# Flexible mental calculation and "Zahlenblickschulung"

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*The study focuses on the development of mental calculation of elementary students who show difficulties in learning math. In total, 20 children in 8 classes were observed during their first year at school. The math education of five classes was based on a special approach called "Zahlenblickschulung", whereas three classes experienced more regular lessons. The collected data allowed a development of a typology of flexibility in mental calculation. Additionally, it was possible to describe the development of each student. The data analysis shows that instruction with "Zahlenblickschulung" also supports less advanced students in developing flexibility in mental calculation. Another result indicates that the recognition of number patterns and numerical relationships is crucial for learning to calculate (beyond counting).*

**Keywords**: Flexible mental calculation, less advanced students in mathematics, elementary arithmetic.

## THEORETICAL FRAMEWORK

For over more than a decade, developing flexible mental calculation has been considered as an important goal in elementary school (Lorenz, 1997; Selter, 2009). Nevertheless, there is still no consensus on instructional approaches and support for less advanced students in learning calculation. The study described below focuses on the development of flexible mental calculation of less advanced students in mathematics. Thereby, we define less advanced children as those who have problems in learning mathematics and need a special support (Schipper, 2005).

### Notions and related research results

Current literature offers different definitions of flexible mental calculation (Rathgeb-Schnierer & Green, 2013; Threlfall, 2009; Verschaffel, Luwel, Torbeyns, & van Dooren, 2009). Most of these definitions involve two common aspects: flexibility and adaptivity. Thereby, flexibility is commonly understood as the ability to switch between different tools of solution (Rathgeb-Schnierer & Green, 2013; Verschaffel et al., 2009), whereas adaptivity "puts more emphasis on selecting the most appropriate strategy" (Verschaffel et al., 2009, 337). What is meant by adaptivity is considered differently (Rechtsteiner-Merz, 2013; Verschaffel et al., 2009):

- adaptivity of solution methods and problem characteristics (Blöte, van der Burg, & Klein, 2001),
- adaptivity of solution methods and speed of obtaining a solution (Torbeyns, Verschaffel, & Ghesquière, 2005; Verschaffel et al., 2009),
- adaptivity of cognitive elements that underlie the solution process (Rathgeb-Schnierer & Green, 2013; Threlfall, 2002, 2009).

In this project, flexible mental calculation involves both aspects: flexibility as mentioned above and adaptivity. Referring to Rathgeb-Schnierer and Green (2013, 2015), this project is based on the assumption that the aspect of adaptivity in flexible mental calculation is related to the recognition of problem characteristics, number patterns and numerical relationships.

### Number sense – structure sense – "Zahlenblick"

Our basic assumption of flexible mental calculation influences the notion of how to teach towards flexibility. If flexible mental calculation is related to problem characteristics, number patterns and numerical relationships, it is necessary to provide activities that encourage students to focus on these aspects. Therefore, the crucial aim is to develop "Zahlenblick" (Schütte, 2004; Rathgeb-Schnierer, 2006; Rechtsteiner-Merz, 2013). To describe the meaning of "Zahlenblick", it is necessary to regard the constructs number sense und structure sense.

The term number sense is connected with two different notions: as a result of experience based development or as an inherent skill.

> "With respect to its origins, some consider number sense to be part of our genetic endowment, whereas others regard it as an acquired skill set that develops with experience." (Berch, 2005, 333f.)

Regarding the construct structure sense, the notions are quite similar. Lükens' definition (2010) of early structure sense reminds us of an inherent competence, whereas Linchevski and Livneh (1999) point out the necessity of its development. "Zahlenblick" is considered a result of development and means the competence to recognize problem characteristics, number patterns and numerical relationships immediately, and to use them for solving problems (Schütte, 2004). Comparing number sense, structure sense and "Zahlenblick" it is obvious that the meaning of number and structure sense as acquired skills that can be developed by special activities is quite similar to our notion of "Zahlenblick". Since there are still discussions about the different definitions, we use the term "Zahlenblick" in the previously described sense of Schütte (see above). To support the development of "Zahlenblick", it is crucial to provide activities, which highlight problem characteristics, patterns and numerical relationships (Rechtsteiner-Merz, 2013; Schütte, 2004). Generally, these activities target the development of number concepts, understanding of operations and strategic means [1]. They encourage students to recognize number patterns, problem characteristics and relations between numbers and problems, and to sort and arrange problems by using structural relations. These activities include cognitively challenging questions to provoke students' thinking and reflection. By combining mathematical topics with challenging questions, an increase of metacognitive competences is intended (Rechtsteiner-Merz, 2013). This can be illustrated by an activity called "Problem-Family" (Figure 1): The students start with one problem, for instance 5+5=10. Then, they were asked to arrange lots of cards with related problems (e.g. 5+6, 6+6, 4+6 etc.) around the first one with the aim of making the relations visible. Subsequently, the students were encouraged to describe their arrangements, and give reasons for their decisions. This activity does not focus on solving problems, but on recognising problem features and relationships.

