of $SL(2, \mathbb{R})$, there exists a bounded invariant trilinear form. However, if we drop the requirement of boundedness and work with smooth representations, then we may also ask the existence of invariant trilinear forms on $\pi_1 \otimes \pi_2 \otimes \pi_3^*$ (with the tensor product defined in proper sense) and the uniqueness of such forms. Such questions are of great interests in automorphic representations [3, 31, 4, 11, 25, 32] and have been studied mostly for the group $SL(2, \mathbb{R})$ or $GL(2, \mathbb{R})$.

One of the main objectives of this paper is to construct trilinear forms on smooth induced principal series representations for real-rank one Lie groups. Such representations can be realized on the unit sphere S in $\mathbb{F}^n$, where $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. Indeed, we will study the formal intertwining operators on the tensor product of the principal series representations and the corresponding trilinear forms. We will prove that those trilinear operators have analytic continuation as operators on the space of smooth functions on $S \times S \times S$ and we find their poles.

To be more specific about our results, let G be an arbitrary real-rank one semisimple Lie group, and let K be a maximal compact subgroup of G . As usual, denote by $P = MAN$ the maximal parabolic subgroup of G , where A is a vector group, M is the centralizer of A in K , and N is a simply connected nilpotent subgroup of G .

For $\nu \in \mathbb{C}$, denote by π_ν the spherical principal series representation $\text{Ind}_P^G(1 \otimes e^{\nu+\rho} \otimes 1)$ of G , where, as usual, ρ is the half sum of positive roots. There are various analytic and algebraic frames to define these representations, and we will consider the smoothly induced ones, i.e., those that can be realized as smooth complex-valued functions on the unit sphere $S = K/M \cong \mathbb{S}^{dn-1}$, where $d = 1, 2, 4$, or 8 according to $G = \text{SO}_o(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$, or $\text{F}_{4(-20)}$ (with $n = 2$).

For $f \in C^\infty(S)$, consider the integral transform, called Knapp-Stein intertwining operator,

$$\mathcal{A}_\nu f(x) = \int_S \kappa(x, y)^{\nu-\rho} f(y) d\sigma(y),$$

where $\kappa(x, y) = |1 - \langle x, y \rangle|$. It is known that when these integrals converge, they satisfy $\mathcal{A}_\nu \pi_\nu(g) = \pi_\nu^*(g) \mathcal{A}_\nu$ for all $g \in G$.

Set $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$. By means of the intertwining operator $\mathcal{A}_\nu$, we construct a bilinear operator $\mathcal{B}_{\underline{\nu}}$ from the tensor product $\pi_{\nu_2+\nu_3-\rho} \otimes \pi_{\nu_1+\nu_3-\rho}$ onto $\pi_{\nu_1+\nu_2-\rho}^*$ (see (4.1)). Then we use it to define an invariant trilinear form $\mathcal{K}_{\underline{\nu}}$ on $\pi_{\nu_2+\nu_3-\rho} \otimes \pi_{\nu_1+\nu_3-\rho} \otimes \pi_{\nu_1+\nu_2-\rho}$ by

$$\mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3) := \langle \mathcal{B}_{\underline{\nu}}(f_1 \otimes f_2), f_3 \rangle_{L^2(S)},$$

which will be the main object of this paper. More explicitly,

$$\mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3) = \int_{(S)^3} f_1(x) f_2(y) f_3(z) \kappa(x, y)^{\nu_3-\rho} \kappa(x, z)^{\nu_2-\rho} \kappa(y, z)^{\nu_1-\rho} d\sigma(x) d\sigma(y) d\sigma(z). \quad (1.1)$$

We study the absolute convergence of the triple integral (1.1). Further, we investigate the meromorphic continuation of the map $\underline{\nu} \mapsto \mathcal{K}_{\underline{\nu}}$ and we locate the hyperplanes containing the poles, where we prove that there are two classes of poles. To achieve these results we use a Heisenberg (or non-compact) type model of $\mathcal{K}_{\underline{\nu}}$, which transfers the problems to the flat setting (S being replaced by a Euclidean space of the same dimension). This transfer is due to the existence of a diffeomorphism from the H -type group $\mathbb{F}^{n-1} \times \text{Im}(\mathbb{F})$ onto $S \setminus \{-e_1\}$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ according to $G = \text{SO}_o(n, 1)$, $\text{SU}(n, 1)$, $\text{Sp}(n, 1)$, or $\text{F}_{4(-20)}$, with $n = 2$ for $\mathbb{F} = \mathbb{O}$. We pin down that if G is a Lie group as above and $G = KAN$ is the Iwasawa decomposition of G , then the H -type group $\mathbb{F}^{n-1} \times \text{Im}(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, is (up to a natural identification) nothing but the nilpotent part N . We refer the reader to the subsection 4.2 for more details.

We then compute the residues of $\mathcal{K}_{\underline{\nu}}$ for only one class of poles. The expression of the residues involves conformally covariant differential operators on the unit sphere S . The computation of the residues at the second class of poles is more involved, and is worth further study.

For the $G = \text{SO}_o(n, 1)$ case, the absolute convergence of (1.1) and the meromorphic continuation were established by Clerc *et al.* in [8] (see also [9]), while the corresponding residue computation was done by Beckmann and Clerc in [6].

It is known that the invariant trilinear form $\mathcal{K}_{\underline{\nu}}$ is, up to isomorphism, unique for $G = \text{SO}_o(n, 1)$ (see [9, Theorem 3.1]). Otherwise, $\mathcal{K}_{\underline{\nu}}$ is in general not unique (see [20]).

We close the paper by showing that $\mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1)$ can be expressed in terms of hypergeometric functions. The strategy of the computation is to use a result due to Johnson and Wallach [17, 18] on the “partial intertwining operator” from π_ν to π_ν^* (see Theorem 3.3). The case $G = \text{SO}_o(n, 1)$ goes back to Clerc *et al.* [8] (the case $n = 1$ was firstly established by Bernstein and Reznikov [3]).

It is worthwhile to mention the very recent preprint [19] where Kobayashi discusses general perspectives related to the main object of the present paper.

We now give a summary of the results in this paper. After a brief recall on semisimple real-rank one Lie groups, we introduce in Section 3 the spherical principal series representation $\pi_\nu = \text{Ind}_P^G(1 \otimes e^{\nu+\rho} \otimes 1)$ and the Knapp-Stein intertwining operator. In Section 4 we define the trilinear invariant form $\mathcal{K}_{\underline{\nu}}$, and in Section 5 we establish the absolute convergence of the triple integral (1.1). Section 6 is devoted to the analytic continuation of the map $\underline{\nu} \mapsto \mathcal{K}_{\underline{\nu}}$, where in Section 7 we compute the residues at only

one class of poles. In Section 8 we evaluate the generalized Bernstein-Reznikov integral $\mathcal{K}_{\underline{z}}(1 \otimes 1 \otimes 1)$ in terms of hypergeometric functions.

2. NOTATION

In this section we state some of the basic results concerning semisimple Lie groups of real-rank one.

Let $\mathbb{F}$ denote one of the fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the quaternions), or the division algebra $\mathbb{O}$ (the octonions) of real dimension $d = 1, 2, 4$, or 8 , respectively. Henceforth $\mathbb{F}_0 := \{q \in \mathbb{F} : q + \bar{q} = 0\}$, where “ $\bar{\cdot}$ ” being the usual conjugation in $\mathbb{F} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\bar{q} = q$ for $q \in \mathbb{R}$.

Consider the vector space $\mathbb{F}^{n+1}$ and let $\{e_1, \dots, e_{n+1}\}$ be its standard basis. On $\mathbb{F}^{n+1} \times \mathbb{F}^{n+1}$ we consider the bilinear form

$$\beta(x, y) = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n - x_{n+1} \bar{y}_{n+1}, \quad (2.1)$$

and we denote by γ the corresponding quadratic form

$$\gamma(x) = |x_1|^2 + \dots + |x_n|^2 - |x_{n+1}|^2.$$

The unit ball in $\mathbb{F}^n$ is

$$\mathcal{B}(\mathbb{F}^n) := \{(x_1, \dots, x_n) \in \mathbb{F}^n : |x_1|^2 + \dots + |x_n|^2 < 1\}.$$

2.1. The classical rank one Lie groups. Let $\mathbb{F}$ be one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Let $n \geq 2$ and G be the connected component of the group of all $\mathbb{F}$ -linear automorphisms of $\mathbb{F}^{n+1}$ which preserve γ with the additional property that if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we consider only automorphisms of determinant 1. Then G is $\mathrm{SO}_o(n, 1)$, $\mathrm{SU}(n, 1)$ or $\mathrm{Sp}(n, 1)$ according to whether $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (recall that $\mathrm{SU}(1, 1)$ is locally isomorphic to $\mathrm{SO}_o(2, 1)$, $\mathrm{Sp}(1, 1)$ is locally isomorphic to $\mathrm{SO}_o(4, 1)$, and $\mathrm{SO}_o(1, 1)$ is not semisimple).

The group G acts on the unit ball $\mathcal{B}(\mathbb{F}^n)$ by fractional transformations,

$$g \cdot x = (ax + b)(cx + d)^{-1}, \quad x \in \mathcal{B}(\mathbb{F}^n) \quad (2.2)$$

whenever $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with a an $n \times n$ matrix, b and c in $\mathbb{F}^n$, and d in $\mathbb{F}$.

On the Lie algebra $\mathfrak{g}$ of G we consider the Cartan involution defined by $\theta(X) := -\bar{X}^t$. Then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ \bar{Y}^t & 0 \end{pmatrix} \mid Y \in \mathbb{F}^n \right\},$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \mid \begin{array}{l} X_1 \text{ is } n \times n \text{ skew hermitian, } X_2 \text{ is a skew scalar,} \\ X_2 + \mathrm{tr}(X_1) = 0 \text{ if } \mathbb{F} = \mathbb{C}, \text{ and } X_2 = 0 \text{ for } \mathbb{F} = \mathbb{R} \end{array} \right\}.$$

Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then K is of the form

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \mid k \in \mathrm{SO}_o(n) \right\} \cong \mathrm{SO}_o(n), \quad \text{if } \mathbb{F} = \mathbb{R},$$

$$\left\{ \begin{pmatrix} u & 0 \\ 0 & c \end{pmatrix} \mid u \in \mathrm{U}(n), |c| = 1, c \det u = 1 \right\} \cong \mathrm{U}(u), \quad \text{if } \mathbb{F} = \mathbb{C},$$

$$\left\{ \begin{pmatrix} w & 0 \\ 0 & u \end{pmatrix} \mid w \in \mathrm{Sp}(n), u \in \mathrm{Sp}(1) \right\} \cong \mathrm{Sp}(n) \times \mathrm{Sp}(1), \quad \text{if } \mathbb{F} = \mathbb{H}.$$

It is well known that the action (2.2) of G on $\mathcal{B}(\mathbb{F}^n)$ is transitive, and it realizes $\mathcal{B}(\mathbb{F}^n)$ as the symmetric space G/K where K is viewed as the stabilizer of the base point $\mathbf{0} \in \mathcal{B}(\mathbb{F}^n)$.

Let $H_0 \in \mathfrak{p}$ be the following element

$$H_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The endomorphism $\mathrm{ad}(H_0)$ is diagonalized with eigenvalues $\{0, \pm 1\}$ if $\mathbb{F} = \mathbb{R}$, and $\{0, \pm 1, \pm 2\}$ if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. The space $\mathfrak{a} := \mathbb{R} \cdot H_0$ is a maximal abelian subspace of $\mathfrak{p}$ and there exists a linear functional $\alpha \in \mathfrak{a}^*$ such that $\alpha(H_0) = 1$ and so the positive restricted roots $\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha, 2\alpha\}$. Then the root spaces are given by

$$\begin{aligned} \mathfrak{g}_\alpha &= \left\{ \begin{pmatrix} 0 & \overline{X}^t & 0 \\ -X & 0 & X \\ 0 & \overline{X}^t & 0 \end{pmatrix} \mid X \in \mathbb{F}^{n-1} \right\}, \\ \mathfrak{g}_{2\alpha} &= \begin{cases} \{0\}, & \text{if } \mathbb{F} = \mathbb{R}, \\ \left\{ \begin{pmatrix} Y & 0 & -Y \\ 0 & 0 & 0 \\ Y & 0 & -Y \end{pmatrix} \mid Y \in \mathbb{F}_0 \right\}, & \text{if } \mathbb{F} = \mathbb{C} \text{ or } \mathbb{H}. \end{cases} \end{aligned}$$

It follows that $\dim_{\mathbb{R}}(\mathfrak{g}_\alpha) = d(n-1)$ and $\dim_{\mathbb{R}}(\mathfrak{g}_{2\alpha}) = d-1$, where $d = \dim_{\mathbb{R}} \mathbb{F}$. Let $\rho = \frac{1}{2}(\dim(\mathfrak{g}_\alpha) + 2\dim(\mathfrak{g}_{2\alpha}))$. Then

$$\rho = (n-1)/2, \ n, \text{ or } 2n+1 \quad (2.3)$$

according to $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

As usual, put $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ and $\overline{\mathfrak{n}} = \theta(\mathfrak{n}) = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$, and let $N, \overline{N}$ and A denote the analytic subgroups of G with Lie algebra $\mathfrak{n}, \overline{\mathfrak{n}}$ and $\mathfrak{a}$, respectively. More precisely,

$$\begin{aligned}
N &= \left\{ \begin{pmatrix} 1 + Y - \frac{1}{2}|X|^2 & \overline{X}^t & -Y + \frac{1}{2}|X|^2 \\ -X & I_{n-1} & X \\ Y - \frac{1}{2}|X|^2 & \overline{X}^t & 1 + Y - \frac{1}{2}|X|^2 \end{pmatrix} \mid X \in \mathbb{F}^{n-1}, Y \in \mathbb{F}_0 \right\}, \\
\overline{N} &= \left\{ \begin{pmatrix} 1 + Y - \frac{1}{2}|X|^2 & \overline{X}^t & Y - \frac{1}{2}|X|^2 \\ -X & I_{n-1} & -X \\ -Y + \frac{1}{2}|X|^2 & -\overline{X}^t & 1 - Y + \frac{1}{2}|X|^2 \end{pmatrix} \mid X \in \mathbb{F}^{n-1}, Y \in \mathbb{F}_0 \right\}, \\
A &= \left\{ \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}.
\end{aligned}$$

