Sharing the road: the economics of autonomous vehicles

Raphaël Lamotte, André de Palma, Nikolas Geroliminis

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Raphaël Lamotte[1], André de Palma[2], Nikolas Geroliminis[1]

[1]School of Architecture, Civil and Environmental Engineering,
Urban Transport Systems Laboratory
Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland

[2]Ecole Normale Supérieure de Cachan,
CES - Université Paris-Saclay, France

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Abstract

Automated cars are likely to change mobility substantially in the coming years. Much research is developed in engineering, about legal and behavioral issues, but the economics of autonomous vehicle remains an open area. In this paper, we consider a single-bottleneck situation, in which the capacity of the freeway is divided between conventional and autonomous vehicles. Users of conventional vehicles freely choose their departure time from home, while users of autonomous vehicles collaborate with a central operator that ensures they do not queue. An individual-specific cooperation cost is integrated in the modeling framework. We address the following key issues: how should infrastructure be allocated to conventional and automated cars? Are there synergies between the two fleets of vehicle? How should each infrastructure be tolled? Should the government be a toll leader? Which regulations are needed?

1 Introduction

Supply- and demand-oriented traffic management techniques are considered the main control approaches to deal with congestion, due to the lack of available space for new infrastructure and increase of mobility needs. From the more supply-oriented to the more demand-oriented, these techniques include: smart control of traffic signals at the network level (see for example Haddad et al., 2013; Ramezani et al., 2015), dynamic usage of urban space (Eichler and Daganzo, 2006; Guler and Cassidy, 2012; Zheng and Geroliminis, 2013), integrated corridor management (Diakaki et al., 2000), preferential treatment of public transport, route guidance (Papageorgiou, 1990; Yildirimoglu et al., 2015), congestion pricing (Vickrey, 1969) and car sharing. Given the huge number of private cars associated with limited network capacity, supply-oriented techniques cannot fully eliminate congestion as the demand for travel in the peak hours of cities is far beyond capacity. Properly integrating demand- and supply-oriented policies remains a challenging direction that would require bringing together realistic physical models of congestion, economic principles and user acceptability. Nowadays, new technologies and intelligent transport systems can further increase the capacity of the roads under the same available space.

A promising direction is the integration of autonomous vehicles in city traffic. This integration is significantly more difficult if conventional and automated vehicles coexist on the same network, as their behaviours would obey different rules. Recently, under the Darpa Urban Challenge (DUC)(DARPA, 2008), different car manufacturers were competing in an environment where autonomous vehicles were capable of driving in traffic, performing complex maneuvers such as recognize blockages, merging, passing, parking and navigating through intersections. A new DUC challenge is under preparation.

While some experts might find this rather optimistic, various executives of many major industry players and the US Secretary of Transportation have announced that fully autonomous cars will be available on the market by horizons ranging from 2017 to 2025[1]. Standardization is likely to take more time but expert IEEE members forecasted in 2012 that by 2040, autonomous vehicles will account for up to 70 percent of cars on the road (IEEE, 2012). The level of automation might improve gradually in conventional vehicles, or fully automated vehicles might be directly deployed in limited but gradually expanding contexts (International Transport Forum, 2015).

Eventually, autonomous vehicles will reshape our relation to mobility. Vehicle automation represents more than simply improved safety, higher speeds, higher densities and the possibility to use commuting time more productively. The ability to drive itself also means that the same vehicle can serve many different users throughout the day instead of staying idle and using parking space. In this

[1]See a list of links to interviews here: `http://www.driverless-future.com/?page_id=384` (accessed on December 1, 2015).

sense, autonomous vehicles are much more similar to a taxi service than to conventional vehicles. We believe that the most important consequences of the rise of autonomous vehicles will not come from the mere technical superiority of robot drivers, but from the transition from an ownership-oriented mobility paradigm to one which is service-oriented.

The cost of congestion is often estimated to be in the order of 1% of the GDP in developed countries. Technically, autonomous vehicles might be able to reduce this cost by reducing the headways between vehicles and increasing the traveling speeds. However, purely capacity-oriented measures have been shown to lead almost systematically to the induced demand phenomenon: as cheaper mobility encourages more trips, congestion levels hardly change on the long term (Goodwin, 1996). Thus, capacity management allows for more mobility but the only sustainable way to reduce congestion on the long term is demand management. Vickrey (1969, 1973) and many other transportation economists have argued for decades for congestion pricing, which is the conversion of the deadweight loss represented by time spent waiting in congestion into a source of revenue for the collectivity. However, public acceptability (Hårsman and Quigley, 2010; Börjesson et al., 2012), heterogeneous travelers, lack of realistic modeling of traffic dynamics (Geroliminis and Levinson, 2009) and technical difficulties have almost systematically prevented such implementations. The emergence of a service-oriented approach to mobility paves the way for a new approach of congestion management. From the customer point of view, what matters is the generalized cost of traveling. If the central operator can guarantee a short travel time and a reasonable arrival time at work, competition for a limited road capacity does not need to translate into long waiting times at the bottleneck.

In this paper, we consider a single-bottleneck situation, in which the capacity of the freeway is divided optimally between conventional and autonomous vehicles. Users of conventional vehicles freely choose their departure time from home, while users of autonomous vehicles collaborate with a central operator towards creating system optimum conditions for the entire freeway. The operator informs autonomous car users when they should depart from home with a guarantee of no congestion cost. This work addresses some of the aforementioned modeling and implementation limitations by considering heterogeneous travelers, partially reserved lanes and the integration of a new scheduling service.

The details of the implementation of this scheduling service remain voluntarily vague at the moment, but the analytical description of this work will also shed some light in this direction. For example, departure times could be allocated randomly, on a first-registered basis, in an equitable way across all cooperative users in the long term, or based on some auction (similarly to railway or airline companies' strategies aimed at filling up their vehicles). However, we acknowledge that the obligation to follow some schedule should be associated to a cost. This cost, named the "cooperation cost", can be thought of as the sum of a (positive) cost associated to scheduling and of a (possibly negative) cost that is specific to the autonomous vehicles' technology. The cooperation cost is assumed to be distributed among individuals but to be constant for one individual over time. This new cost then plays the role of the selecting criterion: at equilibrium, users would choose to use autonomous vehicles only if by doing so they can reduce their cost associated to travel time and schedule penalty by more than their personal cooperation cost. Rather than "autonomous car users", such users will be qualified as "cooperative" while the others will be qualified as "independent", both for brevity and because in terms of congestion, the cooperative behavior matters more than technology. Note also that a similar concept is used to distinguish planning users from not planning users in previous works related to public transit (Tisato, 1992; Fosgerau, 2009). In other words, the service proposed can be considered as some additional alternative between public transit and private vehicles: cooperative users have to comply with the schedule but enjoy their own vehicle, have shorter travel times, and importantly, do not waste time in connecting between different modes of transportation.

While this scheme might be difficult to implement with conventional vehicles, it might emerge naturally as the trends toward autonomous vehicles and car-sharing converge. Indeed, the development of autonomous vehicles is likely to lead to the emergence of separated road networks, to avoid interactions with human drivers. In addition, car-sharing already imposes some form of cooperation since its users often have to reserve in advance a vehicle for a given time. Thus, the convergence of these two trends would create an ideal framework for the implementation of the scheme described.

As this is the first time that the economics of this scheme is studied, many configurations are investigated, sometimes to conclude that some of them should simply not be implemented in the real world. However, the study of the user equilibrium under optimal distribution of capacity and of the social optimum revealed a great potential. It is found that, unlike in the classic theory with a single bottleneck, these two regimes have very similar social costs. This finding is of great importance in practice as significant benefits could be obtained by simply offering this service, without collecting any toll. Furthermore, we prove analytically under mild conditions on the distribution of the cooperation cost that if the capacity split is optimally chosen, the user equilibrium with partial cooperation Pareto-dominates the case with no cooperation. Thus, our approach of cooperation would benefit from a greater public acceptability while implementation would be eased by the absence of toll.