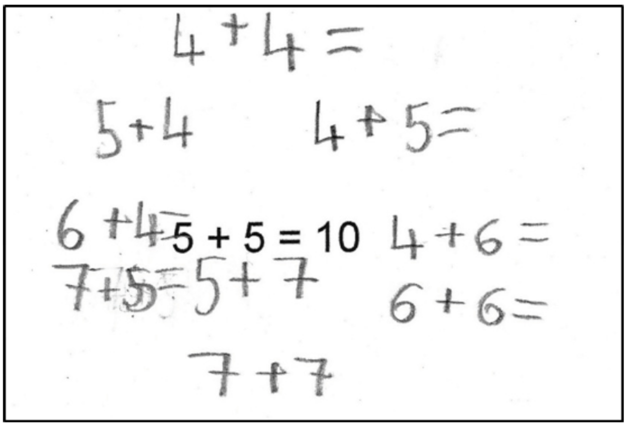


**Figure 1:** A "Problem-family" (Rechtsteiner-Merz, 2013, 113)

"Zahlenblickschulung" is not considered as an additional program. Rather, it can be understood as an essential principle of teaching arithmetic.

## OVERVIEW OF THE PROJECT

### Questions

Referring to prior research, we assume that

- "Zahlenblickschulung" is a good vehicle for developing flexible mental calculation (Rathgeb-Schnierer, 2006; Schütte, 2004) and
- not only middle and high achievers, but also less advanced students can develop flexible mental calculation (Torbeyns et al., 2005; Verschaffel et al., 2009).

These assumptions lead to the following research question: Are first graders with difficulties in learning math (numbers and operations) able to develop flexible mental calculation when educated with "Zahlenblickschulung"?

### Design

Based on the theoretical notion of flexibility introduced above, a qualitative study that focuses on learning processes has been designed. The study included two parts: the instructional approach and the investigation of learning processes (Figure 1).

The investigation started with an extended period of observation to find students with problems in learn-

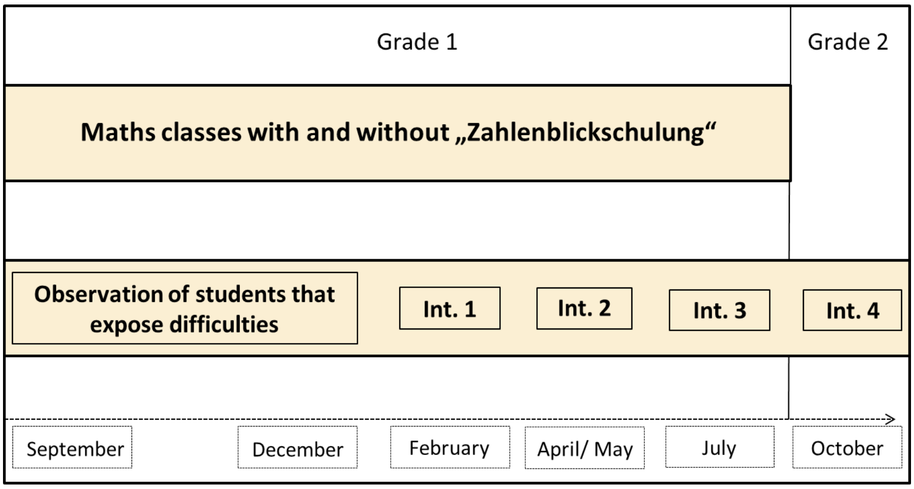


**Figure 2:** Design

ing numbers and operation. Therefore, two different not standardized tests were conducted with all students. Those who performed poorly were subsequently observed over a period of 6 to 8 weeks. As Schipper (2005) describes, approx. 20% of each class develops problems in learning mathematics. Based on this 20% benchmark, we decided to choose 20 students from eight different classes. Twelve students (five classes) experienced "Zahlenblickschulung" in one of four math lessons per week; eight students (three classes) had regular math classes.