Let M be the stabilizer of $e_1 = (1, 0, \dots, 0) \in \mathbb{F}^{n+1}$ in K . Then M is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathrm{SO}_o(n-1) \right\} \cong \mathrm{SO}_o(n-1), \quad \text{if } \mathbb{F} = \mathbb{R},$$

$$\left\{ \begin{pmatrix} c & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & c \end{pmatrix} \mid |c| = 1, \lambda \in U(n-1), c^2 \det \lambda = 1 \right\} \cong \mathbb{Z}_2 \times U(n-1), \quad \text{if } \mathbb{F} = \mathbb{C},$$

$$\left\{ \begin{pmatrix} u & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & u \end{pmatrix} \mid u \in \mathrm{Sp}(1), w \in \mathrm{Sp}(n-1) \right\} \cong \mathrm{Sp}(n-1) \times \mathrm{Sp}(1), \quad \text{if } \mathbb{F} = \mathbb{H}.$$

The set $P := MAN$ is a maximal parabolic subgroup of G and the homogeneous space $G/P = K/M$ may be identified with the unit sphere $S := \mathbb{S}^{dn-1}$ in $\mathbb{F}^n$ with $d = 1, 2$, or 4 according to $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

2.2. The exceptional rank one Lie group. The group $F_{4(-20)}$ is the only one nonclassical simple Lie group of real-rank one. It is a real form of F_4 . Its Lie algebra $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ is isomorphic to the Lie algebra $\mathrm{Der}(V)$ of derivations of the Jordan algebra $(V, \circ)$ of 3×3 octonion matrices A of the form

$$A = \begin{pmatrix} \alpha_1 & a_3 & a_2 \\ \overline{a}_3 & \alpha_2 & a_1 \\ -\overline{a}_2 & -\overline{a}_1 & \alpha_3 \end{pmatrix}, \quad a_i \in \mathbb{O}, \alpha_i \in \mathbb{R}, \quad i = 1, 2, 3,$$

with multiplication given by $A \circ B = \frac{1}{2}(AB + BA)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition with $\mathfrak{k} = \mathfrak{spin}(9)$ and

$$\mathfrak{p} = \left\{ D_Q \in \mathrm{Der}(V) \mid Q = \begin{pmatrix} 0 & X^t \\ \overline{X} & 0 \end{pmatrix}, X \in \mathbb{O}^2 \right\},$$

where $D_Q(A) = QA - AQ$ with $A \in V$. Henceforth, we will identify $\mathfrak{p}$ with $\mathbb{O}^2$. The space

$$\mathfrak{a} = \{D_Q \in \text{Der}(V) \mid Q = tE_{13} + tE_{31}, t \in \mathbb{R}\}$$

is a maximal abelian subspace of $\mathfrak{p}$. We choose $H_0 = D_{Q_0}$ with $Q_0 = E_{13} + E_{31}$, and $\alpha \in \mathfrak{a}^*$ such that $\alpha(H_0) = 1$. The zero root space $\mathfrak{m}$ of H_0 in $\mathfrak{k}$ is the Lie algebra $\mathfrak{spin}(7)$, and the direct sum $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra. The 15-dimensional nilpotent Lie algebra $\bar{\mathfrak{n}} := \theta(\mathfrak{n})$ decomposes into $\bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ with

$$\begin{aligned} \mathfrak{g}_{-\alpha} &= \left\{ D_Q : Q = \begin{pmatrix} 0 & -\bar{X} & 0 \\ X & 0 & X \\ 0 & \bar{X} & 0 \end{pmatrix}, X \in \mathbb{O} \right\}, \\ \mathfrak{g}_{-2\alpha} &= \left\{ D_Q : Q = \begin{pmatrix} Y & 0 & Y \\ 0 & 0 & 0 \\ \bar{Y} & 0 & \bar{Y} \end{pmatrix}, Y \in \mathbb{O}_0 \right\}. \end{aligned}$$

The half sum of positive roots is $\rho = 11$.

Let $K = \text{Spin}(9)$ and $M = \text{Spin}(7)$ be the analytic subgroups of $F_{4(-20)}$ with Lie algebras $\mathfrak{k}$ and $\mathfrak{m}$, respectively. It is known that K/M can be identified with the unit sphere $S = \{x = (x_1, x_2) \in \mathbb{O}^2 : |x_1|^2 + |x_2|^2 = 1\}$. We refer to [30] for more details on the exceptional group $F_{4(-20)}$.

2.3. Decomposition of the spherical harmonics. Let $\hat{K}$ be the set of equivalence classes of finite dimensional unitary representations of K . For $(\tau, V_\tau) \in \hat{K}$ put

$$V_\tau^M := \{v \in V_\tau \mid \tau(m)v = v \text{ for all } m \in M\},$$

and

$$\hat{K}_M = \{\tau \in \hat{K} \mid V_\tau^M \neq \{0\}\}.$$

It is well known from [22] that $L^2(S)$ has a decomposition into irreducible K -invariant subspaces,

$$L^2(S) = \sum_{\tau \in \hat{K}_M} V_\tau. \quad (2.4)$$

For later use, we pin down the following characterizations of V_τ and $\hat{K}_M$.

Fact 2.1. Below $(a)_p = a(a+1) \cdots (a+p-1)$ denotes the Pochhammer symbol.

- (1) For $\mathbb{F} = \mathbb{R}$, V_τ is the space $H^p(\mathbb{R}^n)$ of spherical harmonics on $S = \mathbb{S}^{n-1}$ of degree p , and $\hat{K}_M = \mathbb{N}$. We note that

$$\dim H^p(\mathbb{R}^n) = \frac{(n-2)_p \left(\frac{n}{2}\right)_p}{p! \left(\frac{n-2}{2}\right)_p}. \quad (2.5)$$

- (2) For $\mathbb{F} = \mathbb{C}$, V_τ is the space $H^{p,q}(\mathbb{C}^n)$ of spherical harmonics on $\mathbb{C}^n$ of degree $p+q$, holomorphic of degree p and antiholomorphic of degree q . In this case, we have $\hat{K}_M = \mathbb{N} \times \mathbb{N}$ and

$$\dim H^{p,q}(\mathbb{C}^n) = \frac{(p+q+n-1)(p+1)_{n-2}(q+1)_{n-2}}{\Gamma(n)\Gamma(n-1)}. \quad (2.6)$$

- (3) For $\mathbb{F} = \mathbb{H}$ we will express V_τ in terms of highest weights of $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1) \cong \mathrm{Sp}(n) \times \mathrm{SU}(2)$. The representations of K are of the form $\tau = \tau_1 \times \tau_2$, which will be denote by (τ_1, τ_2) and further identified with their highest weights. The root system of $\mathrm{Sp}(n)$ is of type C_n , and let $\{\alpha_1, \dots, \alpha_{n-1}, \alpha_n\}$ be the set of simple roots with α_n the longest one. Denote by $\{\lambda_1, \dots, \lambda_n\}$ the corresponding fundamental weights with λ_1 the defining representation on $\mathbb{C}^{2n}$. The representation of $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$ on the symmetric tensor power $\odot^q(\mathbb{C}^2) = \mathbb{C}^{q+1}$ will be identified with $q \in \mathbb{N}$ for simplicity. The V_τ in (2.4) are of the form $(q\lambda_1 + \frac{p-q}{2}\lambda_2, q)$ of $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$, and therefore $\widehat{K}_M$ may be identified with the set of pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p - q$ is a nonnegative even integer. In view of the above identification, Weyl's dimensional formula gives

$$\dim V_\tau = c_0(q+1)(2n+2m+q-1) \frac{\Gamma(2n+q-1)}{\Gamma(q+1)} \frac{(2n+q-1)_m (2n-2)_m}{m!(q+2)_m} \quad (2.7)$$

where $c_0 = 16\Gamma(2n) \prod_{j=3}^n (j-1)^2 (2n-j)^2$ and $m = (p-q)/2$.

- (4) Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be a choice of simple roots in Dynkin's diagram for $K = \mathrm{Spin}(9)$. Denote by $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the corresponding fundamental weights. For $p, q \in \mathbb{N}$, the space V_τ may be identified with the spin-representation of K of highest weight $\frac{p-q}{2}\lambda_1 + q\lambda_4$. The corresponding $\widehat{K}_M$ coincides with the set of pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p - q$ is a nonnegative even integer. Define e_1, e_2, e_3, e_4 in terms of dual basis of the α_i 's (cf. [7]). Then $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4$, and $\frac{p-q}{2}\lambda_1 + q\lambda_4 = \frac{p}{2}e_1 + \frac{q}{2}e_2 + \frac{q}{2}e_3 + \frac{q}{2}e_4$. By Weyl's dimensional formula, it follows that

$$\dim V_\tau = \frac{48}{7}(2m+q+7) \frac{(7)_q (6)_q}{q! (3)_q} \frac{(4)_m}{m!} \frac{(q+7)_m}{(q+4)_m}, \quad (2.8)$$

with $m = (p-q)/2$.

3. INDUCED REPRESENTATIONS AND INTERTWINING OPERATORS

3.1. Spherical principal series representations. Let us introduce the automorphy kernel κ defined for $x, y \in S$ by

$$\kappa(x, y) := |\beta((x, 1), (y, 1))| = |1 - \langle x, y \rangle|, \quad (3.1)$$

where β denotes the bilinear form (2.1), and $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product on $\mathbb{F}^n$. The kernel κ has the transformation property

$$\kappa(g \cdot x, g \cdot y) = j(g, x) \kappa(x, y) j(g, y), \quad g \in G, \quad (3.2)$$

where $j(g, x)$ is a positive constant, called the conformal factor of g at x . For the classical rank one groups, $j(g, x) = |cx + d|^{-1}$ whenever $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The function j is a cocycle,

$$j(g_1 g_2, x) = j(g_1, g_2 \cdot x) j(g_2, x), \quad g_1, g_2 \in G \text{ and } x \in S,$$

and satisfies

$$\begin{aligned} j(k, x) &= 1, & \text{for } k \in K \text{ and } x \in S, \\ j(g^{-1}, g \cdot x) &= j(g, x)^{-1}, & \text{for } g \in G \text{ and } x \in S. \end{aligned}$$

Moreover, $j(g, x)$ may be related to the Jacobian $J(g, x)$ of g at x as follows

$$j(g, x) = J(g, x)^{\frac{1}{2\rho}},$$

where $\rho = (n-1)/2$, n , $2n+1$, or 11 according to $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.

It follows that if f is a continuous function of S and $g \in G$, then

$$\int_S f(g^{-1} \cdot x) d\sigma(x) = \int_S f(x) j(g, x)^{2\rho} d\sigma(x), \quad (3.3)$$

where $d\sigma(x)$ is the normalized K -invariant measure on S .

For $\nu \in \mathfrak{a}_{\mathbb{C}}^* \cong \mathbb{C}$ let $X_{\nu}^{\infty}(G)$ to be the space of smooth functions $f : G \rightarrow \mathbb{C}$ such that

$$f(gme^{tH_0}n) = e^{-(\rho+\nu)t} f(g), \quad g \in G, me^{tH_0}n \in MAN.$$

We will denote the action of G on $X_{\nu}^{\infty}(G)$ by

$$(\pi_{\nu}(g)f)(x) = f(g^{-1}x), \quad g, x \in G.$$

In the standard notation π_{ν} is smoothly induced $\text{Ind}_{MAN}^G(1 \otimes e^{\rho+\nu} \otimes 1)$ and this action of G is said to be a spherical principal series representation. Since every $f \in X_{\nu}^{\infty}(G)$ is invariant under the action of M , then π_{ν} may be realized on the space $C^{\infty}(S)$ as following:

$$\pi_{\nu}(g)f(x) = j(g^{-1}, x)^{\rho+\nu} f(g^{-1} \cdot x), \quad g \in G, x \in S. \quad (3.4)$$

By an abuse of notation, we will denote this representation by $(\pi_{\nu}, X_{\nu}^{\infty})$.

The representations π_{ν} and $\pi_{-\nu}$ are dual in the sense that

$$\int_S \pi_{\nu}(g)f_1(x) \overline{\pi_{-\nu}(g)f_2(x)} d\sigma(x) = \int_S f_1(x) \overline{f_2(x)} d\sigma(x)$$

for all $f_1 \in X_{\nu}^{\infty}$ and $f_2 \in X_{-\nu}^{\infty}$.

3.2. Knapp-Stein intertwining operator. For $\nu \in \mathbb{C}$, we define (whenever it makes sense) the operator $\mathcal{A}_{\nu}$ on X_{ν}^{∞} by

$$\mathcal{A}_{\nu}f(x) = \int_S f(y) \kappa(x, y)^{-\rho+\nu} d\sigma(y), \quad (3.5)$$

where $\kappa(x, y)$ is as in (3.1). The integral transform $\mathcal{A}_{\nu}$ are the so-called Knapp-Stein intertwining operator of the spherical principal series (see [24]). Proposition 7.8 in [23] tells us that when $\text{Re } \nu > 0$, the integral

$$\int_S f(y) |1 - \langle x, y \rangle|^{-\rho+\nu} d\sigma(y)$$

converges absolutely for all $x \in S$. In particular, for such ν 's, we have:

Proposition 3.1. *The operator $\mathcal{A}_\nu$ satisfies*

$$\mathcal{A}_\nu \circ \pi_\nu(g) = \pi_{-\nu}(g) \circ \mathcal{A}_\nu, \quad \forall g \in G. \quad (3.6)$$

That is, $\mathcal{A}_\nu : X_\nu^\infty \longrightarrow X_{-\nu}^\infty$ is G -equivariant.

We will normalize the intertwining operator $\mathcal{A}_\nu$ by

$$C_{\mathbb{F}}(\nu) = \int_S \kappa(e_1, y)^{-\rho+\nu} d\sigma(y), \quad (3.7)$$

where $e_1 = \overbrace{(1, 0, \dots, 0)}^n$, with $n = 2$ for $\mathbb{F} = \mathbb{O}$. The following elementary lemma can be derived from the formula of Harish-Chandra's c -function of G [15]. However, to make this paper self-contained, we will give a proof for it.