The remainder of this paper is structured as follows: Section 2 lays the background by introducing the general assumptions, expressions of the different costs, and rapidly addresses the problem without cooperation cost to provide some intuition. Then, the socially optimal demand and capacity splits are studied in Section 3. It is shown in Section 4 that many results obtained for the social optimum remain valid under user equilibrium and that the user equilibrium can be made socially optimal with a simple constant toll (decentralization of the social optimum). It is also shown that unlike the social optimum, the user equilibrium with a socially optimal capacity split Pareto-dominates the case with no cooperation. The management of a route by a private operator is considered in Section 5 to assess the impact of profit-maximizing strategies on the social cost and finally, conclusions are drawn and suggestions for future research are stated in Section 7. The impact of automation is addressed separately at the end of each section.

2 Background

2.1 Impact of automation

While this paper is primarily concerned with the great potential of cooperation, automated vehicles will also impact our transportation systems in many different ways that should not be ignored.

First, although there is no consensus on the scale of this change, automation will impact the capacity of the road network. The impact should be globally positive on highways, with authors estimating capacity improvements ranging from 20% to more than 100% (Varaiya, 1993; Faggio and Silva, 2014). For urban networks, the impact is even more uncertain. Different research groups working on intersection control showed that communication technologies could allow for reductions in delays at intersections (Fajardo et al., 2012; Qian et al., 2014). On the contrary, Le Vine et al. (2015) argued that users of autonomous vehicles would have a lower acceptance to accelerations and decelerations, which would actually slow down traffic in urban areas. Progress in terms of safety could also significantly improve the capacity overall. Indeed, since traffic incidents are responsible for approximately 10-30% of the total congestion delay (see e.g. Skabardonis et al. (2003)) and since human error is the critical factor in the vast majority of crashes, reducing crashes would also increase the average capacity of roadways. Based on the literature above, various values of g between 1 and 2 are used for numerical applications in this paper. Fortunately, the theoretical results obtained remain valid for any value of the automation factor $g > 0.5$, which seems to be an extremely reasonable assumption.[2]

Second, automation would radically change the way transportation is experienced. While some users will certainly always prefer conventional cars because they wish to keep their autonomy or simply enjoy driving, others might appreciate not having to drive or might favor automated vehicles for others reasons (e.g. safety or travel time reliability). This preference a priori for one mode of transportation or the other is also captured here by the cooperation cost, introduced in Section 2.4.

Additionally, the value of the in-vehicle time might also be impacted as users can have other activities instead of driving. If autonomous and conventional vehicles have to share the same road, this is likely to push conventional users away from the common desired arrival time and to reduce the cost of autonomous vehicle users (van den Berg and Verhoef, 2015). However, it is well known that if all users are homogeneous, the individual costs do not depend on the value of the in-vehicle

[2] It can be easily shown that such a scheme cannot reduce the social cost if g is smaller than $\frac{1}{2}$ - see Section 2.6.1.2.

time (Arnott et al., 1990b). In the case at hand, conventional and autonomous cars are physically separated, so the users of each route are homogeneous and the value of in-vehicle time does not impact the individual congestion cost. While the cost of the free-flow part of the trip might still depend on the type of car used, this cost is independent of the traffic conditions and hence can also be captured by the constant individual specific cooperation cost considered in this work[3]. The different components of the individual costs are described in more details in Section 2.4.

2.2 Comparison with fast lanes

There are several evidences suggesting that partial space allocation can solve the acceptability issues associated with congestion pricing. Empirically, the contrast between the ordeals experienced for all full congestion pricing projects and the extremely rapid emergence of managed lanes (e.g. the High-Occupancy Toll (HOT) lanes) in the US and around the world is unequivocal. On the theoretical side, second-best tax rules for situations where not all links can be tolled have been studied extensively in the static case (Lévy-Lambert, 1968; Verhoef, 2002) but fewer authors have looked at the dynamic case. One important contribution on this topic was made by Fosgerau (2011), who showed analytically that temporarily reserved lanes can be Pareto-improving with Vickrey's bottleneck model, as the costs of prioritized users can be reduced without increasing the cost of the others. The fundamental assumption behind this result is that "when the capacity is not used, it is available for the nonprioritized travelers". Note that although the work of Fosgerau (2011) was not specifically focused on autonomous vehicles, prioritized vehicles could be considered to be autonomous vehicles without requiring any additional change (except maybe an automation factor similar to the variable g considered in our work).

Nevertheless, the present work has very specific characteristics that result from its focus on autonomous vehicles. First, the scheduling of departures introduced here allows entirely removing queues on the cooperative route. While this could also be achieved in theory with fast lanes by creating as many levels of priority as users, this would either lead to a significant amount of wasted capacity during transitions, or to a need for users to schedule, which would have to be associated to a cost, like in the present work. Second, as autonomous vehicles might require specific road infrastructures, a permanent capacity split was considered here. It is striking that this still allows for a Pareto improvement, while fast lanes require real-time adaptive capacity allocation to have this property. Third, thanks to its longer time horizon, the scheme proposed here permits reducing the costs of all users. Indeed, as the purchase of a vehicle is a long-term decision, the coexistence of different vehicles or different types of ownership requires considering the average cost over a long time horizon. Thus, unlike with fast lanes, trips can be scheduled over a longer time period on the cooperative route than on the independent one,

[3] Note that this does not stand if the bottleneck capacity depend on the queue length, as for instance in urban networks with a Macroscopic Fundamental Diagram representation (Geroliminis and Levinson, 2009).

in such a way that all average individual costs are reduced. Finally, the arbitrary selection of the users allowed on fast lanes is politically difficult and might even be regressive if it is based on the capacity to buy some expensive vehicle. With scheduling however, users choose themselves to be cooperative or independent based on their own cooperation cost. Hence, this selection is entirely endogenous.

2.3 Problem description

Let us consider a single origin/single destination situation with only one route and a bottleneck of capacity S and a total (inelastic) demand N. This route can be divided into two parallel sub-routes, which have bottlenecks at the same location as the original route and which are reserved for independent and cooperative users, respectively. The proportion of the demand that is cooperative and the proportion of the bottleneck capacity that is allocated to them are denoted by x and $y \in [0, 1]$. However, for the reasons given above, the effective capacity of the cooperative sub-route is likely to differ as it is used by autonomous vehicles. Thus, a cooperative infrastructure with a "traditional-vehicle-capacity" of yS would have an effective capacity of gyS, where the automation factor g would characterize how much more/less efficiently the facility is used (g would be equal to 1 if the cooperative service was implemented with conventional cars). Table 1 summarizes the notations utilized in the paper.

In this work, the only heterogeneity considered among users is their cooperation cost, in case they choose to be cooperative. This is in agreement with an important part of the literature that considers only homogeneous users (Arnott et al., 1990b). In particular, all users value their time in the same way and have the same desired arrival time. Thus, in a classic equilibrium with no cooperation, all users should have the same cost. In order to have a similar property for cooperative users, it is assumed that the decision to be independent or cooperative is taken only once for a long time horizon and that the allocation mechanism is assumed to be fair on the long term, i.e. all cooperative users have in average (over long enough periods) the same cost (excluding their personal cooperation cost). This can be obtained for instance with a variable allocation of departure times or by compensating any difference in schedule penalty by other means (financial for instance).