From February until October 2008, each student was interviewed four times. The interviews were guideline-based, problem-orientated, and documented by videotape. Due to different moments in the academic year, each interview included different activities, except the one that was finally analysed. This special activity did not change and contained two parts: First, students were asked to sort addition problems and to talk about their reasons for sorting. Second, students were encouraged to solve the problems and to describe their solution procedures. The first three interviews included addition problems with single-digit numbers. In the last interview, several problems, up to 100, were added additionally.

### Data analysis

For data analysis, interviews were transcribed. With the aim to reveal solution procedures and cognitive elements that sustain these procedures, two coding systems were designed based on the "Qualitative Inhaltsanalyse" of Mayring (2008): To classify the solution procedures, an a priori system was used. The analyses of sorting and reasoning were done with an inductively developed coding system (Rechtsteiner-Merz, 2013). Since there was a huge difference in the quality of students' reasoning, it was necessary to judge the value of arguments. Based on a theory called "Argumentationsanalyse" (analyses of arguments) (Fetzer, 2011; Toulmin, 1996) and the theory of proof (Almeida, 2001; Sowder & Harel, 1998), it was possible to judge different arguments according to the theory of flexible mental calculation (see above).

According to Kelle and Kluge (2010), types were constructed by interrelating three dimensions: (1) the amount of correct solutions, (2) the solution procedures, and (3) the reasoning for sorting and solving (Rechtsteiner-Merz, 2013). Therefore, it was necessary to build two feature spaces: First, the dimension "amount of correct solution" and the dimension "solution procedures" were combined. At this level, it was possible to construct pre-types which describe calculation in first grade. In the second step these pre-types were linked to the dimension "reasoning for sorting and solving". On this level, it was possible to develop a typology of flexible mental calculation in first grade (Figure 2).

## RESULTS AND OUTLOOK

Finally, nine types could be derived from the data, four main types and five temporary types (Rechtsteiner-Merz, 2013) (Figure 3). The main types focus on a typical phase at the beginning of first grade (*counting strategies*) or on an intended phase at the end of first grade (*consistent use of procedural mastery*, *partly basic facts with relational expertise* or *basic facts extended with relational expertise*). The temporary types represent stages of developments when students learn calculating (beyond counting).

The arrangement of the types in Figure 3 must be understood as the combination of the two dimensions "counting subsumed by calculating" (horizontal dimension) and "reliance on numerical relationship in argumentation" (vertical dimension).

Subsequently, we describe the main types, followed by the temporary types: Students with *counting strategies* [2] solve each problem by counting, usually starting from the large number. Students with *consistent use of procedural mastery* are able to solve most of the problems up to twenty by calculating. Therefore, they always use the same solution procedure without noticing any problem characteristics. They argue for example "I do always like this" or "like always up to ten and then the rest". Students who exhibit *partly basic facts with relational expertise* use different strategic means by relying on problem characteristics. They are able to describe the solution process and give reasons for their strategic means in an elaborate way as the following example shows: "These problems are easy (*points to* 8+5 *and* 4+9), because here it's one less and here it's one more (*points to* 4 *and* 9)". Students who depict *basic facts extended with relational expertise* relay on basic facts with addition problems up to twenty. Additionally, they are able to solve problems with two digit numbers (higher than twenty) based on recognized characteristics and numerical relationships (even if this is not a topic in first grade). This type is special for first grade since all addition problems up to twenty can be memorized by heart.

Students who solve problems *predominantly by counting* are divided in two groups: those who *rely on procedures (temporary type 1)* and those who *rely sometimes on numerical relationships (temporary type 3)*. Students who belong to the *temporary type 1* predominantly practice counting as a rule. Sometimes, they suddenly use strategic means or number facts, although they cannot describe their approach or give a reason for it. Students who belong to the *temporary type 3* also use predominantly counting to solve addition problems, but sometimes they notice numerical relationships, and they are able to describe and reason their approach.