Lemma 3.2. *For $\nu \in \mathbb{C}$ such that $\operatorname{Re} \nu > 0$ we have*

$$C_{\mathbb{F}}(\nu) = \Gamma\left(\frac{dn}{2}\right) \frac{\Gamma(\nu)}{\Gamma\left(\frac{\rho+\nu}{2}\right) \Gamma\left(\frac{\rho+\nu-d+2}{2}\right)}, \quad (3.8)$$

where $d = 1, 2, 4$ or 8 and $\rho = (n-1)/2, n, 2n+1$, or 11 for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, respectively. The function $\nu \mapsto C_{\mathbb{F}}(\nu)$ has a meromorphic extension to $\mathbb{C}$ with at most simple poles along the hyperplanes $P_k = \{\nu \in \mathbb{C} : \nu = -k\}$, with $k \in \mathbb{N}$. Its zeros occur at most along the hyperplanes $Z_k = \{\nu \in \mathbb{C} : \nu = -2k - \rho \text{ or } \nu = -2k - \rho + d - 2\}$, with $k \in \mathbb{N}$.

Proof. For $\mathbb{F} = \mathbb{R}$, the value of $C_{\mathbb{F}}(\nu)$ follows from the following fact : If f is a continuous function on the sphere $\mathbb{S}^{n-1}$ such that $f(x_1, \dots, x_n) = F(x_1)$, where F is a continuous function on $[-1, 1]$, then

$$\int_{\mathbb{S}^{n-1}} f(x) d\sigma(x) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)} \int_{-1}^1 F(t) (1-t^2)^{\frac{n-3}{2}} dt.$$

For $\mathbb{F} = \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$, we have

$$\int_S f(\operatorname{Re} y_1 + i|\operatorname{Im} y_1|) d\sigma(y) = \int_0^1 \int_0^\pi f(re^{i\theta}) dm_{\alpha,\beta}(r, \theta), \quad (3.9)$$

where f is a function on the upper half unit disk and

$$dm_{\alpha,\beta}(r, \theta) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma\left(\beta+\frac{1}{2}\right)} (1-r^2)^{\alpha-\beta-1} (\sin \theta)^{2\beta} r^{2\beta+1} dr d\theta$$

with $\alpha = n-1$ and $\beta = 0$ for $\mathbb{F} = \mathbb{C}$, $\alpha = 2n-1$ and $\beta = 1$ for $\mathbb{F} = \mathbb{H}$, and finally $\alpha = 7$ and $\beta = 3$ for $\mathbb{F} = \mathbb{O}$. Let us consider the case $\mathbb{F} = \mathbb{C}$. The remaining cases hold in the

same manner. The integration formula (3.9) implies

$$\begin{aligned}
& \int_S |1 - \langle e_1, y \rangle|^{-\rho+\nu} d\sigma(y) \\
&= \frac{2(n-1)}{\pi} \int_0^1 \int_0^\pi |1 - re^{i\theta}|^{-\rho+\nu} (1-r^2)^{n-2} r dr d\theta \\
&= \frac{2(n-1)}{\pi} \int_0^1 r (1-r^2)^{n-2} \left(\int_0^\pi (1 - 2r \cos \theta + r^2)^{-\frac{\rho+\nu}{2}} d\theta \right) dr \\
&= 2(n-1) \int_0^1 r (1-r^2)^{n-2} {}_2F_1 \left(\frac{\rho-\nu}{2}, \frac{\rho-\nu}{2}; 1; r^2 \right) dr \\
&= {}_2F_1 \left(\frac{\rho-\nu}{2}, \frac{\rho-\nu}{2}; n; 1 \right) \\
&= \frac{\Gamma(\nu)\Gamma(n)}{\Gamma\left(\frac{n+\nu}{2}\right)^2}.
\end{aligned}$$

Above we have used the following facts (see [1, 12]):

$$\int_0^\pi (\sin \theta)^{2b-1} (1 - 2r \cos \theta + r^2)^{-a} d\theta = \frac{\sqrt{\pi}\Gamma(b)}{\Gamma\left(b + \frac{1}{2}\right)} {}_2F_1 \left(a, a - b + \frac{1}{2}; b + \frac{1}{2}; r^2 \right),$$

and

$$\int_0^1 r^{c-1} (1-r)^{\delta-1} {}_2F_1(a, b; c; \gamma r) dr = \frac{\Gamma(c)\Gamma(\delta)}{\Gamma(\delta+c)} {}_2F_1(a, b; c+\delta; \gamma).$$

□

Now, if we normalize the intertwining operator $\mathcal{A}_\nu$ by the constant $C_{\mathbb{F}}(\nu)$ then the map

$$\nu \longmapsto \mathcal{N}_\nu := \frac{1}{C_{\mathbb{F}}(\nu)} \mathcal{A}_\nu$$

extends to a meromorphic function on $\mathbb{C}$ and defines an intertwining operator between π_ν and $\pi_{-\nu}$ whenever ν is not a pole. Moreover, in the light of the K -decomposition $L^2(S) = \sum_{\tau \in \widehat{K}_M} V_\tau$ (see (2.4)), and, since $\mathcal{N}_\nu$ commutes with the action of K on functions on the sphere S , it follows from Schur's lemma that $\mathcal{N}_\nu$ acts on each V_τ by a scalar,

$$\mathcal{N}_{\nu|_{V_\tau}} = a_\nu(\tau) \text{id}, \tag{3.10}$$

where the coefficients $a_\nu(\tau)$ are calculated by Johnson and Wallach [17] for the classical groups and by Johnson [18] for the exceptional group. The following statement is nothing other than Theorem 6.1 in [17] and Theorem 5.2 in [18], with changes appropriate to the fact that they use a different normalization of principal series representations.

Theorem 3.3. *Retaining the notation of Fact 2.1, we have:*

1) For $\mathbb{F} = \mathbb{R}$ and $p \in \mathbb{N}$,

$$a_\nu(p) = \frac{\left(\frac{n-1}{2} - \nu\right)_p}{\left(\frac{n-1}{2} + \nu\right)_p}. \tag{3.11}$$

2) For $\mathbb{F} = \mathbb{C}$ and $(p, q) \in \mathbb{N} \times \mathbb{N}$,

$$a_\nu(p, q) = \frac{\left(\frac{n-\nu}{2}\right)_p \left(\frac{n-\nu}{2}\right)_q}{\left(\frac{n+\nu}{2}\right)_p \left(\frac{n+\nu}{2}\right)_q}. \quad (3.12)$$

3) For $\mathbb{F} = \mathbb{H}$ and $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p - q$ is a non-negative even integer,

$$a_\nu(p, q) = \frac{\left(\frac{2n-\nu-1}{2}\right)_{\frac{p-q}{2}} \left(\frac{2n-\nu+1}{2}\right)_{\frac{p+q}{2}}}{\left(\frac{2n+\nu-1}{2}\right)_{\frac{p-q}{2}} \left(\frac{2n+\nu+1}{2}\right)_{\frac{p+q}{2}}}. \quad (3.13)$$

4) For $\mathbb{F} = \mathbb{O}$ and $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p - q$ is a non-negative even integer,

$$a_\nu(p, q) = \frac{\left(\frac{5-\nu}{2}\right)_{\frac{p-q}{2}} \left(\frac{11-\nu}{2}\right)_{\frac{p+q}{2}}}{\left(\frac{5+\nu}{2}\right)_{\frac{p-q}{2}} \left(\frac{11+\nu}{2}\right)_{\frac{p+q}{2}}}. \quad (3.14)$$

4. AN INVARIANT TRILINEAR FORM

In this section we will define certain bilinear intertwining operators and the corresponding trilinear forms. Most of the general perspectives and motivations have been explained in [8].

4.1. A construction of a trilinear form. Let ν_1 and ν_2 be two complex numbers, and consider the representations π_{ν_1} and π_{ν_2} . For $f \in C^\infty(S \times S)$ and $g \in G$, we have

$$\pi_{\nu_1} \otimes \pi_{\nu_2}(g)f(x, y) = j(g^{-1}, x)^{\rho+\nu_1} j(g^{-1}, x)^{\rho+\nu_2} f(g^{-1} \cdot x, g^{-1} \cdot y).$$

The evaluation of the product of two functions in $C^\infty(S)$ defines an intertwining operator

$$\Delta : X_{-\nu_2}^\infty \otimes X_{-\nu_1}^\infty \longrightarrow X_{-\nu_1-\nu_2+\rho}^\infty, \quad (\Delta f)(x) \mapsto f(x, x).$$

The tensor product $X_{-\nu_2}^\infty \otimes X_{-\nu_1}^\infty$ here is defined just to be the space $C^\infty(S \times S)$ with the tensor product action of $G \times G$. The formal intertwining relation is obvious. Thus the composition of Δ and the tensor product $\mathcal{A}_{\nu_1} \otimes \mathcal{A}_{\nu_2}$ defines an intertwining operator from $X_{\nu_1}^\infty \otimes X_{\nu_2}^\infty$ to $X_{-\nu_1-\nu_2+\rho}^\infty$. To incorporate another complex parameter ν_3 in the intertwining relations, we observe that the multiplication operator

$$\mathcal{M}_{\nu_3} : X_{\nu_2+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_3-\rho}^\infty \longrightarrow X_{\nu_2}^\infty \otimes X_{\nu_1}^\infty$$

defined by

$$\mathcal{M}_{\nu_3} f(x, y) = \kappa(x, y)^{-\rho+\nu_3} f(x, y),$$

where $\kappa(x, y)$ is as in (3.1), is an intertwining operator. Indeed,

$$\begin{aligned} & \mathcal{M}_{\nu_3} \circ (\pi_{\nu_2+\nu_3-\rho} \otimes \pi_{\nu_1+\nu_3-\rho})(g)f(x, y) \\ &= \kappa(x, y)^{-\rho+\nu_3} j(g^{-1}, x)^{\nu_2+\nu_3} j(g^{-1}, y)^{\nu_1+\nu_3} f(g^{-1} \cdot x, g^{-1} \cdot y) \\ &= \kappa(g^{-1}x, g^{-1}y)^{-\rho+\nu_3} j(g^{-1}, x)^{\nu_2+\rho} j(g^{-1}, y)^{\nu_1+\rho} f(g^{-1} \cdot x, g^{-1} \cdot y) \\ &= j(g^{-1}, x)^{\nu_2+\rho} j(g^{-1}, y)^{\nu_1+\rho} \mathcal{M}_{\nu_3} f(g^{-1} \cdot x, g^{-1} \cdot y) \\ &= (\pi_{\nu_2} \otimes \pi_{\nu_1})(g)(\mathcal{M}_{\nu_3} f)(x, y). \end{aligned}$$

Now, for $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$, we consider the operator

$$\mathcal{B}_{\underline{\nu}} := \Delta \circ (\mathcal{A}_{\nu_1} \otimes \mathcal{A}_{\nu_2}) \circ \mathcal{M}_{\nu_3}. \quad (4.1)$$

Then

$$\mathcal{B}_{\underline{\nu}} : X_{\nu_2+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_3-\rho}^\infty \longrightarrow X_{-\nu_1-\nu_2+\rho}^\infty,$$

and its integral representation is given by

$$\mathcal{B}_{\underline{\nu}} f(z) = \int_{S \times S} f(x, y) K_{\underline{\nu}}(x, y, z) d\sigma(x) d\sigma(y), \quad (4.2)$$

where

$$K_{\underline{\nu}}(x, y, z) = \kappa(x, y)^{\nu_3-\rho} \kappa(x, z)^{\nu_2-\rho} \kappa(y, z)^{\nu_1-\rho}. \quad (4.3)$$

Since for each $\nu \in \mathbb{C}$ there is a G -invariant pairing between (π_ν, X_ν^∞) and $(\pi_{-\nu}, X_{-\nu}^\infty)$ given by $(f_1, f_2) \mapsto \int_S f_1(x) f_2(x) d\sigma(x)$, we may study the bilinear operator $\mathcal{B}_{\underline{\nu}}$ defined above by considering the trilinear form

$$\mathcal{K}_{\underline{\nu}} : (\pi_{\nu_2+\nu_3-\rho}, X_{\nu_2+\nu_3-\rho}^\infty) \otimes (\pi_{\nu_1+\nu_3-\rho}, X_{\nu_1+\nu_3-\rho}^\infty) \otimes (\pi_{\nu_1+\nu_2-\rho}, X_{\nu_1+\nu_2-\rho}^\infty) \longrightarrow \mathbb{C},$$

defined

$$\mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3) = \langle \mathcal{B}_{\underline{\nu}}(f_1 \otimes f_2), f_3 \rangle_{L^2(S)}.$$

That is

$$\mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3) = \int_{S \times S \times S} f_1(x_1) f_2(x_2) f_3(x_3) K_{\underline{\nu}}(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3), \quad (4.4)$$

where $K_{\underline{\nu}}(x_1, x_2, x_3)$ is as in (4.3). Due to the transformation property (3.2) of κ , together with the fact that $j(k, x) = 1$ for all $k \in K$ and all $x \in S$, it follows that $\mathcal{K}_{\underline{\nu}}$ is $K \times K \times K$ -invariant. That is

$$\mathcal{K}_{\underline{\nu}} : X_{\nu_2+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_2-\rho}^\infty \longrightarrow \mathbb{C}$$

is well-defined whenever the integral converges absolutely. Concerning the invariance of $\mathcal{K}_{\underline{\nu}}$, one may say more:

Theorem 4.1. *Let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ and define $\underline{\nu} = (\nu_1, \nu_2, \nu_3)$ by*

$$\begin{aligned} \nu_1 &= (-\lambda_1 + \lambda_2 + \lambda_3 + \rho)/2, \\ \nu_2 &= (-\lambda_2 + \lambda_1 + \lambda_3 + \rho)/2, \\ \nu_3 &= (-\lambda_3 + \lambda_1 + \lambda_2 + \rho)/2. \end{aligned} \quad (4.5)$$

Then

$$\mathcal{K}_{\underline{\nu}}(\pi_{\lambda_1}(g)f_1 \otimes \pi_{\lambda_2}(g)f_2 \otimes \pi_{\lambda_3}(g)f_3) = \mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3), \quad \forall g \in G, \quad (4.6)$$

whenever the integrals on both sides exist.