The long time horizon considered is supported by the fact that the vehicles are different for both routes and that the purchase of a vehicle is only required for independent users. While a vehicle owner could easily decide to stop being cooperative whenever she is given a departure time that she does not like, a car-sharing user does not have this flexibility as she needs to have access to a vehicle. This distinction matters as it means that the cooperative route could potentially be used longer than the independent one, which would allow reducing the cost of independent users as well. Hence, it was chosen in this work to consider cooperation as a long-term decision. Ability of cooperative users to

Variables	Unit	Description
α	\$/h	unit cost of travel time
β	\$/h	unit cost of arriving early
γ	\$/h	unit cost of arriving late
δ	\$/h	$\triangleq \frac{\beta\gamma}{\beta+\gamma}$
t^*	h	desired departure timall users aree from the bottleneck
S	veh/h	capacity of the bottleneck
N	veh/h	demand
x	0	proportion of the demand that is cooperative
y	0	proportion of the capacity that is reserved for cooperative users
g	0	automation factor (characterizes the impact on the effective capacity)
θ	0	type of user or (normalized) cooperation cost
$\hat{\theta}$	0	critical value of θ separating cooperative from independent users
κ	\$	unit cooperation cost
f, F	0	probability (pdf) and cumulative (cdf) density function of θ
Θ	0	support of f
$\underline{\theta}, \bar{\theta}$	0	infimum and supremum of Θ
τ	\$	a toll imposed for each trip on the independent route
$\tau_{pc}, \tau_{pi}, \tau_{gc}, \tau_{gi}$	\$	toll imposed for each trip by the government (g) or the private operator (p) on the cooperative (c) or on the independent (i) route
c_c, c_i	\$	individual congestion cost for cooperative and independent users
r	0	cost ratio equal to $\frac{\delta N}{\kappa S}$
$y^o(\hat{\theta}), \hat{\theta}^o(y), x^o(y)$	0	socially optimal values, given the variables in parentheses
$\hat{\theta}(y), x(y)$	0	user equilibrium values (no toll) for a given capacity split y
y^p, x^p, c_c^p, c_i^p		profit-maximizing values
y_e	0	$\triangleq F((1-\frac{1}{2g})r)$, capacity split such that at equilibrium $x = y$
$SC(\hat{\theta}, y)$	\$	social cost as a function of $\hat{\theta}$ and y
$SC(x, y)$	\$	social cost as a function of x and y
$SC(\hat{\theta})$	\$	$\triangleq SC(\hat{\theta}, y^o(\hat{\theta}))$, social cost function with a socially optimal capacity split
$SC(y)$	\$	$\triangleq SC(\hat{\theta}^o(y), y)$, social cost function with a socially optimal demand split
SC^{ref}, c^{ref}	\$	social and individual cost in the reference scenario (no cooperation)
$\widetilde{SC}$	0	$\triangleq \frac{SC}{SC^{ref}}$; the same tilde notation is used for other variables and it always indicates the ratio of the variable divided by its value in the reference scenario (no cooperation).

Table 1: Notations

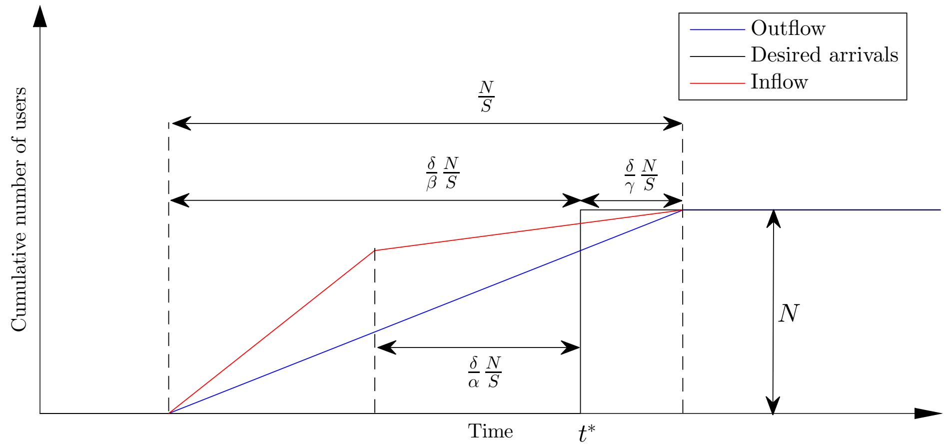

Figure 1: Bottleneck dynamics for the independent route

partially deny the offered time of departure will be considered in the future, as the mathematical derivations become tedious.

2.4 Individual costs

Keeping the original assumptions of Vickrey (1969) and Arnott et al. (1993), we consider that users have a personal generalized cost related to congestion that is the sum of a travel time cost (assumed to be proportional to the travel time with a coefficient α) and of a schedule penalty, that accounts for the inconvenience of arriving too early or too late (proportional to the time early with coefficient β or to the time late with coefficient γ). All users share the same value of α, β, γ and the same desired arrival time t^*.

Then, since N users want to pass a bottleneck with capacity S, the duration the bottleneck is used is given by $\frac{N}{S}$. If all individuals are independent and identical, then they must all have the same cost at equilibrium. If there is no toll, this cost is the sum of a schedule penalty and a travel time cost due to congestion. For the first and last users, there is no delay caused by congestion so the entire cost is their schedule penalty. If the first user has an advance T_1 and the last user a lateness T_N, then $T_1 + T_N = \frac{N}{S}$ and since they must have the same total cost $\beta T_1 = \gamma T_N$. By combining these two equations, the individual equilibrium cost can be expressed as the sum of a congestion cost and of a fixed cost: $c = \delta \frac{N}{S} + \alpha t_0$, where $\delta = \frac{\beta\gamma}{(\beta+\gamma)}$ and t_0 is the free-flow travel time. Without loss of generality, it is assumed from now on that $t_0 = 0$, i.e. that there is no fixed cost. The dynamics are illustrated in Fig. 1.

Thus, with the notations defined above, the equilibrium cost c_i for a user of the independent route is:

$$c_i = \begin{cases} \frac{\delta N}{S}\frac{1-x}{1-y} & \text{if} \quad y \in [0,1[\\ \infty & \text{if} \quad y = 1. \end{cases} \tag{1}$$

Since their departure times are properly scheduled, cooperative users do not have any queuing time. However, they still incur a schedule penalty cost, which is a uniformly distributed random variable taking values between 0 and $\frac{\delta N}{S}\frac{x}{gy}$. Thus, their average schedule penalty is $\frac{\delta N}{S}\frac{x}{2gy}$. In addition, cooperative users also incur a cooperation cost, which is a characteristic of each individual. The probability density function (pdf) of this cost in the entire population is assumed to be known and to verify the following condition:

Condition 1. *The support of the probability density function of the cooperation cost is an interval including* 0*, positive for at least some users, bounded below but not necessarily above.*

Note that this condition allows the cooperation cost to be negative for some users. This might happen for instance if travel time reliability is highly valued or if the technology used for cooperative vehicles has significant advantages. Since the cooperation cost is the only source of heterogeneity, users can be characterized by their individual cooperation cost, relatively to the entire population: an individual of type θ has the cooperation cost $\kappa\theta$, where κ is referred to as the unit cooperation cost. The pdf of the type θ is denoted by f, its support by Θ, and its infimum and supremum by $\underline{\theta}$ and $\bar{\theta}$. Condition 1 imposes that $\underline{\theta} \in \mathbb{R}^-$ and $\bar{\theta} \in \mathbb{R}^{+*} \cup \{+\infty\}$. With these notations, the average cost for a cooperative individual of type θ is:

$$c_c + \kappa\theta = \begin{cases} \frac{\delta N}{S}\frac{x}{2gy} + \kappa\theta & \text{if} \quad y \in]0,1] \\ \infty & \text{if} \quad y = 0, \end{cases} \tag{2}$$

where c_c is referred to as the congestion cost for cooperative users.