There are also students who solve problems *predominantly by calculating relying usually on procedures (temporary type 2)*. Based on procedures, they can solve many problems up to twenty. Exceptionally, they rely sometimes on numerical relationships when solving a problem or giving reasons for the sorting. On the other hand, there are students from the *temporary type 5* who solve problems *predominantly by*

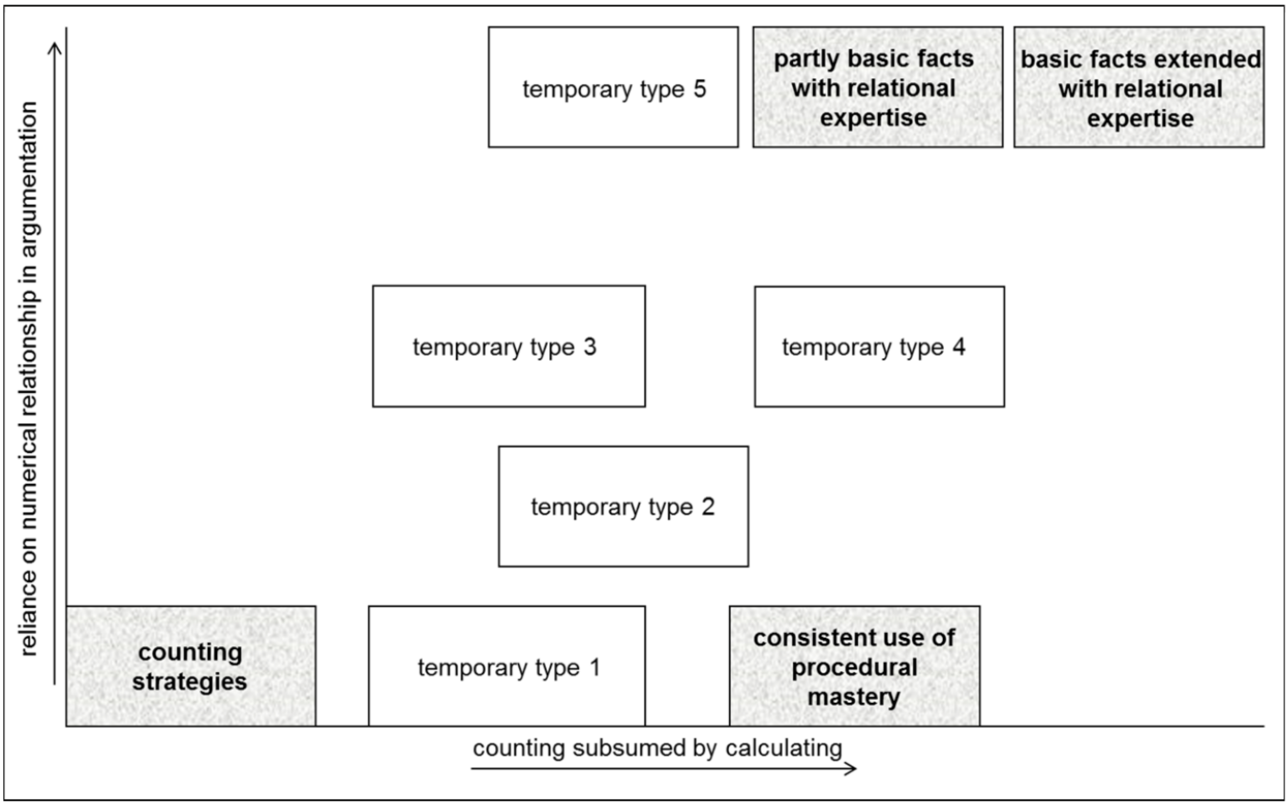


**Figure 3:** Typology of flexible mental calculation (Rechtsteiner-Merz, 2013)

*calculating relying on numerical relationships*. They are also able to solve a lot of problems up to twenty, but they recognize and use problem characteristics and numerical relationships.

Students who solve problems by *calculating relying sometimes on numerical relationships (temporary type 4)* are able to solve all the problems up to 20 by going beyond counting. Some solutions rely on procedures, others on numerical relationships. Whenever a solution is based on numerical relationships, it can be described and reasoned.

Finally, we give examples of students' development that were apparent in the timeframe from January in first grade to October in second grade.