Proof. By means of the integral formula (3.3) and the fact that $j(g^{-1}, g \cdot x) = j(g, x)^{-1}$, the left-hand side of (4.6) becomes

$$\int_{S \times S \times S} f_1(x_1) f_2(x_2) f_3(x_3) j(g, x_1)^{\rho-\lambda_1} j(g, x_2)^{\rho-\lambda_2} j(g, x_3)^{\rho-\lambda_3} K_{\underline{\nu}}(g \cdot \mathbf{x}) d\sigma(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, x_3)$. Now the statement follows from

$$K_{\underline{\nu}}(g \cdot \mathbf{x}) = j(g, x_1)^{-2\rho+\nu_2+\nu_3} j(g, x_2)^{-2\rho+\nu_1+\nu_3} j(g, x_3)^{-2\rho+\nu_1+\nu_2} K_{\underline{\nu}}(\mathbf{x}).$$

□

Remark 4.2. It is known that the invariant trilinear form $\mathcal{K}_{\underline{\nu}}$ is, up to isomorphism, unique for $G = \mathrm{SO}_o(n, 1)$ (see [9, Theorem 3.1]). Otherwise, $\mathcal{K}_{\underline{\nu}}$ is in general not unique (see [20]).

Next we will develop the basic properties of the trilinear form $\mathcal{K}_{\underline{\nu}}$, namely the existence of the integral (4.4) and the analytic continuation of the map $\underline{\nu} \mapsto \mathcal{K}_{\underline{\nu}}$. For that purpose we shall establish a Heisenberg-type realization of the trilinear form $\mathcal{K}_{\underline{\nu}}$, which it turned out to be much easier to handle.

4.2. A Heisenberg-type model of the trilinear form. It is known that the unit sphere S can be seen as an H -type group [21]. Starting from this point of view we will give a Heisenberg-type version of the trilinear form $\mathcal{K}_{\underline{\nu}}$. We refer to [5] for more details on the theory of H -type groups.

Recall that $\mathbb{F}_0 = \{q \in \mathbb{F} : q + \bar{q} = 0\}$. For $n \geq 2$, the set $\mathbb{F}^{n-1} \times \mathbb{F}_0$ endowed with the product

$$(Z_1, t_1)(Z_2, t_2) = (Z_1 + Z_2, t_1 + t_2 + \frac{1}{2}[Z_1, Z_2]),$$

for $Z_1, Z_2 \in \mathbb{F}^{n-1}$ and $t_1, t_2 \in \mathbb{F}_0$, is an H -type group. Here $[\cdot, \cdot]$ denotes the commutator. This is a two-step nilpotent, hence unimodular, group with Haar measure $dZdt$.

For every $r > 0$ the map δ_r defined by $\delta_r(Z, t) = (rZ, r^2t)$ is an automorphism of $\mathbb{F}^{n-1} \times \mathbb{F}_0$. The non-homogeneous norm gauge of an element (Z, t) is defined by

$$\mathcal{N}(Z, t) = \left(|t|^2 + \frac{1}{4}\|Z\|^4 \right)^{1/4}, \quad (4.7)$$

where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{F}^{n-1}$. It satisfies $\mathcal{N}(\delta_r(Z, t)) = r\mathcal{N}(Z, t)$.

The map c defined by

$$c : (Z, t) := (z_1, \dots, z_{n-1}, t) \mapsto \begin{pmatrix} \frac{1+2t-\|Z\|^2}{1-2t+\|Z\|^2} \\ \frac{-2z_1}{1-2t+\|Z\|^2} \\ \frac{-2z_2}{1-2t+\|Z\|^2} \\ \vdots \\ \frac{-2z_{n-1}}{1-2t+\|Z\|^2} \end{pmatrix} \quad (4.8)$$

is a diffeomorphism from $\mathbb{F}^{n-1} \times \mathbb{F}_0$ onto $S \setminus \{-e_1\}$, with $n = 2$ for $\mathbb{F} = \mathbb{O}$. The corresponding integration formula is given by

$$\int_S f(x) d\sigma(x) = c_{\mathbb{F}} \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} f(c(Z, t)) |1 + \|Z\|^2 - 2t|^{-2\rho} dZdt, \quad (4.9)$$

where

$$\begin{aligned}
c_{\mathbb{F}}^{-1} &= \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} |1 + \|Z\|^2 - 2t|^{-2\rho} dZ dt \\
&= \begin{cases} \frac{\pi^{n/2}}{2^{n-2} \Gamma(\frac{n}{2})} & \text{for } \mathbb{F} = \mathbb{R}, \\ \frac{\pi^{dn/2}}{2^{dn-1} \Gamma(\frac{dn}{2})} & \text{with } \begin{cases} d = 2 & \text{for } \mathbb{F} = \mathbb{C}, \\ d = 4 & \text{for } \mathbb{F} = \mathbb{H}, \\ d = 8 \text{ and } n = 2 & \text{for } \mathbb{F} = \mathbb{O}. \end{cases} \end{cases} \quad (4.10)
\end{aligned}$$

The rest of this section is devoted to the Heisenberg version of the trilinear form $\mathcal{K}_{\underline{\nu}}$. To simplify the notation we will denote the element (Z, t) by $\mathbf{n}$.

For $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{F}^{n-1} \times \mathbb{F}_0$, one may check that

$$\langle c(\mathbf{n}_1), c(\mathbf{n}_2) \rangle = \frac{-4(t_1 + \bar{t}_2 + \frac{1}{2}[Z_1, Z_2]) + 2\|Z_1 - Z_2\|^2}{(1 + \|Z_1\|^2 - 2t_1)(1 + \|Z_2\|^2 - 2\bar{t}_2)}.$$

Thus

$$\begin{aligned}
\kappa(c(\mathbf{n}_1), c(\mathbf{n}_2))^2 &= |1 - \langle c(\mathbf{n}_1), c(\mathbf{n}_2) \rangle|^2 \\
&= 16 \frac{|t_1 - t_2 - \frac{1}{2}[Z_1, Z_2]|^2 + \frac{1}{4}\|Z_1 - Z_2\|^4}{|1 + \|Z_1\|^2 - 2t_1|^2 |1 + \|Z_2\|^2 - 2\bar{t}_2|^2} \\
&= 16 \frac{\mathcal{N}(\mathbf{n}_2^{-1} \mathbf{n}_1)^4}{\mathcal{P}(\mathbf{n}_1) \mathcal{P}(\mathbf{n}_2)},
\end{aligned}$$

where

$$\mathcal{P}(\mathbf{n}) := |1 + \|Z\|^2 - 2t|^2 = 4|t|^2 + (1 + \|Z\|^2)^2 \quad (4.11)$$

is the Poisson kernel on $\mathbb{F}^{n-1} \times \mathbb{F}_0$.

For $\mathbf{n}_j = (Z_j, t_j)$ with $j = 1, 2, 3$, and for $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$, we set

$$J_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) := \mathcal{N}(\mathbf{n}_2^{-1} \mathbf{n}_1)^{2(\nu_3 - \rho)} \mathcal{N}(\mathbf{n}_3^{-1} \mathbf{n}_2)^{2(\nu_1 - \rho)} \mathcal{N}(\mathbf{n}_3^{-1} \mathbf{n}_1)^{2(\nu_2 - \rho)}.$$

We therefore define (whenever it makes sense) the distribution $\mathcal{J}_{\underline{\nu}}$ by

$$\mathcal{J}_{\underline{\nu}}(\varphi) := \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^3} \varphi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) J_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3,$$

for a smooth function φ on $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^3$. Moreover, let us introduce

$$\Upsilon_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) := \mathcal{P}(\mathbf{n}_1)^{-(\nu_2 + \nu_3)/2} \mathcal{P}(\mathbf{n}_2)^{-(\nu_1 + \nu_3)/2} \mathcal{P}(\mathbf{n}_3)^{-(\nu_1 + \nu_2)/2}, \quad (4.12)$$

where $\mathcal{P}$ is as in (4.11). Hence, in view of the integral formula (4.9), we obtain:

Proposition 4.3. *For $f \in C^\infty(S \times S \times S)$, we have*

$$\mathcal{K}_{\underline{\nu}}(f) = 4^{\nu_1 + \nu_2 + \nu_3 - 3\rho} c_{\mathbb{F}}^3 \mathcal{J}_{\underline{\nu}}((f \circ c) \Upsilon_{\underline{\nu}}),$$

whenever the left hand side is well defined. The constant $c_{\mathbb{F}}$ is as in (4.10).

For $g \in G$, let $\tilde{g} := c^{-1} \circ g \circ c$. Denote by $\tilde{\kappa}(\tilde{g}, \mathbf{n})$ its conformal factor at $\mathbf{n} \in \mathbb{F}^{n-1} \times \mathbb{F}_0$. The non-compact realization of the spherical principal series (3.4) is given by

$$\tilde{\pi}_\nu(g)\varphi(\mathbf{n}) = \tilde{\kappa}(\tilde{g}^{-1}, \mathbf{n})^{\nu+\rho}\varphi(\tilde{g}^{-1} \cdot \mathbf{n})$$

for a function φ in the space $\tilde{X}_\nu^\infty$ of smooth functions on $\mathbb{F}^{n-1} \times \mathbb{F}_0$. Observe that for $f \in X_\nu^\infty$, the map $\iota_\nu : X_\nu^\infty \rightarrow \tilde{X}_\nu^\infty$ defined by

$$\iota_\nu(f)(\mathbf{n}) := \tilde{\kappa}(c, \mathbf{n})^{\nu+\rho}f \circ c(\mathbf{n})$$

satisfies

$$\tilde{\pi}_\nu(g)\iota_\nu(f) = \iota_\nu(\pi_\nu(g)f), \quad \forall g \in G.$$

That is ι_ν intertwines the two models π_ν and $\tilde{\pi}_\nu$.

Retaining the notation of Theorem 4.1, it follows, from Proposition 4.3 and the G -invariance (4.6) of $\mathcal{K}_{\underline{\nu}}$, that

$$\mathcal{J}_{\underline{\nu}}(\tilde{\pi}_{\lambda_1}(g)\varphi_1 \otimes \tilde{\pi}_{\lambda_2}(g)\varphi_2 \otimes \tilde{\pi}_{\lambda_3}(g)\varphi_3) = \mathcal{J}_{\underline{\nu}}(\varphi_1 \otimes \varphi_2 \otimes \varphi_3), \quad \forall g \in G.$$

We close this section by mentioning that the integral (4.4) defining $\mathcal{K}_{\underline{\nu}}$ is finite either if the three functions f_1 , f_2 , and f_3 are pairwise disjoint supports, or if

$$\int_{S \times S \times S} |K_{\underline{\nu}}(x_1, x_2, x_3)| d\sigma(x_1, x_2, x_3) < \infty.$$

5. THE EXISTENCE OF THE TRILINEAR FORM

Let $\{U_1, \dots, U_r\}$ be a finite open cover of the compact set $S \times S \times S$. Assume that U_1 is a neighborhood of (e_1, e_1, e_1) in $(S)^3$. Pick a partition of unity $\{\phi_1, \dots, \phi_r\}$ on $(S)^3$ subordinate to the cover $\{U_1, \dots, U_r\}$. Then

$$\int_{(S)^3} K_{\underline{\nu}}(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3) = \sum_i \int_{U_i} (K_{\underline{\nu}}\phi_i)(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3).$$

The integrals under the summation-sign are of the same nature. Thus, the problem is reduced to the convergence of the integral

$$\int_{U_1} K_{\underline{\nu}}(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3). \quad (5.1)$$

In the light of Proposition 4.3, the integral (5.1) has the same nature as

$$\mathcal{I}_{\underline{\nu}}^{(1)} := \int J_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3$$

where the integral is taken over the set of points $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ such that, for δ small enough,

$$\mathcal{N}(\mathbf{n}_1) < \delta, \quad \mathcal{N}(\mathbf{n}_2) < \delta, \quad \mathcal{N}(\mathbf{n}_3) < \delta.$$

Put

$$\mathbf{n}_1^{-1}\mathbf{n}_2 = \tilde{\mathbf{n}}_1, \quad \mathbf{n}_1^{-1}\mathbf{n}_3 = \tilde{\mathbf{n}}_2.$$

After integrating with respect to $\mathbf{n}_1$, the integral $\mathcal{I}_{\underline{\nu}}^{(1)}$ becomes, up to a scalar,

$$\mathcal{I}_{\underline{\nu}}^{(2)} = \int \mathcal{N}(\tilde{\mathbf{n}}_2^{-1}\tilde{\mathbf{n}}_1)^{2(\nu_1-\rho)} \mathcal{N}(\tilde{\mathbf{n}}_2)^{2(\nu_2-\rho)} \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3-\rho)} d\tilde{\mathbf{n}}_1 d\tilde{\mathbf{n}}_2$$

where the integral is taken over the set of points $(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2)$ such that $\mathcal{N}(\tilde{\mathbf{n}}_1) < \delta$ and $\mathcal{N}(\tilde{\mathbf{n}}_2) < \delta$, for δ small enough.

Let $\Lambda(\mathbb{F})$ be the unit sphere in $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$ with respect to the norm $\mathcal{N}$. By [13, Proposition 1.15] there exists a Radon measure η on $\Lambda(\mathbb{F})$ such that for all f in $L^1((\mathbb{F}^{n-1} \times \mathbb{F}_0)^2)$

$$\int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2} f(\mathbf{n}_1, \mathbf{n}_2) d\mathbf{n}_1 d\mathbf{n}_2 = \int_0^\infty \int_{\Lambda(\mathbb{F})} f(\delta_r \sigma_1, \delta_r \sigma_2) r^{4\rho-1} d\eta(\sigma_1, \sigma_2) dr. \quad (5.2)$$

Thus, the integral formula $\mathcal{I}_{\underline{\nu}}^{(2)}$ takes the form

$$\left\{ \int_0^\delta r^{2(\nu_1 + \nu_2 + \nu_3 - 3\rho)} r^{4\rho-1} dr \right\} \left\{ \int_{\Lambda(\mathbb{F})} \mathcal{N}(\sigma_2^{-1} \sigma_1)^{2(\nu_1 - \rho)} \times \mathcal{N}(\sigma_1)^{2(\nu_3 - \rho)} \mathcal{N}(\sigma_2)^{2(\nu_2 - \rho)} d\eta(\sigma_1, \sigma_2) \right\}.$$

The first integral between brackets converges if and only if $\operatorname{Re}(\nu_1 + \nu_2 + \nu_3) > \rho$. It remains to discuss the convergence of the integral over $\Lambda(\mathbb{F})$, which we will denote by $\mathcal{I}_{\underline{\nu}}^{(3)}$.