2.5 Primary analysis of the social cost

Since individuals differ only by their cooperation cost, the cooperative population consists only of the individuals with the smallest value of θ, both under user equilibrium and under social optimum. Thus, there exists a critical type denoted by $\hat{\theta}$ such that all individuals of type $\theta < \hat{\theta}$ are cooperative, while all individuals of type $\theta > \hat{\theta}$ are independent. Note that $\hat{\theta}$ might potentially be equal to $\underline{\theta}$ or $\bar{\theta}$, in which case all users belong to the same category. The proportion of the demand that is cooperative is simply given by the cumulative distribution function (cdf) F of the cooperation cost θ evaluated at $\hat{\theta}$ (see Fig. 2):

$$x\left(\hat{\theta}\right) = F\left(\hat{\theta}\right). \tag{3}$$

By assumption (cf Condition 1), $f(u) > 0$ for all $u \in]\underline{\theta}, \bar{\theta}[$ so x is a strictly increasing function of

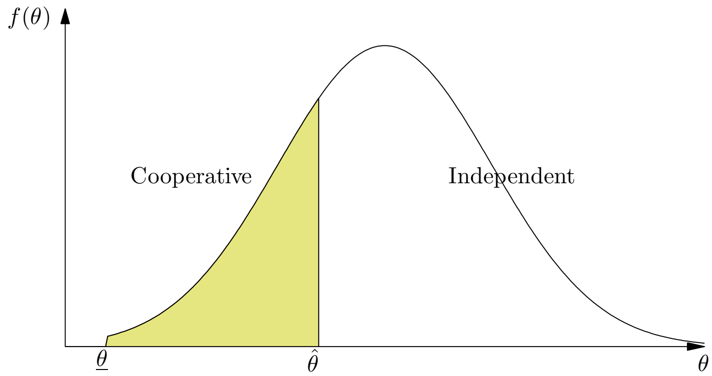

Figure 2: Example of a distribution of the cooperation cost among the population and separation between cooperative and independent users

$\hat{\theta}$. Consequently, $\hat{\theta} \to x\left(\hat{\theta}\right)$ is a bijection from $[\underline{\theta}, \bar{\theta}]$ to $[0,1]$. Then, the social cost is expressed as a function of the demand and capacity splits by:

$$SC(\hat{\theta}, y) = \begin{cases} N\kappa \int_{\underline{\theta}}^{\bar{\theta}} u f(u) du + \frac{\delta N^2}{S} \frac{1}{2g} & \text{if} \quad (\hat{\theta}, y) = (\bar{\theta}, 1) \\ N\kappa \int_{\underline{\theta}}^{\hat{\theta}} u f(u) du + \frac{\delta N^2}{S} \frac{x^2}{2gy} + \frac{\delta N^2}{S} \frac{(1-x)^2}{1-y} & \text{if} \quad (\hat{\theta}, y) \in [\underline{\theta}, \bar{\theta}] \times]0, 1[\\ \frac{\delta N^2}{S} & \text{if} \quad (\hat{\theta}, y) = (\underline{\theta}, 0), \end{cases} \tag{4}$$

where $x = x\left(\hat{\theta}\right)$ according to Eq. (3). The social cost is infinite for $\hat{\theta} \neq \underline{\theta}$ and $y = 0$ and for $\hat{\theta} \neq \bar{\theta}$ and $y = 1$.

Besides the demand and capacity splits x and y, Eq. (4) also involves the exogenous variables N, S, κ and δ. To further simplify this expression and the expressions that are derived thereafter, we now introduce the relative social cost, that is the social cost divided by the social cost under a reference scenario. The reference scenario chosen here is the situation with no cooperation at all ($x = 0$, $y = 0$). In this case, the social cost is simply $SC^{ref} = \frac{\delta N^2}{S}$. Similarly, we will also consider individual costs relative to the individual cost in the reference scenario $c^{ref} = \frac{\delta N}{S}$ and the duration the network is used, relative to $\frac{N}{S}$. With this transformation, the relative social cost is given by:

$$\widetilde{SC}(\hat{\theta}, y) \triangleq \frac{SC(\hat{\theta}, y)}{SC^{ref}} = \begin{cases} \frac{1}{r}\int\limits_{\underline{\theta}}^{\hat{\theta}} uf(u)du + \frac{1}{2g} & \text{if} \quad (\hat{\theta}, y) = (\bar{\theta}, 1) \\ \frac{1}{r}\int\limits_{\underline{\theta}}^{\hat{\theta}} uf(u)du + \frac{x^2}{2gy} + \frac{(1-x)^2}{1-y} & \text{if} \quad (\hat{\theta}, y) \in [\underline{\theta}, \bar{\theta}] \times]0, 1[\\ 1 & \text{if} \quad (\hat{\theta}, y) = (\underline{\theta}, 0), \end{cases} \tag{5}$$

where $r = \frac{\delta N}{\kappa S}$. Note that r has a physical interpretation: it represents the ratio of the congestion costs (if all users are independent, $\frac{\delta N}{S}$ is the individual cost, i.e. the sum of the schedule delay penalty and of the travel time cost) divided by the unit cooperation cost. Note that the relative social cost is only a function of the capacity y, the automation factor g, the cost ratio r and the critical cooperation cost $\hat{\theta}$. If both the demand split and the capacity split are optimally chosen, the relative social cost depends only on g and r. Similarly, it is shown in Section 4 that the demand split under user equilibrium is only a function of g, r and y and that hence, the relative social cost under user equilibrium with a socially optimal capacity split is also a function only of g and r.

We will also denote by $\widetilde{SC}$ the relative social cost functions that use other input arguments. However, to avoid any ambiguity, the relevant input arguments will always be mentioned. For instance, since there is no cooperation cost in Section 2.5, we will use the function $\widetilde{SC}(x, y)$. Later in this work, the functions $\widetilde{SC}(\hat{\theta}) = \widetilde{SC}(\hat{\theta}, y^o(\hat{\theta}))$ and $\widetilde{SC}(y) = \widetilde{SC}(\hat{\theta}^o(y), y)$ will be used to refer to relative social cost functions of one variable only, assuming that the other split is fixed (or optimal).

2.5.1 Optimal capacity split with fixed demands

Before studying the impact of the demand split at the social optimum or at the user equilibrium, one can consider the problem of determining the socially optimal capacity, given an arbitrary demand split. Such a problem would occur for instance if the cooperation scheme was implemented with autonomous vehicles of a given fleet size. Since the demands for both sub-networks are fixed, the cooperation cost plays no role in this problem. However, this result will be useful when both the capacity and demand splits will be optimized, in Section 3.2. Since the demand split is fixed, $\widetilde{SC}$ is considered in this section as a function of y only.

Proposition 1. *For any given demand split $x \in [0, 1]$, there exists a unique capacity split that minimizes the social cost. It is independent of the cooperation cost and congestion parameter δ, and it is given by:*

$$y^o(x) = \frac{x}{\sqrt{2g}(1-x) + x}. \tag{6}$$

This capacity split is continuous and strictly increasing on $[0,1]$. *The cooperative route is used* $\sqrt{2g}$ *times longer than the independent route.*

Proof. If $x=0$ (or $x=1$), the socially optimal capacity split is trivially given by $y^o=0$ (resp. $y^o=1$).