Two students who used *counting strategies* at the beginning of second grade exhibited a *predominance of counting relying on procedures (temporary type 1)* in January. Finally, they showed a kind of regression, since their use of basic facts or strategic means were higher in the middle of first grade than at the beginning of second grade.

Four students who used *counting strategies* in January switched to the *temporary type 1* and solved problems by *predominantly counting relying on procedures* between April and July (end of first grade). After this change, no further development was obvious; it seems that they were trapped in counting.

Two students who reached the type *consistent use of procedural mastery* showed different ways of development. However, both exhibited relying on numerical relationships by calculating at least in one interview.

Five students who belong to the type *partly basic facts with relational expertise* at beginning of second grade started in January from *temporary type 1*, and solved problems *predominantly by counting relying on procedures*. They relied obviously on numerical relationships in April; some still by counting, others overcame counting. Lena, for example, solved the same number of problems predominantly by counting in January and April. But, there was a big difference in her reasoning: In January she did not recognize any problem characteristics, in April she used numerical relationships in solving and reasoning at least sometimes.

When comparing students' developments with and without math education based on "Zahlenblickschulung", some crucial differences can be described. Most students (only two exceptions) who did not experience "Zahlenblickschulung" stuck with their counting strategies based on procedures. Actually, those students did not show any progress in the second term of first grade. In contrast, all students who experienced "Zahlenblickschulung" (except Yannik) were able to overcome their counting strategies at least until the end of first grade. Additionally, all these students (except Amelie) used numerical relationships for solving problems, and they were able to reason sorting procedures in very elaborate ways.

**Developed hypotheses: Conclusions**

Focusing on students who have difficulties in learning addition, data analysis suggests the development of four central hypotheses:

- Relying on numerical relationships is an absolute condition for developing calculation strategies that go beyond counting.
- "Zahlenblickschulung" supports the development of conceptual knowledge.
- "Zahlenblickschulung" supports the development of flexible mental calculation.
- Activities in "Zahlenblickschulung" are a fundamental condition for developing calculation strategies and flexible mental calculation.

Subsequently, two hypotheses will be reported in detail:

**Relying on numerical relationships is an absolute condition for developing calculation strategies that go beyond counting.**

The knowledge of basic facts and strategic means seems to be insufficient for the development of a deep understanding of calculation that goes beyond counting. Therefore, the focus on numerical relationships and structures is essential. All students who overcame their counting strategies were able to "calculate without counting" at the beginning of second grade, and relied on numerical relationships at least in one stage of development. On the other hand, all students who were predominantly counting relying on procedures remained in this stage and could not progress. This *temporary type 1* seems to be like a dead-end road. Thus, the recognition and use of number patterns and numerical relations seems to be a crucial prerequisite for going beyond counting.

## Activities in "Zahlenblickschulung" are a fundamental condition for developing calculation strategies and flexible mental calculation.

In order to develop flexible mental calculation in elementary school, Rathgeb-Schnierer (2006) and Schütte (2004) emphasized the necessity of "Zahlenblickschulung". Focusing on calculation competence of middle and high-achieving first graders, Torbeyns and colleagues (2005) showed that they are much more flexible than students who are considered as low-achieving peers. This observation indicates that middle- and high-achiever students develop a minimum of number patterns and numerical relationships for going beyond counting independently. However, students who have difficulties in learning arithmetic benefit from the "Zahlenblickschulung" approach; first to overcome counting, and second to develop an appropriate degree of flexibility in mental calculation.

The study reveals that less advanced first grade students are also able to develop competences in flexible mental calculation. Thereby, "Zahlenblickschulung" is an important and supportive vehicle. Especially for less advanced students, the recognition of number patterns and numerical relationships is the key for learning to calculate (beyond counting) and developing flexibility.

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## ENDNOTES

1. Strategic means are distinct devices to modify problems to them easier. They can be flexibly combined in a solution process, and include for instance composing and decomposing, modifying a problem, deriving the solution from a known fact, and using analogies (i.a., Rathgeb-Schnierer, 2006)

2. For calculating you can use different tools for solution: counting, basic facts, strategic means. Counting can be distinguished if it's with or without models and in there are counting-all or counting-on strategies used (Carpenter & Moser, 1982).