For $\delta > 0$ small enough, we consider the following open subsets of $\Lambda(\mathbb{F})$,

$$\begin{aligned} \Lambda_1(\mathbb{F}) &= \{(\sigma_1, \sigma_2) \in \Lambda(\mathbb{F}) : \mathcal{N}(\sigma_1) < \delta\}, \\ \Lambda_2(\mathbb{F}) &= \{(\sigma_1, \sigma_2) \in \Lambda(\mathbb{F}) : \mathcal{N}(\sigma_2) < \delta\}, \\ \Lambda_3(\mathbb{F}) &= \{(\sigma_1, \sigma_2) \in \Lambda(\mathbb{F}) : \mathcal{N}(\sigma_2^{-1} \sigma_1) < \delta\}. \end{aligned}$$

Since $(\sigma_1, \sigma_2) \in \Lambda(\mathbb{F})$, it follows that for $\delta \ll 1$, the above three sets are disjoint. Moreover, on $\Lambda(\mathbb{F}) \setminus \Lambda_1(\mathbb{F}) \cup \Lambda_2(\mathbb{F}) \cup \Lambda_3(\mathbb{F})$ the integrated function is bounded from below by positive constants. Then, the convergence of the integral $\mathcal{I}_{\underline{\nu}}^{(3)}$ is equivalent to that of

$$\mathcal{I}_{\underline{\nu}}^{(3,j)} = \int_{\Lambda_j(\mathbb{F})} \mathcal{N}(\sigma_2^{-1} \sigma_1)^{2(\nu_1 - \rho)} \mathcal{N}(\sigma_1)^{2(\nu_3 - \rho)} \mathcal{N}(\sigma_2)^{2(\nu_2 - \rho)} d\eta(\sigma_1, \sigma_2),$$

for all $j = 1, 2, 3$.

On $\Lambda_1(\mathbb{F})$, we have $\mathcal{N}(\sigma_2^{-1} \sigma_1) > \delta$ and $\mathcal{N}(\sigma_2) > \delta$. Thus, $\mathcal{I}_{\underline{\nu}}^{(3,1)}$ converges if and only if the integral

$$\int_{\Lambda_1(\mathbb{F})} \mathcal{N}(\sigma_1)^{2(\nu_3 - \rho)} d\eta(\sigma_1, \sigma_2)$$

is finite. That is when $\operatorname{Re} \nu_3 > 0$ (see [13, Corollary 1.17]).

A similar study can be done for $\mathcal{I}_{\underline{\nu}}^{(3,2)}$ and $\mathcal{I}_{\underline{\nu}}^{(3,3)}$. Thus we have proved the following:

Theorem 5.1. *The kernel $K_{\underline{\nu}}$ is integrable on $S \times S \times S$ if and only if*

$$\operatorname{Re} \nu_j > 0 \quad (\text{for } j = 1, 2, 3), \quad \text{and} \quad \operatorname{Re}(\nu_1 + \nu_2 + \nu_3) > \rho. \quad (5.3)$$

Let λ_1, λ_2 and λ_3 be three complex numbers such that

$$0 \leq \operatorname{Re} \lambda_j < \rho \quad \text{for } j = 1, 2, 3, \quad (5.4)$$

and consider $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ such that

$$\begin{aligned}\nu_1 &= (-\lambda_1 + \lambda_2 + \lambda_3 + \rho)/2, \\ \nu_2 &= (-\lambda_2 + \lambda_1 + \lambda_3 + \rho)/2, \\ \nu_3 &= (-\lambda_3 + \lambda_1 + \lambda_2 + \rho)/2.\end{aligned}\tag{5.5}$$

Observe that

$$2(\nu_1 + \nu_2 + \nu_3 - \rho) = \lambda_1 + \lambda_2 + \lambda_3 + \rho.$$

From (5.4) it has become clear that $\underline{\nu} = (\nu_1, \nu_2, \nu_3)$ satisfies $\operatorname{Re} \nu_j > 0$, for $j = 1, 2, 3$, and $\operatorname{Re}(\nu_1 + \nu_2 + \nu_3) - \rho > 0$. Thus we have:

Corollary 5.2. *Let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ such that $0 \leq \operatorname{Re} \lambda_j < \rho$ for all $j = 1, 2, 3$, and let $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ subject to the construction (5.5). Then the kernel $K_{\underline{\nu}}$ is integrable, and the corresponding trilinear form $\mathcal{K}_{\underline{\nu}}$ is invariant with respect to the representations $(\pi_{\lambda_1}, \pi_{\lambda_2}, \pi_{\lambda_3})$.*

Fact 5.3. The above corollary includes two important spherical unitary representations, namely the unitary principal series when $\operatorname{Re} \lambda = 0$, and the complementary series when $0 < \lambda < \rho$.

6. AN ANALYTIC CONTINUATION OF THE TRILINEAR FORM

Let $f \in C^\infty(S \times S \times S)$ and consider the trilinear form

$$\mathcal{K}_{\underline{\nu}}(f) = \int_{(S)^3} f(x_1, x_2, x_3) K_{\underline{\nu}}(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3),$$

where the kernel $K_{\underline{\nu}}$ is as in (4.3). We will assume that f has its support contained in a small neighborhood of $(e_1, e_1, e_1) \in (S)^3$. The goal of this section is to study the analytic continuation of the map $\underline{\nu} \mapsto \mathcal{K}_{\underline{\nu}}(f)$. Our approach uses the Heisenberg-type model $\mathcal{J}_{\underline{\nu}}$ of $\mathcal{K}_{\underline{\nu}}$. By means of Proposition 4.3, to prove the continuity of $\mathcal{K}_{\underline{\nu}}(f)$ as a function of $\underline{\nu}$, it is enough to study the analytic continuation of the map

$$\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}((f \circ c) \Upsilon_{\underline{\nu}}),\tag{6.1}$$

where c is the transformation (4.8) and $\Upsilon_{\underline{\nu}}$ is as in (4.12). The function $f \circ c$ is smooth with compact support in $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^3$, and $\underline{\nu} \mapsto \Upsilon_{\underline{\nu}}$ is an entire function on $\mathbb{C}^3$. Thus, the meromorphic continuation of the map (6.1) blows down to the meromorphic continuation of $\mathcal{J}_{\underline{\nu}}$ as a distribution on $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^3$. The proof follows the same approach of [9].

Recall that for $\varphi \in C_c^\infty((\mathbb{F}^{n-1} \times \mathbb{F}_0)^3)$, with $n = 2$ for $\mathbb{F} = \mathbb{O}$, we have

$$\mathcal{J}_{\underline{\nu}}(\varphi) = \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^3} \varphi(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) J_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) d\mathbf{n}_1 d\mathbf{n}_2 d\mathbf{n}_3,$$

where

$$J_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \mathcal{N}(\mathbf{n}_2^{-1} \mathbf{n}_1)^{2(\nu_3 - \rho)} \mathcal{N}(\mathbf{n}_3^{-1} \mathbf{n}_2)^{2(\nu_1 - \rho)} \mathcal{N}(\mathbf{n}_3^{-1} \mathbf{n}_1)^{2(\nu_2 - \rho)}.$$

Using the change of variables:

$$\mathbf{n}_2^{-1} \mathbf{n}_1 = \tilde{\mathbf{n}}_1^{-1} \quad \text{and} \quad \mathbf{n}_3^{-1} \mathbf{n}_1 = \tilde{\mathbf{n}}_2^{-1},$$

we obtain

$$\mathcal{J}_{\underline{\nu}}(\varphi) = \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2} \psi(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1)^{2(\nu_1 - \rho)} \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3 - \rho)} \mathcal{N}(\tilde{\mathbf{n}}_2)^{2(\nu_2 - \rho)} d\tilde{\mathbf{n}}_1 d\tilde{\mathbf{n}}_2, \quad (6.2)$$

where

$$\psi(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) := \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)} \varphi(\mathbf{n}_1, \mathbf{n}_1 \tilde{\mathbf{n}}_1, \mathbf{n}_1 \tilde{\mathbf{n}}_2) d\mathbf{n}_1.$$

The function ψ is compactly supported in $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$. Hereafter we will use the following notation

$$L_{\underline{\nu}}(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) = \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1)^{2(\nu_1 - \rho)} \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3 - \rho)} \mathcal{N}(\tilde{\mathbf{n}}_2)^{2(\nu_2 - \rho)}.$$

For $r > 0$, we consider the following subsets in $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$,

$$\Omega_0 = \left\{ (\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mid \mathcal{N}(\tilde{\mathbf{n}}_1) < r, \mathcal{N}(\tilde{\mathbf{n}}_2) < r, \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1) < r \right\},$$

$$\Omega_1 = \left\{ (\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mid \mathcal{N}(\tilde{\mathbf{n}}_1) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2) < r/2, \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1) > r/2 \right\},$$

$$\Omega_2 = \left\{ (\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mid \mathcal{N}(\tilde{\mathbf{n}}_1) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1) < r/2 \right\},$$

$$\Omega_3 = \left\{ (\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mid \mathcal{N}(\tilde{\mathbf{n}}_1) < r/2, \mathcal{N}(\tilde{\mathbf{n}}_2) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1) > r/2 \right\},$$

$$\Omega_\infty = \left\{ (\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mid \mathcal{N}(\tilde{\mathbf{n}}_1) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2) > r/2, \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1) > r/2 \right\}.$$

Since the family $\{\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_\infty\}$ forms a covering of $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$, we are reduced to the following cases :

case 1: Assume that ψ is supported in Ω_∞ . Since the kernel function $L_{\underline{\nu}}$ has no singularities in Ω_∞ , it follows that $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ extends to an entire function on $\mathbb{C}^3$.

case 2: Assume that ψ is supported in Ω_3 . In this circumstance, let us rewrite the integral (6.2) as

$$\mathcal{J}_{\underline{\nu}}(\varphi) = \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \phi_{\nu_1, \nu_2}(\tilde{\mathbf{n}}_1) \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3 - \rho)} d\tilde{\mathbf{n}}_1,$$

where

$$\phi_{\nu_1, \nu_2}(\tilde{\mathbf{n}}_1) := \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \psi(\tilde{\mathbf{n}}_1, \tilde{\mathbf{n}}_2) \mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1)^{2(\nu_1 - \rho)} \mathcal{N}(\tilde{\mathbf{n}}_2)^{2(\nu_2 - \rho)} d\tilde{\mathbf{n}}_2.$$

The map $(\nu_1, \nu_2) \mapsto \phi_{\nu_1, \nu_2}(\tilde{\mathbf{n}}_1)$ is analytic on $\mathbb{C}^2$. This is due to the fact that $\mathcal{N}(\tilde{\mathbf{n}}_2)$ and $\mathcal{N}(\tilde{\mathbf{n}}_2^{-1} \tilde{\mathbf{n}}_1)$ are bounded from below on Ω_3 . On the other hand, by [10, Theorem 2.7], the distribution-valued function $\nu_3 \mapsto \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3 - \rho)}$ extends meromorphically into $\mathbb{C}$ with simple poles when ν_3 is a non-positive integer. Thus the function $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ admits a meromorphic continuation into $\mathbb{C}^3$ with simple poles along the hyperplanes

$$\mathcal{H}_k^{(3)} = \{ \underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3 : \nu_3 = -k \} \text{ with } k \in \mathbb{N}.$$

A similar analysis on the open sets Ω_1 and Ω_2 implies that $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ extends meromorphically into $\mathbb{C}^3$ with simple poles along the hyperplanes

$$\mathcal{H}_k^{(1)} = \{\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3 : \nu_1 = -k\},$$

and

$$\mathcal{H}_k^{(2)} = \{\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3 : \nu_2 = -k\},$$

with $k \in \mathbb{N}$.

case 3: Assume that ψ is supported in Ω_0 . By a general argument [28], the map $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ admits a meromorphic continuation and the poles are located on a locally finite family of hyperplanes. Let $\mathcal{H}^{(0)}$ be such an hyperplane with the additional property $\mathcal{H}^{(0)} \cap \mathcal{H}_k^{(j)}$ is empty for all $j = 1, 2, 3$ and for all $k \in \mathbb{N}$. Let $\underline{\nu}^0 = (\nu_1^0, \nu_2^0, \nu_3^0) \in \mathcal{H}^{(0)}$. The Laurent coefficients at $\underline{\nu}^0$ are distributions on $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$, and by [9, Proposition 2.6], their supports have to be contained in $\{(0, 0) \in (\mathbb{F}^{n-1} \times \mathbb{F}_0)^2\}$. Hence, by classical arguments, they are a finite linear combinations of Dirac's function $\delta_{(0,0)}$ and its derivatives. Thus, the map $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ does have a pole at $\underline{\nu}^0$, and there exists a smooth function Φ with compact support in $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$ which is identically equal to 1 in a neighborhood of $(0, 0)$, and a homogeneous¹ polynomial p on $(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2$ of degree m with respect to $\{\delta_r\}$ such that $\mathcal{J}_{\underline{\nu}}(\Phi p)$ does not extend holomorphically at $\underline{\nu}^0$. Thus, for each $\underline{\nu} \neq \underline{\nu}^0$, we have

$$\begin{aligned} \mathcal{J}_{\underline{\nu}}((\Phi \circ \delta_r)p) &= \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2} \Phi \circ \delta_r(\tilde{n}_1, \tilde{n}_2) p(\tilde{n}_1, \tilde{n}_2) L_{\underline{\nu}}(\tilde{n}_1, \tilde{n}_2) d\tilde{n}_1 d\tilde{n}_2 \\ &= |r|^{-4\rho} \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2} \Phi(\tilde{n}_1, \tilde{n}_2) p \circ \delta_{1/r}(\tilde{n}_1, \tilde{n}_2) L_{\underline{\nu}} \circ \delta_{1/r}(\tilde{n}_1, \tilde{n}_2) d\tilde{n}_1 d\tilde{n}_2 \\ &= |r|^{-2(\nu_1 + \nu_2 + \nu_3 - \rho)} r^{-m} \mathcal{J}_{\underline{\nu}}(\Phi p). \end{aligned}$$

This implies

$$(1 - |r|^{-2(\nu_1 + \nu_2 + \nu_3 - \rho)} r^{-m}) \mathcal{J}_{\underline{\nu}}(\Phi p) = \mathcal{J}_{\underline{\nu}}((\Phi - \Phi \circ \delta_r)p), \quad \forall r \in \mathbb{R}^*.$$

On the other hand, from the definition of Φ , the pair $(0, 0)$ does not belong to the support of $(\Phi - \Phi \circ \delta_r)p$. Hence the map $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}((\Phi - \Phi \circ \delta_r)p)$ extends to a meromorphic function on $\cup_{i=1}^3 \mathcal{H}_k^{(i)}$ for all $k \in \mathbb{N}$. The assumption that $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\Phi p)$ does have a pole at $\underline{\nu}^0$ forces the condition

$$(1 - |r|^{-2(\nu_1 + \nu_2 + \nu_3 - \rho)} r^{-m}) = 0, \quad \forall r \in \mathbb{R}^*.$$

This amounts to

$$\nu_1 + \nu_2 + \nu_3 = \rho - \frac{m}{2} \quad \text{with } m \in 2\mathbb{N}.$$

We summarize the consequence of our preceding discussion.