Now, if $x\in]0,1[$, the choices $y=0$ and $y=S$ lead to infinite values of the social cost. Thus, we can restrict the search to $y\in]0,1[$. By differentiating Eq. (5), we get for $y\in]0,1[$:

$$\frac{d\widetilde{SC}}{dy}(y)=-\frac{x^2}{2gy^2}+\frac{(1-x)^2}{(1-y)^2}.$$

Thus, $\frac{d\widetilde{SC}}{dy}(y^o)=0$ is equivalent to:

$$\sqrt{2g}\frac{1-x}{1-y^o}=\frac{x}{y^o}, \tag{7}$$

so that we obtain Eq. (6) for $x\in]0,1[$.

It is trivial to show that this solution is interior for $x\in]0,1[$ and that $\frac{d^2\widetilde{SC}}{dy^2}(y)$ is strictly positive for $y\in]0,1[$. Therefore, the social cost reaches its global minimum for $y=y^o(x)$.

Finally, the function y^o is clearly continuous on $]0,1[$ while:

$$\lim_{x\to 0^+}(y^o(x))=0=y^o(0)\ ;\ \lim_{x\to 1^-}(y^o(x))=1=y^o(1).$$

Hence, y^o is continuous on $[0,1]$. □

As highlighted in Proposition 1, Eq. (7) has the intuitive interpretation that the cooperative route is used $\sqrt{2g}$ times longer than the independent route. Since the total cooperation cost is a constant, the social optimum simply minimizes congestion costs and as routes are always either used at capacity or not used at all, the duration a route is used is simply the demand to capacity ratio. It is natural that the social optimal requires a higher ratio on the cooperative route as there is no queueing, only schedule penalties.

2.6 No cooperation cost

The cooperation cost introduced in this work has important consequences on the user equilibrium and social optimum. In order to better highlight the need for such a cost, we first address such a system with a cooperative subnetwork without cooperation cost ($\kappa=0$). This intermediate step is also useful for comparison purposes and gives a quick overview of the problem with few calculations.

2.6.1 Social optimum

The objective of this first sub-section is to determine the smallest social cost that can be achieved, when only the demand split or both the capacity and demand splits can be set optimally. Besides their theoretical interest, these two problems also correspond to practical situations. On one hand, at the tactical level, transportation authorities often consider the capacities of different facilities as given and seek to optimize the demand only. On the other hand, the strategic level involves not only demand management but also longer term decisions such as the construction/reconversion of new/existing roads. Thus, these two mathematical problems are representative of real world challenges, which make the insights derived here of practical importance.

2.6.1.1 *Optimal demand split with fixed capacities*

With fixed capacities, the social cost is only a function of the demand split (and of the exogenous factors g and r). For $y = 0$ (resp. $y = 1$), the socially optimal demand split $x^o(y)$ is trivially $x^o = 0$ (resp. $x = 1$).

Consider now $y \in]0, 1[$. By differentiating Eq. (5), we get that the optimal demand split x^o should satisfy $\frac{d\widetilde{SC}}{dx}(x^o) = 0$, i.e.

$$\frac{x^o}{gy} - \frac{2(1-x^o)}{1-y} = 0, \tag{8}$$

or

$$x^o = \frac{2gy}{1+(2g-1)y}. \tag{9}$$

Note that Eq. (8) imposes that the ratio of demand to capacity (or equivalently, the duration the route is used) is twice as big for the cooperative route as for the independent one. This is consistent with the classical results of Arnott et al. (1990b), who showed that queueing time represents exactly half of the social cost at equilibrium when all users are perfectly identical. Cooperation removes queuing and as by definition marginal costs must be equal on both routes at the social optimum, the cooperative route should be used twice as long as the independent one.

2.6.1.2 *Socially optimal demand and capacity splits*

Instead of considering $\widetilde{SC}$ as a function of two variables, we can take advantage of one of the closed-form expressions giving the socially optimal value of one split as a function of the other. Let us consider for instance the function $\widetilde{SC}(y)$ that associates to a capacity split its social cost assuming a socially optimal demand split (see Eq. 5):

$$\widetilde{SC}(y) = \frac{(x^o)^2}{2gy} + \frac{(1-x^o)^2}{1-y}.$$

By using Eq. (8):

$$\widetilde{SC}(y) = \frac{(x^o)^2}{2gy} + (1-x^o)\frac{x^o}{2gy} = \frac{x^o}{2gy}.$$

Finally, by using the expression of x^o from Eq. (9):

$$\widetilde{SC}(y) = \frac{1}{1+(2g-1)y}.$$

For $g > \frac{1}{2}$, $\widetilde{SC}(y)$ is a decreasing function of the capacity split y. Hence, the social cost is minimized for $y = 1$, i.e. when cooperation is the only alternative (and then $x^o = 1$).

2.6.2 User equilibrium

Proposition 2. *For any given capacity split, the socially optimal demand split and the equilibrium demand split are identical if there is no cooperation cost.*[4]

Proof. At equilibrium, no user can reduce her cost by changing her decision. If only one network is used, the cost on this network is necessarily positive while the other network has zero cost for users with no cooperation cost (i.e. as assumed in this section). Hence, at equilibrium both sub-networks should be used and the equilibrium demand split N_c^e should satisfy the following condition:

$$c_c = c_i \Leftrightarrow \frac{x(y)}{2gy} = \frac{1 - x(y)}{1 - y}, \tag{10}$$

which is also the equation of the social optimum (Eq. (8)). □

Thus, without a cooperation cost, both the social optimum and the user equilibrium lead to a system that is entirely cooperative, with no alternative (see Section 2.6.1.2). Of course, such a system would be unacceptable for the users who wish to keep the control of their departure time. We will see in Section 3 how these results are affected by the introduction of a cooperation cost.

2.7 Specific distributions and numerical values

An effort was made throughout this work to keep a general scope and assumptions about specific distributions or numerical values were avoided when appropriate. However, some analytical expressions and the graphical illustrations require assumptions. When necessary (and it is always mentioned), a uniform distribution is assumed for the cooperation cost. Some numerical applications are also replicated with a log-normal or an exponential distribution with the same expected value.

Based on Eq. 5, the results depend only on dimensionless relative quantities that are used in all graphical illustrations. However, some intuition about reasonable values of $r = \frac{\delta N}{\kappa S}$ is critical to assess the scale of the benefits that should be expected. The numerical evaluation of the reference individual cost $\frac{\delta N}{S}$ is relatively common. Typical values of earliness ($\beta = 0.5\alpha$) and lateness ($\gamma = 2\alpha$) (Small, 1982) lead to $\delta = \frac{\beta\gamma}{\beta+\gamma} = 0.4\alpha$, where α is the value of time at home while $\frac{N}{S}$ is simply the length of the peak period ($\sim$ 2h for instance).

Concerning the scale κ of the cooperation cost, mode choice models can suggest an educated guess since most of them also require that users plan their trips in advance (e.g. carpooling, buses with long

[4] Arnott et al. (1990a) found the same result for two simple routes in parallel (with non-cooperative users).

headways, trains, and planes). Bhat (1995) for instance proposed different models for mode choice between cities, including a multinomial logit. The ratio of the mode-specific parameter and of the parameter associated to in-vehicle travel time leads to the following estimates: the train has a mode-specific cost that is approximately equal to the cost of 51 min of travel time and the mode-specific cost of taking the plane is about 62 min (compared to the car). However, this intrinsic utility does not only account for the cooperation cost but also for other characteristics of the mode which penalize public transit (e.g. comfort). Thus, the cooperation cost would most likely have a smaller value if personal vehicles were used, say around 30 min. If the time unit is 1h, then the average value of $\kappa\theta$ should be around 0.5α so with a uniform distribution on $[0,1]$, this leads to $\kappa = \alpha$. Altogether, these estimates lead to the best guess $r = \frac{\delta N}{\kappa S} \sim 0.8$.