¹A polynomial $p = p(Z, t)$ is said to be homogeneous of degree m if $p \circ \delta_r = r^m p$ where $\delta_r(Z, t) = (rZ, r^2t)$ for $r \in \mathbb{R}^*$.

Theorem 6.1. *The map $\underline{\nu} \mapsto \mathcal{K}_{\underline{\nu}}$ initially defined for $\underline{\nu}$ as in (5.3), extended meromorphically into $\mathbb{C}^3$ with at most simple poles along the hyperplanes defined by the equations*

$$\nu_j = -k, \quad \forall j = 1, 2, 3, \forall k \in \mathbb{N}, \quad (6.3)$$

$$\nu_1 + \nu_2 + \nu_3 = \rho - k, \quad \forall k \in \mathbb{N}. \quad (6.4)$$

The following is then immediate.

Theorem 6.2. *Let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ satisfying*

$$-\lambda_1 + \lambda_2 + \lambda_3 \notin -\rho - 2\mathbb{N},$$

$$\lambda_1 - \lambda_2 + \lambda_3 \notin -\rho - 2\mathbb{N},$$

$$\lambda_1 + \lambda_2 - \lambda_3 \notin -\rho - 2\mathbb{N},$$

$$\lambda_1 + \lambda_2 + \lambda_3 \notin -\rho - 2\mathbb{N}.$$

Define the multiparameter $\underline{\nu} = (\nu_1, \nu_2, \nu_3)$ by the relations

$$\nu_1 = (-\lambda_1 + \lambda_2 + \lambda_3 + \rho)/2,$$

$$\nu_2 = (-\lambda_2 + \lambda_1 + \lambda_3 + \rho)/2,$$

$$\nu_3 = (-\lambda_3 + \lambda_1 + \lambda_2 + \rho)/2.$$

Then the map $(f_1, f_2, f_3) \mapsto \mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3)$ is a well defined non trivial trilinear form with the property

$$\mathcal{K}_{\underline{\nu}}(\pi_{\lambda_1}(g)f_1 \otimes \pi_{\lambda_2}(g)f_2 \otimes \pi_{\lambda_3}(g)f_3) = \mathcal{K}_{\underline{\nu}}(f_1 \otimes f_2 \otimes f_3), \quad \forall g \in G.$$

Recall from Section 4 that $\mathcal{K}_{\underline{\nu}}$ is defined via the operator $\mathcal{B}_{\underline{\nu}} : X_{\nu_2+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_3-\rho}^\infty \longrightarrow X_{-\nu_1-\nu_2+\rho}^\infty$. The following statement is an immediate consequence of the above theorem.

Corollary 6.3. *The intertwining operator*

$$\mathcal{B}_{\underline{\nu}} : X_{\nu_2+\nu_3-\rho}^\infty \otimes X_{\nu_1+\nu_3-\rho}^\infty \longrightarrow X_{-\nu_1-\nu_2+\rho}^\infty$$

has a meromorphic continuation into $\mathbb{C}^3$ with at most simple poles along the hyperplanes given in (6.3)-(6.4).

7. RESIDUES AND INVARIANT BILINEAR DIFFERENTIAL OPERATORS

In the previous section we have proved that the singularities of the map $\underline{\nu} \mapsto \mathcal{J}_{\underline{\nu}}(\varphi)$ belong to

$$\Sigma_1 := \{\underline{\nu} = (\nu_1, \nu_2, \nu_3) : -\nu_i \in \mathbb{N} \text{ for } 1 \leq i \neq j, k \leq 3\} \quad (7.1)$$

and

$$\Sigma_2 := \{\underline{\nu} = (\nu_1, \nu_2, \nu_3) : -\nu_1 - \nu_2 - \nu_3 + \rho \in \mathbb{N}\}. \quad (7.2)$$

Next we will study the trilinear form $\mathcal{J}_{\underline{\nu}}$ when $\underline{\nu}$ belongs to $\Sigma_2 \setminus \Sigma_1$. We will prove that the residues are given by intertwining bilinear differential operators.

On functions of the form $\phi \otimes \psi$, with $\phi \in C_c^\infty(\mathbb{F}^{n-1} \times \mathbb{F}_0)$ and $\psi \in C_c^\infty((\mathbb{F}^{n-1} \times \mathbb{F}_0)^2)$, the identity (6.2) takes the form

$$\mathcal{J}_{\underline{\nu}}(\phi \otimes \psi) = \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \phi(\mathbf{n}) \mathcal{C}_{\underline{\nu}} \psi(\mathbf{n}) d\mathbf{n},$$

where

$$\mathcal{C}_{\underline{\nu}}\psi(\mathbf{n}) := \int_{(\mathbb{F}^{n-1} \times \mathbb{F}_0)^2} \psi(\mathbf{n}\tilde{\mathbf{n}}_1, \mathbf{n}\tilde{\mathbf{n}}_2) \mathcal{N}(\tilde{\mathbf{n}}_1)^{2(\nu_3-\rho)} \mathcal{N}(\tilde{\mathbf{n}}_2)^{2(\nu_2-\rho)} \mathcal{N}(\tilde{\mathbf{n}}_2^{-1}\tilde{\mathbf{n}}_1)^{2(\nu_1-\rho)} d\tilde{\mathbf{n}}_1 d\tilde{\mathbf{n}}_2. \quad (7.3)$$

By a general argument (see e.g. [16] or [6]), we have

$$\operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=-k \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{J}_{\underline{\nu}}(\phi \otimes \psi)) = \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \phi(\mathbf{n}) \operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=-k \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{C}_{\underline{\nu}}\psi(\mathbf{n})) d\mathbf{n}. \quad (7.4)$$

Next we will compute the residue of $\mathcal{C}_{\underline{\nu}}\psi(\mathbf{n})$ at $\nu_1 + \nu_2 + \nu_3 - \rho = -k$. We must point out that in view of the construction (5.5) of $\underline{\nu} = (\nu_1, \nu_2, \nu_3)$ via $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, saying that $\nu_1 + \nu_2 + \nu_3 - \rho = -k$ is equivalent to $\lambda_1 + \lambda_2 + \lambda_3 + \rho = -2k$.

Recall from Section 4 that $(\tilde{\pi}_{\nu}, \tilde{X}_{\nu}^{\infty})$ denotes the non-compact model of the spherical principal series $(\pi_{\nu}, X_{\nu}^{\infty})$ of G . The rest of this section is devoted to the proof of the following statement.

Theorem 7.1. *Let $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ and let $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ in such a way that*

$$\begin{aligned} \nu_1 &= (-\lambda_1 + \lambda_2 + \lambda_3 + \rho)/2, \\ \nu_2 &= (-\lambda_2 + \lambda_1 + \lambda_3 + \rho)/2, \\ \nu_3 &= (-\lambda_3 + \lambda_1 + \lambda_2 + \rho)/2. \end{aligned}$$

There exists a bilinear intertwining differential operator

$$\mathcal{D}_k^{(\nu_1, \nu_2)} : \tilde{X}_{\lambda_1}^{\infty} \otimes \tilde{X}_{\lambda_2}^{\infty} \longrightarrow \tilde{X}_{\lambda_1+\lambda_2+\rho+2k}^{\infty}; \quad \tilde{\pi}_{\lambda_1} \otimes \tilde{\pi}_{\lambda_2} \longrightarrow \tilde{\pi}_{\lambda_1+\lambda_2+\rho+2k}$$

so that

$$\operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=-k \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{C}_{\underline{\nu}}\psi(\mathbf{n})) = \mathcal{D}_k^{(\nu_1, \nu_2)}(\psi)(\mathbf{n}). \quad (7.5)$$

Let us first pin down the known formula for the meromorphic continuation of the distribution r^s on $\mathbb{R}^+$ (see e.g. [14]). Let $C_c^{\infty}(\mathbb{R}^+)$ be the space of smooth functions f with all derivatives being continuous and of compact support on $\mathbb{R}^+$. The integral $(r^s, f) := \int_0^{\infty} r^s f(r) dr$ is well defined for $\operatorname{Re} s > -1$ and has meromorphic continuation in $s \in \mathbb{C}$ with simple poles at $s = -1 - 2k$ with $k \in \mathbb{N}$. The residues at these poles are given by

$$\operatorname{Res}_{s=-1-2k} (r^s, f) = \frac{1}{\Gamma(2k+1)} \left(\frac{d}{dr} \right)^{2k} f(r) \Big|_{r=0}. \quad (7.6)$$

Now let us go back to the proof of Theorem 7.1. In view of the integration formula (5.2), the integral (7.3) can be written as

$$\mathcal{C}_{\underline{\nu}}\psi(\mathbf{n}) = \int_0^{\infty} r^{2(\nu_1+\nu_2+\nu_3-\rho)-1} \mathcal{I}_{\underline{\nu}}(\psi_r)(\mathbf{n}) dr, \quad (7.7)$$

where

$$\mathcal{I}_{\underline{\nu}}(\psi_r)(\mathbf{n}) := \int_{\Lambda(\mathbb{F})} \psi(\mathbf{n}\delta_r\sigma_1, \mathbf{n}\delta_r\sigma_2) \mathcal{N}(\sigma_1)^{2(\nu_3-\rho)} \mathcal{N}(\sigma_2)^{2(\nu_2-\rho)} \mathcal{N}(\sigma_2^{-1}\sigma_1)^{2(\nu_1-\rho)} d\eta(\sigma_1, \sigma_2).$$

Observe that $\mathcal{I}_{\underline{\nu}}$ is well defined when $\underline{\nu} \notin \Sigma_1$.

The integral (7.7) has meromorphic continuation as a function of $\nu_1 + \nu_2 + \nu_3 - \rho$ with simple poles at $\nu_1 + \nu_2 + \nu_3 - \rho = -k$ with $k \in \mathbb{N}$. By (7.6), the residue at this point is

$$\operatorname{Res}_{\substack{\nu_1 + \nu_2 + \nu_3 - \rho = -k \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{C}_{\underline{\nu}} \psi(\mathbf{n})) = \frac{1}{\Gamma(2k+1)} \left(\frac{d}{dr} \right)^{2k} \mathcal{I}_{\underline{\nu}}(\psi_r)(\mathbf{n})|_{r=0}.$$

For $\mathbf{n} \in \mathbb{F}^{n-1} \times \mathbb{F}_0$, denote by $\nabla_{\mathbf{n}}$ the left-invariant directional derivative along $\mathbf{n}$,

$$\nabla_{\mathbf{n}} f(\tilde{\mathbf{n}}) = \frac{d}{dr} f(\mathbf{n} \delta_r \tilde{\mathbf{n}})|_{r=0}, \quad \tilde{\mathbf{n}} \in \mathbb{F}^{n-1} \times \mathbb{F}_0.$$

By an abuse of notation, whenever the element $\mathbf{n}$ is as $(X, 0)$ or $(0, s)$ we will simply denote $\nabla_{\mathbf{n}}$ by ∇_X or ∇_s , respectively.

By writing $\sigma_1 = (X, s)$ and $\sigma_2 = (Y, t)$, we have $\mathbf{n} \delta_r \sigma_1 = \mathbf{n}(rX, 0)(0, r^2s)$ and $\mathbf{n} \delta_r \sigma_2 = \mathbf{n}(rY, 0)(0, r^2t)$ as product of commuting elements in $\mathbb{F}^{n-1} \times \mathbb{F}_0$. We shall write the integrand $\left(\frac{d}{dr} \right)^{2k} \psi(\mathbf{n} \delta_r \sigma_1, \mathbf{n} \delta_r \sigma_2)|_{r=0}$ as a polynomial of right differentiations ∇_X , ∇_s , ∇_Y , and ∇_t . Indeed,

$$\frac{d}{dr} h(\mathbf{n}(rX, 0)(0, r^2s)) = \underbrace{(\nabla_X + 2r\nabla_s)}_{:=p_1(\nabla_X, \nabla_s; r)} h(\mathbf{n}(rX, 0)(0, r^2s)),$$

with the two operators ∇_X and ∇_s being commuting. Thus

$$\frac{d^2}{dr^2} h(\mathbf{n}(rX, 0)(0, r^2s)) = \underbrace{\{(\nabla_X + 2r\nabla_s)^2 + 2\nabla_s\}}_{:=p_2(\nabla_X, \nabla_s; r)} h(\mathbf{n}(rX, 0)(0, r^2s)),$$

and for $j \geq 3$,

$$\frac{d^j}{dr^j} h(\mathbf{n}(rX, 0)(0, r^2s)) = p_j(\nabla_X, \nabla_s; r) h(\mathbf{n}(rX, 0)(0, r^2s)),$$

where

$$\begin{aligned} p_j(\nabla_X, \nabla_s; r) &= (\nabla_X + 2r\nabla_s)^j \\ &\quad + 2(\nabla_X + 2r\nabla_s)^{j-2} \nabla_s \left\{ (j-1) + (j-2) + (j-3) \right\} \\ &\quad + 2(\nabla_X + 2r\nabla_s)^{j-4} \nabla_s^2 \left\{ (j-1) + (j-2) + (j-3) + (j-4) \right\} \\ &\quad + 2(\nabla_X + 2r\nabla_s)^{j-6} \nabla_s^3 \left\{ (j-1) + (j-2) + \cdots + (j-5) \right\} \\ &\quad \vdots \\ &\quad + 2(\nabla_X + 2r\nabla_s)^{j-2\lceil \frac{j}{2} \rceil} \nabla_s^{\lceil \frac{j}{2} \rceil} \left\{ (j-1) + (j-2) + \cdots + \left(j - \left\lceil \frac{j}{2} \right\rceil - 2 \right) \right\}. \end{aligned}$$

It follows that

$$\left(\frac{d}{dr} \right)^{2k} \psi(\mathbf{n} \delta_r \sigma_1, \mathbf{n} \delta_r \sigma_2)|_{r=0} = \sum_{j=0}^{2k} \binom{2k}{j} p_j(\nabla_X, \nabla_s) p_j(\nabla_Y, \nabla_t) \psi(\mathbf{n}, \mathbf{n}),$$

where $p_j(\cdot, \cdot) := p_j(\cdot, \cdot; 0)$ is homogeneous of degree j with respect to δ_t .

Finally, we arrive at

$$\operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=-k \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{C}_{\underline{\nu}} \psi(\mathbf{n})) = \mathcal{D}_k^{(\nu_1, \nu_2)}(\psi)(\mathbf{n}),$$

where

$$\begin{aligned} \mathcal{D}_k^{(\nu_1, \nu_2)}(\psi)(\mathbf{n}) &= \frac{1}{\Gamma(2k+1)} \int_{\Lambda(\mathbb{F})} \left(\sum_{j=0}^{2k} \binom{2k}{j} p_j(\nabla_X, \nabla_s) p_j(\nabla_Y, \nabla_t) \psi(\mathbf{n}, \mathbf{n}) \right) \times \\ &\quad \times \mathcal{N}(\sigma_1)^{2(\nu_3-\rho)} \mathcal{N}(\sigma_2)^{2(\nu_2-\rho)} \mathcal{N}(\sigma_2^{-1} \sigma_1)^{2(\nu_1-\rho)} d\eta(\sigma_1, \sigma_2) \end{aligned} \quad (7.8)$$

is an integration of polynomial differential operators, and is its self a differential operator. This finishes the proof of Theorem 7.1.

Remark 7.2. Consider the case $k = 0$, *i.e.* $\nu_1 + \nu_2 + \nu_3 - \rho = 0$. In view of (7.4) together with (7.5) and (7.8), we have

$$\operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=0 \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{J}_{\underline{\nu}}(\phi \otimes \psi)) = c(\underline{\nu}) \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \phi(\mathbf{n}) \psi(\mathbf{n}, \mathbf{n}) d\mathbf{n} \quad (7.9)$$

for some constant $c(\underline{\nu})$. To find the value of $c(\underline{\nu})$, it is enough to compute the residue of $\mathcal{J}_{\underline{\nu}}$ for freely chosen function $\phi \otimes \psi$. To do so, recall from (4.12) the function

$$\Upsilon_{\underline{\nu}}(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3) = \mathcal{P}(\mathbf{n}_1)^{-(\nu_2+\nu_3)/2} \mathcal{P}(\mathbf{n}_2)^{-(\nu_1+\nu_3)/2} \mathcal{P}(\mathbf{n}_3)^{-(\nu_1+\nu_2)/2},$$

where $\mathcal{P}$ is the Poisson kernel on $\mathbb{F}^{n-1} \times \mathbb{F}_0$ (see (4.11)). Observe that if $\nu_1 + \nu_2 + \nu_3 - \rho = 0$, then $\Upsilon_{\underline{\nu}}(\mathbf{n}, \mathbf{n}, \mathbf{n}) = \mathcal{P}(\mathbf{n})^{-\rho}$. Thus, on the one hand, by (7.9), we have

$$\begin{aligned} \operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=0 \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{J}_{\underline{\nu}}(\Upsilon_{\underline{\nu}})) &= c(\underline{\nu}) \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} \mathcal{P}(\mathbf{n})^{-\rho} d\mathbf{n} \\ &= c(\underline{\nu}) \int_{\mathbb{F}^{n-1} \times \mathbb{F}_0} |1 + \|z\|^2 - 2t|^{-2\rho} dz dt \\ &= c(\underline{\nu}) c_{\mathbb{F}}^{-1}, \end{aligned} \quad (7.10)$$

where $c_{\mathbb{F}}$ is as in (4.10). On the other hand, in view of Proposition 4.3, we have

$$\mathcal{J}_{\underline{\nu}}(\Upsilon_{\underline{\nu}}) = c_{\mathbb{F}}^{-3} 4^{-(\nu_1+\nu_2+\nu_3-3\rho)} \mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1).$$

Thus,

$$\operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=0 \\ \underline{\nu} \notin \Sigma_1}} (\mathcal{J}_{\underline{\nu}}(\Upsilon_{\underline{\nu}})) = c_{\mathbb{F}}^{-3} 4^{2\rho} \operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=0 \\ \underline{\nu} \notin \Sigma_1}} \mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1). \quad (7.11)$$

Now, comparing the two identities (7.10) and (7.11) we conclude that

$$c(\underline{\nu}) = 4^{2\rho} c_{\mathbb{F}}^{-2} \operatorname{Res}_{\substack{\nu_1+\nu_2+\nu_3-\rho=0 \\ \underline{\nu} \notin \Sigma_1}} \mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1),$$

where $c_{\mathbb{F}}$ is as in (4.10). In the next section we will find an expression for $\mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1)$ in terms of hypergeometric functions.

8. GENERALIZED BERNSTEIN-REZNIKOV INTEGRALS

The aim of this section is to express the triple integral

$$\mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1) = \int_{(S)^3} \kappa(x_2, x_3)^{\nu_1 - \rho} \kappa(x_3, x_1)^{\nu_2 - \rho} \kappa(x_1, x_2)^{\nu_3 - \rho} d\sigma(x_1) d\sigma(x_2) d\sigma(x_3) \quad (8.1)$$

in terms of special functions for $\underline{\nu}$ satisfying the conditions in (5.3). When $S = \mathbb{S}^{n-1}$, the above triple integral was evaluated in [8] in terms of Gamma functions (the case $\mathbb{S}^1$ was firstly established by Bernstein and Reznikov in [4]).

The generalized hypergeometric function ${}_pF_q$ is defined by means of a power series as

$${}_pF_q \left[\begin{matrix} (\alpha_i) \\ (\beta_i) \end{matrix}; z \right] = \sum_{m \geq 0} \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m} \frac{z^m}{m!}$$

with appropriate conditions. The function ${}_{p+1}F_p$ is called *well poised* if

$$\alpha_1 + 1 = \alpha_2 + \beta_1 = \alpha_3 + \beta_2 = \cdots = \alpha_{p+1} + \beta_p.$$

If ${}_{p+1}F_p$ is well poised and $\alpha_2 = 1 + \frac{1}{2}\alpha_1$, then it is called *very well poised*. The following identity for the very well poised ${}_5F_4$ is needed for latter use (see [2])

$$\begin{aligned} {}_5F_4 \left[\begin{matrix} a, & 1 + \frac{1}{2}a, & c, & d, & e \\ \frac{1}{2}a, & a - c + 1, & a - d + 1, & a - e + 1 \end{matrix}; 1 \right] \\ = \frac{\Gamma(a - c + 1)\Gamma(a - d + 1)\Gamma(a - e + 1)\Gamma(a - c - d - e + 1)}{\Gamma(a + 1)\Gamma(a - d - e + 1)\Gamma(a - c - e + 1)\Gamma(a - c - d + 1)}. \end{aligned} \quad (8.2)$$

In the sequel we will also need the generalized Kampé-de-Fériet hypergeometric function

$$F_{qsv}^{pru} \left[\begin{matrix} (a_i): & (\alpha_i); & (\lambda_i) \\ (b_i): & (\beta_i); & (\mu_i) \end{matrix}; x, y \right] = \sum_{m,n \geq 0} \frac{\prod_{i=1}^p (a_i)_{m+n} \prod_{i=1}^r (\alpha_i)_m \prod_{i=1}^u (\lambda_i)_n}{\prod_{i=1}^q (b_i)_{m+n} \prod_{i=1}^s (\beta_i)_m \prod_{i=1}^v (\mu_i)_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

We also have the following expansion

$$\begin{aligned} F_{qsv}^{pru} \left[\begin{matrix} (a_i): & (\alpha_i); & (\lambda_i) \\ (b_i): & (\beta_i); & (\mu_i) \end{matrix}; x, y \right] \\ = \sum_{m \geq 0} \frac{\prod_{i=1}^p (a_i)_m \prod_{i=1}^r (\alpha_i)_m}{\prod_{i=1}^q (b_i)_m \prod_{i=1}^s (\beta_i)_m} {}_{p+u}F_{q+v} \left[\begin{matrix} (a_i + m): & (\lambda_i) \\ (b_i + m): & (\mu_i) \end{matrix}; y \right] \frac{x^m}{m!}. \end{aligned} \quad (8.3)$$

Some special cases of F_{qsv}^{pru} are the four Appell series. We refer to [29] for more details on Kampé-de-Fériet hypergeometric functions.

We shall need the following result later.

Lemma 8.1. *For $a, b, c \in \mathbb{R}$, the series*

$$\sum_{p,q \geq 1} p^a q^b (p+q)^c$$

converges if and only if

$$a + c + 1 < 0, \quad b + c + 1 < 0, \quad \text{and} \quad a + b + c + 2 < 0.$$

Proof. The statement follows easily from the asymptotic formula $\Gamma(z + \alpha)/\Gamma(z + \beta) \sim z^{\alpha-\beta}$, if $z \rightarrow \infty$. $\square$

Now we turn our attention to the calculation of the integral (8.1). Recall from Section 2 that $\widehat{K}_M$ denotes the subset of the dual object of K consisting of these equivalence classes of representations (τ, V_τ) which contain non zero M -fixed vectors. For $\nu \in \mathbb{C}$ with positive real part, the expansion below is nothing other than a reformulation of the diagonalization (2.4) of the normalized intertwining operator $\mathcal{N}_\nu$,

$$\frac{1}{C_{\mathbb{F}}(\nu)} \kappa(x, y)^{\nu-\rho} = \sum_{\tau \in \widehat{K}_M} a_\nu(\tau) h_\tau(x, y),$$

where $h_\tau(x, y)$ is the reproducing kernel of the space V_τ in $L^2(S)$. We point out that h_τ can be expressed in terms of Jacobi functions. For the coefficients $a_\nu(\tau)$ we refer to Theorem 3.3, and for the constant $C_{\mathbb{F}}(\nu)$ see Lemma 3.2.

For $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ satisfying the condition (5.3), we set

$$\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{F}) = \frac{1}{C_{\mathbb{F}}(\nu_1)C_{\mathbb{F}}(\nu_2)C_{\mathbb{F}}(\nu_3)} \int_{(S)^3} \kappa(x_1, x_2)^{\nu_3-\rho} \kappa(x_2, x_3)^{\nu_1-\rho} \kappa(x_1, x_3)^{\nu_2-\rho} d\sigma(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, x_3)$. That is

$$\mathcal{K}_{\underline{\nu}}(1 \otimes 1 \otimes 1) = \{C_{\mathbb{F}}(\nu_1)C_{\mathbb{F}}(\nu_2)C_{\mathbb{F}}(\nu_3)\} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{F}),$$

where the constant $C_{\mathbb{F}}(\nu)$ is given by (3.2).

Proposition 8.2. *For $\underline{\nu}$ as above, the following expansion holds true,*

$$\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{F}) = \sum_{\tau \in \widehat{K}_M} \dim(V_\tau) a_{\nu_1}(\tau) a_{\nu_2}(\tau) a_{\nu_3}(\tau). \quad (8.4)$$

Proof. Under the conditions of convergence,

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{F}) &= \int_{(S)^3} \left(\sum_{\tau_1 \in \widehat{K}_M} a_{\nu_1}(\tau_1) h_{\tau_1}(x_2, x_3) \right) \left(\sum_{\tau_2 \in \widehat{K}_M} a_{\nu_2}(\tau_2) h_{\tau_2}(x_1, x_3) \right) \\ &\quad \times \left(\sum_{\tau_3 \in \widehat{K}_M} a_{\nu_3}(\tau_3) h_{\tau_3}(x_1, x_2) \right) d\sigma(x_1, x_2, x_3) \\ &= \sum_{\tau_1, \tau_3 \in \widehat{K}_M} a_{\nu_1}(\tau_1) a_{\nu_2}(\tau_1) a_{\nu_3}(\tau_3) \int_{(S)^2} h_{\tau_1}(x_1, x_2) h_{\tau_3}(x_1, x_2) d\sigma(x_1) d\sigma(x_2) \\ &= \sum_{\tau_1 \in \widehat{K}_M} a_{\nu_1}(\tau_1) a_{\nu_2}(\tau_1) a_{\nu_3}(\tau_1) \int_S h_{\tau_1}(x_1, x_1) d\sigma(x_1) \\ &= \sum_{\tau_1 \in \widehat{K}_M} \dim(V_{\tau_1}) a_{\nu_1}(\tau_1) a_{\nu_2}(\tau_1) a_{\nu_3}(\tau_1). \end{aligned}$$

$\square$

Theorem 8.3. Let $\underline{\nu} = (\nu_1, \nu_2, \nu_3) \in \mathbb{C}^3$ such that

$$\operatorname{Re}(\nu_1 + \nu_2 + \nu_3) > \rho, \quad \text{for } \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}. \quad (8.5)$$

Then:

1) For $\mathbb{F} = \mathbb{R}$, we have

$$\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{R}) = \frac{\Gamma\left(\nu_1 + \frac{n-1}{2}\right)\Gamma\left(\nu_2 + \frac{n-1}{2}\right)\Gamma\left(\nu_3 + \frac{n-1}{2}\right)\Gamma\left(\nu_1 + \nu_2 + \nu_3 - \frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma(\nu_1 + \nu_2)\Gamma(\nu_1 + \nu_3)\Gamma(\nu_2 + \nu_3)}.$$

2) For $\mathbb{F} = \mathbb{C}$, we have

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{C}) &= \frac{\Gamma\left(\frac{\nu_1 + n}{2}\right)\Gamma\left(\frac{\nu_2 + n}{2}\right)\Gamma\left(\frac{\nu_3 + n}{2}\right)\Gamma\left(\frac{\nu_1 + \nu_2 + \nu_3 - n}{2}\right)}{\Gamma(n)\Gamma\left(\frac{\nu_1 + \nu_2}{2}\right)\Gamma\left(\frac{\nu_1 + \nu_3}{2}\right)\Gamma\left(\frac{\nu_2 + \nu_3}{2}\right)} \\ &\quad \times {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_j}{2}\right)_{j=1}^3, n-1 \\ \left(\frac{n+\nu_j}{2}\right)_{j=1}^3 \end{matrix} ; 1 \right]. \end{aligned}$$

The above hypergeometric function ${}_4F_3$ is well poised.