3 Social optimum

Similarly to what was done in Section 2.5 for the case $\kappa = 0$, the aim of this section is to determine the configuration that leads to the smallest possible social cost when (i) only the demand split varies, and (ii) when the capacity split varies as well. As explained in Section 2.5.1 (Proposition 1), the case with fixed demand split does not depend on the cooperation cost and thus does not need to be addressed again.

3.1 Optimal demand split with fixed capacities

In this part, the relative social cost is seen as a function of the demand split only since the capacity split is fixed. However, since the cooperation cost plays a crucial role in this section, we will consider $\widetilde{SC}$ as a function of $\hat{\theta}$ rather than x.

Before studying potential optimal demand splits, we state the following technical lemma, whose proof is given in A and that will be useful for different propositions.

Lemma 1. *Let $G(\hat{\theta}) = \hat{\theta} + aF(\hat{\theta}) - b$, where $a, b > 0$ and F denotes the cumulative distribution function (cdf) of the cooperation cost. Then $G(\hat{\theta}) = 0$ has a unique interior solution denoted by $\hat{\theta}^{sol}$ if and only if $\bar{\theta} + a - b > 0$. If the probability density function f is continuous, then $\hat{\theta}^{sol}$ is a locally continuous and differentiable function of a and b, decreasing with a and increasing with b.*

Proposition 3. *For a given capacity split and for a cooperation cost verifying Condition 1, there exists a unique demand split that minimizes the social cost. If $y \in]0,1[$, then the optimal demand split is interior and is the unique solution of*

$$\frac{1}{r}\hat{\theta}^o + \frac{1+(2g-1)y}{gy(1-y)}x(\hat{\theta}^o) - \frac{2}{1-y} = 0. \tag{11}$$

In addition, $\hat{\theta}^o$ is a continuous function of the automation factor (for $g \in [\frac{1}{2}, +\infty[$), the capacity split (for $y \in [0,1]$) and the cost ratio (for $r \in]0, +\infty[$), increasing with g and y. It increases (resp. decreases) with r if $\hat{\theta}^o > 0$ (resp. $\hat{\theta}^o < 0$).

Proof. If $y \in \{0, 1\}$, there is only one demand split that yields a finite social cost, so the result is trivial. Let us now consider $y \in]0,1[$. By differentiating Eq. (5): $\forall \hat{\theta} \in \Theta$,

$$\begin{aligned}\frac{d\widetilde{SC}}{d\hat{\theta}}(\hat{\theta}) &= \frac{1}{r}\hat{\theta} f\left(\hat{\theta}\right) + \frac{x\left(\hat{\theta}\right)}{gy} f\left(\hat{\theta}\right) - \frac{2(1-x\left(\hat{\theta}\right))}{1-y} f\left(\hat{\theta}\right) \\ &= \frac{1}{r} f\left(\hat{\theta}\right) G(\hat{\theta}),\end{aligned}$$

where we define:$G(\hat{\theta}) \triangleq \hat{\theta} + r\left(\frac{1}{gy} + \frac{2}{1-y}\right) x\left(\hat{\theta}\right) - \frac{2r}{1-y}$. G has the form of the function required by Lemma 1 with:

$$\bar{\theta} + a - b = \bar{\theta} + r\left(\frac{1}{gy} + \frac{2}{1-y}\right) - \frac{2r}{1-y} = \bar{\theta} + \frac{r}{gy} > 0.$$

Thus, there exists a unique demand split $\hat{\theta}^o \in]\underline{\theta}, \bar{\theta}[$ satisfying $G(\hat{\theta}^o) = 0$, i.e. satisfying Eq. (11).

Finally, since $\frac{f(\hat{\theta})}{r} > 0$, $\forall \hat{\theta} \in S_\theta$, $\frac{d\widetilde{SC}}{d\hat{\theta}}(\hat{\theta})$ has the same sign as $G(\hat{\theta})$, i.e. $\frac{d\widetilde{SC}}{d\hat{\theta}}(\hat{\theta}) \gtreqless 0$ for $\theta \gtreqless \hat{\theta}^o$ so $\widetilde{SC}$ reaches its global minimum for $\hat{\theta} = \hat{\theta}^o$.

The second part of Lemma 1 implies that $\hat{\theta}^o$ is locally continuous and increases with g. Since this is valid for every $g \in [\frac{1}{2}, +\infty[$, $\hat{\theta}^o$ is continuous and increasing everywhere. Similarly, by the implicit function theorem, $\hat{\theta}^o$ is continuous and increases (resp. decreases) with r on $]0, +\infty[$ if $\hat{\theta}^o > 0$ (resp. $\hat{\theta}^o < 0$) and is continuous and increases with y on $]0,1[$. To generalize this last result to the close interval $y \in [0,1]$, note first that $\hat{\theta}^o = \underline{\theta}$ for $y = 0$ and $\hat{\theta}^o = \bar{\theta}$ for $y = 1$. Then Eq. (11) can be rewritten as:$\frac{x(\hat{\theta}^o)}{gy(1-y)} = -\frac{\hat{\theta}^o}{r} - \frac{2g-1}{g(1-y)}x(\hat{\theta}^o) + \frac{2}{1-y}$. As the left-hand term is clearly positive, so must be the right-hand term. In addition, the right-hand term is bounded above by $-\frac{\underline{\theta}}{r} + \frac{2}{1-y}$. Consequently, $x(\hat{\theta}^o)$ is non-negative and bounded above by $gy(1-y)\left[-\frac{\underline{\theta}}{r} + \frac{2}{1-y}\right]$, and thus converges toward 0 when y tends towards 0, which ensures continuity in 0. Similarly, Eq. (11) can also be rewritten as $1 - y = \frac{r}{\hat{\theta}^o}\left[2 - 2x(\hat{\theta}^o) - \frac{1-y}{gy}x(\hat{\theta}^o)\right]$. The left-hand term clearly converges towards 0 when y tends towards 1, so the right-hand term must do so as well. The right-hand term however is the product of two terms that are related: if the first is small (i.e. $\hat{\theta}^o$ is big), then y must be close to 1 so the second term must be small as well, and vice versa. Thus it is trivial to show that both terms tend towards 0, or, equivalently, that y converges towards 1. □

Note that this is consistent with the results of Section 2.6.1.1 as Eq. (11) reduces to Eq. (9) with no cooperation cost.

3.1.1 Special case: uniform distribution

In order to find a closed-form expression of $\hat{\theta}^o$, we assume a uniform distribution of the type θ: $f(\theta) = 1$ if $\theta \in [0, 1]$ and $f(\theta) = 0$ elsewhere. Hence, $\forall \hat{\theta} \in [0, 1], x = \hat{\theta}$ and Eq. (11) has an explicit solution given by $gy(1-y)\hat{\theta}^o + r(1 + (2g-1)y)\hat{\theta}^o = 2rgy$, or

$$\hat{\theta}^o = \frac{2rgy}{gy(1-y) + r(1 + (2g-1)y)}. \tag{12}$$

It can be easily verified that this solution is interior, i.e. that $\forall y \in]0, 1[, \hat{\theta}^o \in]0, 1[$. The expression of $\hat{\theta}$ given by Eq. (12) is plotted for different values of r in Fig. 3a. In agreement with Proposition 3, $\hat{\theta}^o$ increases with y, regardless of the value of r, and with r, regardless of y.

Note that since $x = \hat{\theta}$, dividing Eq. (12) by the capacity split y yields the demand to capacity ratio for the cooperative route, i.e. the duration this route is used (relatively to the reference scenario). Similarly, the duration the independent route is used is given by:

$$\frac{1-x}{1-y} = \frac{1-\hat{\theta}^o}{1-y} = \frac{gy + r}{gy(1-y) + r(1 + (2g-1)y)}.$$

The optimal demand split is illustrated as a function of the capacity split y for different cost ratios r in Fig. 3a. As argued in Proposition 3, the optimal demand split increases with both r and y. Note that besides the very specific cases where the network is entirely allocated to one route ($x = 0$ and $x = 1$), the curves for different cost ratios r are quite different: those associated to small values of r have a S shape while those associated to large values of r appear to be concave. Nevertheless, all curves are similar for the range of small capacity splits y, which is a natural consequence of the limited number of cooperative users. Indeed, as only the users with almost no cooperation cost are cooperative when the capacity split y is small, such situations can all be approximated by the case with no cooperation studied in Section 2.6.1.1. Hence, as the independent users have a demand to capacity ratio close to 1, the demand to capacity ratio of the cooperative user (i.e. the slopes of the curves here) must tend towards 2 as the capacity split y tends towards 0, regardless of the cost ratio r.

The relative durations are represented in Figs. 3b. While Fig. 3a highlighted the similarity between all cases and the case with no cooperation for small capacity splits, Fig. 3b shows that this similarity also exists for larger capacity splits, but only for large values of the cost ratio r, i.e. when cooperation is relatively less costly. Indeed, for $r = 2$, the cooperative sub-route is used approximately twice longer than the independent sub-route, regardless of the capacity split. However, for other values of r, the ratio between these two durations is highly variable, which suggests that the network might not be optimally used.

In order to assess the social impact of the scheme proposed, the relative social cost obtained for the optimal demand split is plotted in Fig. 3c for different values of r. Fig. 3c shows that even with an

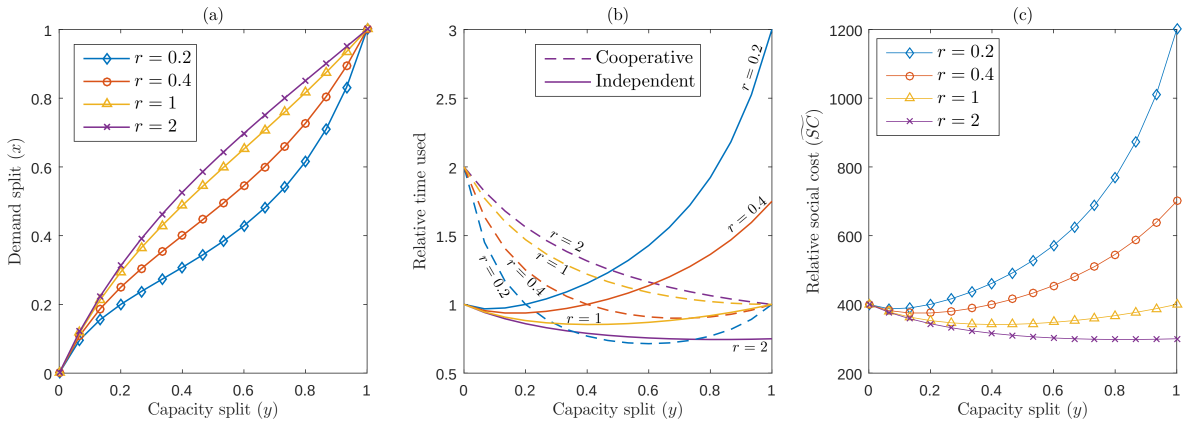

Figure 3: Graphical representations with a uniform distribution of the cooperation cost of (a) the critical cooperation cost, (b) the relative durations each sub-route is used and (c) the relative Social Cost as functions of the capacity ratio for different values of r in the social optimum case ($g = 1$).

optimal demand split, some values of the capacity split may lead to a social cost that is much higher than the social cost in the reference case.

3.2 Optimal demand and capacity splits

We now take a longer-term perspective and look for the pair of demand and capacity splits that minimizes the social cost. Unlike in the case with no cooperation cost, there is no general closed-form relationship between a capacity split and the socially optimal demand split. However, we will build on the relationship defined in Eq. (6), that relates a demand split to the corresponding socially optimal capacity split. Thus, we consider here the function $\widetilde{SC}\left(\hat{\theta}\right) \triangleq \widetilde{SC}\left(\hat{\theta}, y^o\left(x\left(\hat{\theta}\right)\right)\right)$, where we use the same notation with a slight abuse of notation. Before looking for potential minima, we first show the continuity of this function.

Lemma 2. *The relative social cost with a socially optimal capacity split $\widetilde{SC}\left(\hat{\theta}\right)$ is a continuous function on $[\underline{\theta}, \bar{\theta}]$ and, with $x = x\left(\hat{\theta}\right)$,*

$$\widetilde{SC}(\hat{\theta}) = \frac{1}{r}\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{\left(\sqrt{2g}(1-x)+x\right)^2}{2g}. \tag{13}$$

Proposition 4. *If the cooperation cost satisfies Condition 1, there exists a unique pair of demand and capacity splits that minimizes the social cost. This solution is interior if and only if $r < \bar{\theta}\frac{g}{\sqrt{2g}-1}$. In this case, the demand split $x\left(\hat{\theta}\right)$ is given implicitly by:*

$$\frac{1}{r}\hat{\theta} + \frac{\left(\sqrt{2g}-1\right)^2}{g}x\left(\hat{\theta}\right) - \left(\sqrt{2g}-1\right)\sqrt{\frac{2}{g}} = 0. \tag{14}$$

Both the demand split x and capacity split y increase with the cost ratio r and with the automation factor g. If $r \geq \bar{\theta}\frac{g}{\sqrt{2g}-1}$, at the social optimum all vehicles should cooperate.

Thus, there is an interior solution if and only if the maximum cooperation cost within the population ($\kappa\bar{\theta}$) is more than $\frac{\sqrt{2g}-1}{g}$ times the individual cost when all users are independent. Since $\frac{\sqrt{2g}-1}{g} \in]0, 0.5]$ for $g > 0.5$, the following sufficient condition holds:

Corollary 1. *If the maximum cooperation cost within the population is greater than half the individual cost with no cooperation ($\kappa\hat{\theta} > \frac{\delta N}{2S}$), the optimal demand and capacity splits are interior.*

The assumption included in Condition 1 that the closure of the support of the pdf of the cooperation cost includes 0 is critical here. Thanks to this assumption, allocating at least some part of the capacity to cooperative users is always socially beneficial. Else, allocating capacity to cooperative users would be detrimental unless there is at least some minimum level of congestion, which would slightly complicate the derivations and properties. However, as for homogeneous independent users waiting time represents in average half the individual congestion cost, it suffices that some users have a cooperation cost smaller than half the individual congestion cost without cooperation to ensure that the optimum features a cooperative route.