3) For $\mathbb{F} = \mathbb{H}$, we have

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H}) &= c \left\{ 2 F_{413}^{424} \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2}\right)_{i=1}^3, 2n : 2, 2 : \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3, 2n-2 \\ \left(\frac{2n+\nu_i+1}{2}\right)_{i=1}^3, 2 : 1 : \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3 \end{matrix} ; 1, 1 \right] - \right. \\ &\quad \left. F_{423}^{434} \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2}\right)_{i=1}^3, 2n-1 : 2n, 2, 2 : \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3, 2n-2 \\ \left(\frac{2n+\nu_i+1}{2}\right)_{i=1}^3, 2 : 2n-1, 1 : \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3 \end{matrix} ; 1, 1 \right] \right\}, \end{aligned}$$

where $c = 16\Gamma(2n)^2 \prod_{j=3}^n (j-1)^2 (2n-j)^2$.

4) For $\mathbb{F} = \mathbb{O}$, we have

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{O}) &= 48 \left\{ 2 F_{413}^{424} \left[\begin{matrix} \left(\frac{11-\nu_i}{2}\right)_{i=1}^3, 8 : 4, 6 : \left(\frac{5-\nu_i}{2}\right)_{i=1}^3, 4 \\ \left(\frac{11+\nu_i}{2}\right)_{i=1}^3, 4 : 3 : \left(\frac{5+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1, 1 \right] - \right. \\ &\quad \left. F_{423}^{434} \left[\begin{matrix} \left(\frac{11-\nu_i}{2}\right)_{i=1}^3, 7 : 4, 6, 8 : \left(\frac{5-\nu_i}{2}\right)_{i=1}^3, 4 \\ \left(\frac{11+\nu_i}{2}\right)_{i=1}^3, 4 : 3, 7 : \left(\frac{5+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1, 1 \right] \right\}. \end{aligned}$$

Proof. Due to Lemma 8.1, the series expansion $\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{F})$ converges if and only if $\underline{\nu}$ satisfies the condition (8.5).

1) Recall from (2.5) the dimension of $V_\tau \cong H^p(\mathbb{R})$, and from (3.11) the coefficient $a_\nu(\tau)$. Therefore,

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{R}) &= \sum_{p=0}^{\infty} \frac{(n-2)_p \left(\frac{n}{2}\right)_p}{p! \left(\frac{n-2}{2}\right)_p} \prod_{i=1}^3 \frac{\left(\frac{n-1}{2} - \nu_i\right)_p}{\left(\frac{n-1}{2} + \nu_i\right)_p} \\ &= {}_5F_4 \left[\begin{matrix} \left(\frac{n-1}{2} - \nu_i\right)_{i=1}^3, & n-2, & \frac{n}{2} \\ \left(\frac{n-1}{2} + \nu_i\right)_{i=1}^3, & \frac{n-2}{2} \end{matrix} ; 1 \right] \\ &= \frac{\Gamma\left(\nu_1 + \frac{n-1}{2}\right) \Gamma\left(\nu_2 + \frac{n-1}{2}\right) \Gamma\left(\nu_3 + \frac{n-1}{2}\right) \Gamma\left(\nu_1 + \nu_2 + \nu_3 - \frac{n-1}{2}\right)}{\Gamma(n-1) \Gamma(\nu_1 + \nu_2) \Gamma(\nu_1 + \nu_3) \Gamma(\nu_2 + \nu_3)}. \end{aligned}$$

Above we have used the identity (8.2).

2) In the complex case, the set $\widehat{K}_M$ is indexed by pairs of non-negative integers (p, q) . Substitution of the expressions (2.6) and (3.12) for $\dim(V_\tau)$ and $a_\nu(\tau)$ in (8.4) yields:

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{C}) &= \frac{\Gamma(n-1)}{\Gamma(n)} \sum_{p,q \geq 0} (p+q+n-1) \frac{(n-1)_p (n-1)_q}{p!q!} \prod_{i=1}^3 \frac{\left(\frac{n-\nu_i}{2}\right)_p \left(\frac{n-\nu_i}{2}\right)_q}{\left(\frac{n+\nu_i}{2}\right)_p \left(\frac{n+\nu_i}{2}\right)_q} \\ &= \sum_{p,q \geq 0} \frac{(n)_p (n-1)_q}{p!q!} \prod_{i=1}^3 \frac{\left(\frac{n-\nu_i}{2}\right)_p \left(\frac{n-\nu_i}{2}\right)_q}{\left(\frac{n+\nu_i}{2}\right)_p \left(\frac{n+\nu_i}{2}\right)_q} \\ &\quad + \frac{\Gamma(n-1)}{\Gamma(n)} \sum_{p \geq 0, q \geq 1} \frac{(n-1)_p (n-1)_q}{p!(q-1)!} \prod_{i=1}^3 \frac{\left(\frac{n-\nu_i}{2}\right)_p \left(\frac{n-\nu_i}{2}\right)_q}{\left(\frac{n+\nu_i}{2}\right)_p \left(\frac{n+\nu_i}{2}\right)_q} \\ &= {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \left\{ {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \right. \\ &\quad \left. + \prod_{i=1}^3 \frac{\left(\frac{n-\nu_i}{2}\right)}{\left(\frac{n+\nu_i}{2}\right)} {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2} + 1\right)_{i=1}^3, & n \\ \left(\frac{n+\nu_i}{2} + 1\right)_{i=1}^3 \end{matrix} ; 1 \right] \right\}. \end{aligned}$$

Next we will use the following contiguous relation

$$\begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; z \right] &= {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_{p-1}, \alpha_p - 1 \\ \beta_1, \dots, \beta_q \end{matrix} ; z \right] \\ &\quad + z \frac{\alpha_1 \cdots \alpha_{p-1}}{\beta_1 \cdots \beta_q} {}_pF_q \left[\begin{matrix} \alpha_1 + 1, \dots, \alpha_{p-1} + 1, \alpha_p \\ \beta_1 + 1, \dots, \beta_q + 1 \end{matrix} ; z \right]. \quad (8.6) \end{aligned}$$

For the proof, which is straightforward, we leave it to the reader. Thus, $\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{C})$ simplifies to

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{C}) &= 2 {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \\ &\quad - {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right]^2 \\ &= {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \left\{ 2 {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \right. \\ &\quad \left. - {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] \right\}. \end{aligned}$$

Since $2(n)_p - (n-1)_p = (n-1)_p \left(\frac{n+1}{2}\right)_p / \left(\frac{n-1}{2}\right)_p$, it follows that

$$\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{C}) = {}_4F_3 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3 \end{matrix} ; 1 \right] {}_5F_4 \left[\begin{matrix} \left(\frac{n-\nu_i}{2}\right)_{i=1}^3, & \frac{n+1}{2}, & n-1 \\ \left(\frac{n+\nu_i}{2}\right)_{i=1}^3, & \frac{n-1}{2} \end{matrix} ; 1 \right].$$

Now the result follows from (8.2).

3) In the quaternionic case, $\widehat{K}_M$ is indexed by pairs $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p - q$ is a non-negative even integer. Put $m = (p - q)/2$, and $c_0 = 16\Gamma(2n) \prod_{j=3}^n (j-1)^2 (2n-j)^2$. Then by the expressions (2.7) and (3.14) of $\dim(V_\tau)$ and $a_\nu(\tau)$, respectively, we have,

$$\begin{aligned} \mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H}) &= c_0 \sum_{q, m \geq 0} (2n + 2m + q - 1)(q + 1) \frac{\Gamma(2n + q - 1)}{q!} \frac{(2n + q - 1)_m}{m!} \frac{(2n - 2)_m}{(q + 2)_m} \\ &\quad \times \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i-1}{2}\right)_m}{\left(\frac{2n+\nu_i-1}{2}\right)_m} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2}\right)_{m+q}}{\left(\frac{2n+\nu_i+1}{2}\right)_{m+q}} \\ &= c_0 \sum_{q, m \geq 0} (2n + 2m + q - 1)(q + 1) \frac{\Gamma(2n + q - 1)}{q!} \frac{(2n + q - 1)_m}{m!} \frac{(2n - 2)_m}{(q + 2)_m} \\ &\quad \times \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i-1}{2}\right)_m}{\left(\frac{2n+\nu_i-1}{2}\right)_m} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2}\right)_q}{\left(\frac{2n+\nu_i+1}{2}\right)_q} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2} + q\right)_m}{\left(\frac{2n+\nu_i+1}{2} + q\right)_m}. \end{aligned}$$

The summation with respect to the parameter m yields

$$\begin{aligned}
\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H}) = & c_0 \Gamma(2n) \sum_{q \geq 0} \frac{(2)_q}{(1)_q} \frac{(2n)_q}{q!} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2}\right)_q}{\left(\frac{2n+\nu_i+1}{2}\right)_q} \\
& \times {}_8F_7 \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2} + q\right)_{i=1}^3, & 2n-1+q, & \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3, & 2n-2 \\ \left(\frac{2n+\nu_i+1}{2} + q\right)_{i=1}^3, & 2+q, & \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3 & \end{matrix} ; 1 \right] \\
& + 2c_0(2n-2) \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i-1}{2}\right)}{\left(\frac{2n+\nu_i-1}{2}\right)} \sum_{q \geq 0} \frac{(q+1)}{(q+2)} \frac{\Gamma(2n+q)}{q!} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2} + q\right)}{\left(\frac{2n+\nu_i+1}{2} + q\right)} \\
& \times \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2}\right)_q}{\left(\frac{2n+\nu_i+1}{2}\right)_q} {}_8F_7 \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2} + q + 1\right)_{i=1}^3, & 2n+q, & \left(\frac{2n-\nu_i-1}{2} + 1\right)_{i=1}^3, & 2n-1 \\ \left(\frac{2n+\nu_i+1}{2} + q + 1\right)_{i=1}^3, & 3+q, & \left(\frac{2n+\nu_i-1}{2} + 1\right)_{i=1}^3 & \end{matrix} ; 1 \right].
\end{aligned}$$

By the contiguous relation (8.6) and by the expansion (8.3) of F_{qsv}^{pru} , we obtain

$$\begin{aligned}
\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H}) = & c_0 \Gamma(2n) \left\{ 2 F_{413}^{424} \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2}\right)_{i=1}^3, & 2n: 2, 2 : \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3, & 2n-2 \\ \left(\frac{2n+\nu_i+1}{2}\right)_{i=1}^3, & 2 : 1 : \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3 & \end{matrix} ; 1, 1 \right] \right. \\
& \left. - F_{423}^{434} \left[\begin{matrix} \left(\frac{2n-\nu_i+1}{2}\right)_{i=1}^3, & 2n-1: 2, 2, & 2n: \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3, & 2n-2 \\ \left(\frac{2n+\nu_i+1}{2}\right)_{i=1}^3, & 2 : 1, 2n-1 : \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3 & \end{matrix} ; 1, 1 \right] \right\}.
\end{aligned}$$

4) Similar to the $\mathbb{H}$ -case. $\square$

Remark 8.4. We may rewrite $\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H})$ as

$$\begin{aligned}
\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H}) = & c_0 \Gamma(2n) \sum_{q \geq 0} \frac{(2)_q}{(1)_q} \frac{(2n)_q}{q!} \prod_{i=1}^3 \frac{\left(\frac{2n-\nu_i+1}{2}\right)_q}{\left(\frac{2n+\nu_i+1}{2}\right)_q} \\
& \times {}_9F_8 \left[\begin{matrix} 2n+q-1, & \frac{2n+q+1}{2}, & 2n-2, & \left(\frac{2n-\nu_i+1}{2} + q\right)_{i=1}^3, & \left(\frac{2n-\nu_i-1}{2}\right)_{i=1}^3 \\ \frac{2n+q-1}{2}, & q+1, & \left(\frac{2n+\nu_i-1}{2}\right)_{i=1}^3, & \left(\frac{2n+\nu_i+1}{2} + q\right)_{i=1}^3 & \end{matrix} ; 1 \right].
\end{aligned}$$

where $c_0 = 16\Gamma(2n) \prod_{j=3}^n (j-1)^2(2n-j)^2$. Observe that the ${}_9F_8(1)$ involved above is *very well poised*, i.e. of type

$${}_9F_8 \left[\begin{matrix} a, & 1 + \frac{1}{2}a, & b, & c, & \dots, & \dots, & h \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & \dots, & \dots, & 1+a-h & \end{matrix} ; 1 \right].$$

Unfortunately a closed expression for very well poised ${}_9F_8(1)$, as is the case for ${}_5F_4(1)$ (see (8.2)), does not exist (we thank T. H. Koornwinder for confirming this fact to us). This would allow us to express $\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{H})$ in terms of the hypergeometric functions ${}_pF_q$ instead of the Kampé-de-Fériet ones. A similar remark holds for $\mathcal{K}_{\underline{\nu}}^{\text{nor}}(\mathbb{O})$.

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