3.2.1 Special case: uniform distribution

Proposition 5. *If the cooperation cost is uniformly distributed on $[0, 1]$ and if $r < \bar{\theta}\frac{g}{\sqrt{2g}-1}$, then the socially optimal demand and capacity splits are given by:*

$$\hat{\theta}^o = \frac{r\sqrt{2g}\left(\sqrt{2g}-1\right)}{g + r\left(\sqrt{2g}-1\right)^2} \tag{15a}$$

$$y^o = r\frac{\sqrt{2g}-1}{g}. \tag{15b}$$

Proof. In the case of a uniform distribution and assuming that the optimum is interior (i.e. $\bar{\theta} > r\frac{\sqrt{2g}-1}{g}$), the optimality condition is $H(\hat{\theta}^o) = 0$, so

$$\hat{\theta} + \frac{r}{g}\left(\left(\sqrt{2g}-1\right)^2\hat{\theta} - \left(2g - \sqrt{2g}\right)\right) = 0,$$

which is equivalent to Eq. (15a). Then, with a uniform distribution Eq. (6) can be rewritten:

$$y^o = \frac{\hat{\theta}^o}{\sqrt{2g} - (\sqrt{2g}-1)\hat{\theta}^o},$$

and by using Eq. (15a) for $\hat{\theta}^o$, we obtain the required expression Eq. (15b). $\square$

Note that the absolute value of the capacity that is reserved to cooperative users is equal to $\frac{\delta N}{\kappa}\frac{\sqrt{2g}-1}{g}$ and does not depend on S. Alternatively, the criterion for the existence of an interior solution in Proposition 4 can be rewritten as $S > \frac{\delta N}{\kappa\theta}\frac{\sqrt{2g}-1}{g}$, i.e either the total capacity is too small and 100% of the capacity is reserved for cooperative users, or the total capacity is big enough and the capacity that is reserved for cooperative users does not depend on S. This surprising result is the outcome of two processes canceling out: as the total capacity increases, the proportion of the capacity that is reserved to cooperative users y decreases (the total peak hour is shorter, so users have less reasons to bear the cooperation cost). With a uniform distribution, the product of the two processes is constant but numerical applications with other distributions of the cooperation cost show that this result should not be expected in general.

The relative social cost also has a rather simple expression for a uniform distribution of the cooperation cost on $[0,1]$. Indeed, Eq. (13) becomes:

$$\widetilde{SC}\left(\hat{\theta}\right) = \frac{\hat{\theta}^2}{2r} + \frac{\left(\sqrt{2g}\left(1-\hat{\theta}\right)+\hat{\theta}\right)^2}{2g}.$$

Then, by combining it with the expression of the optimal demand split in Eq. (15a), one obtains the expression of the optimal social cost:

$$\widetilde{SC}^o\left(\hat{\theta}^o\right) = \frac{g}{g + r\left(\sqrt{2g}-1\right)^2}. \tag{16}$$

The socially optimal demand and capacity splits obtained with the Eqs. (15a) and (15b) are plotted as functions of the cost ratio r in Fig. 4a, together with the relative social cost and the ratio $\frac{1-\widetilde{SC}^o}{1-\widetilde{SC}^{min}}\frac{1}{y}$, where $\widetilde{SC}^{min}$ is the minimum relative social cost that can be obtained with such a capacity (with no cooperation cost) and $\widetilde{SC}^o$ is given by Eq. (16). As g is the only exogenous parameter here, this graph has a very general scope. It shows that for the entire spectrum of possible states, the socially optimal capacity split y and the demand split x are relatively close, although in most cases the cooperative users have a demand to capacity ratio that is slightly bigger than on the independent route ($y < x$). Note that independent users are never worse off in such conditions. Note as well that the relative social cost is a decreasing function of r and although it is not visible here, it converges to 0.5 when r tends towards infinity [5]. This is conform to intuition since for given δ, N, and S, reducing the cooperation cost (κ) reduces the cost of cooperative solutions. Finally, the ratio $\frac{1-\widetilde{SC}^o}{1-\widetilde{SC}^{min}}\frac{1}{y}$ is a good indicator of performance of the system proposed. $1-\widetilde{SC}^{min}$ represents the ideal gain that could be obtained if there was no cooperation cost and all the capacity was used. Thus, the ratio $\frac{1-\widetilde{SC}^o}{1-\widetilde{SC}^{min}}$ represents how close the system gets from this ideal gain and to be fair, it is divided by the capacity split y that is used to obtain this gain. Hence, this ratio allows us to answer to the following question: if the optimal capacity split is n% of S, is the gain in terms of social cost more or less than n% of the ideal maximum

[5] This is the classic result for Vickrey's bottleneck model that can be obtained without cooperation cost and with a fine toll on all the network - cf Arnott et al. (1990b).

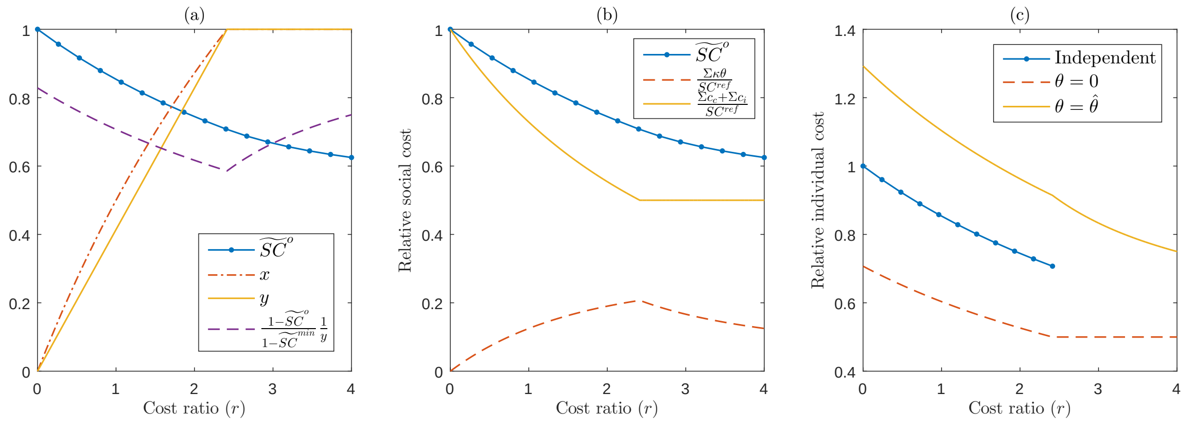

Figure 4: Representations as functions of the cost ratio r for the case of a uniform distribution with $g = 1$ of: (a) the socially optimal demand and capacity splits as well as the relative social cost and the ratio $\frac{1-\widetilde{SC}^o}{1-\widetilde{SC}^{min}}\frac{1}{y}$, where SC^{min} is the relative minimum possible social cost, obtained without any cooperation cost (b) the decomposition of the relative social cost obtained for the social optimum into strict congestion costs and costs of cooperation and (c) the relative individual cost of independent users and the minimum and maximum relative individual costs of cooperative users.

gain with no cooperation cost? Because of the cooperation cost, the gain is smaller than $n\%$, but graphically, it is never smaller than approximately $0.6n\%$.

Second, the different components of the relative social cost are displayed in Fig. 4b. Note that for small values of r, the part of the cost that is not due to cooperation (i.e. the schedule penalty and the waiting time: $\frac{\sum c_c+\sum c_i}{SC^{ref}}$) is reduced almost twice as much as the total social cost $\widetilde{SC}^o$. Since there are external costs that are related only to the presence of vehicles on the road (e.g. pollution and travel time variability) that are not taken into account here, this indicates that the real gain in social cost will be even higher in real conditions.

Finally, Fig. 4c shows the relative individual costs of independent users and of the extreme cooperative users ($\underline{\theta} = 0$ and $\hat{\theta}$). As expected, these costs decrease with r. More remarkably, the relative individual cost for some cooperative users is greater than 1 for small values of r, i.e. the individual cost for these users is bigger than in the reference case. In such a situation, some cooperative users are "sacrificed" to reach the social optimum. This statement is confirmed by the analysis of the cost for independent users (identical for all of them). One can see that it is always smaller than in the reference case and that it is decreasing with r, as the number of cooperative users being "sacrificed" increases (cf. Fig. 4a) (although their individual sacrifices are reduced). Note that this "sacrifice" is somehow justified as for realistic values of r (close to 1), the maximum cost is only 15% higher than in the reference case, while the minimum cost is 65% of the reference case. Nevertheless, as described

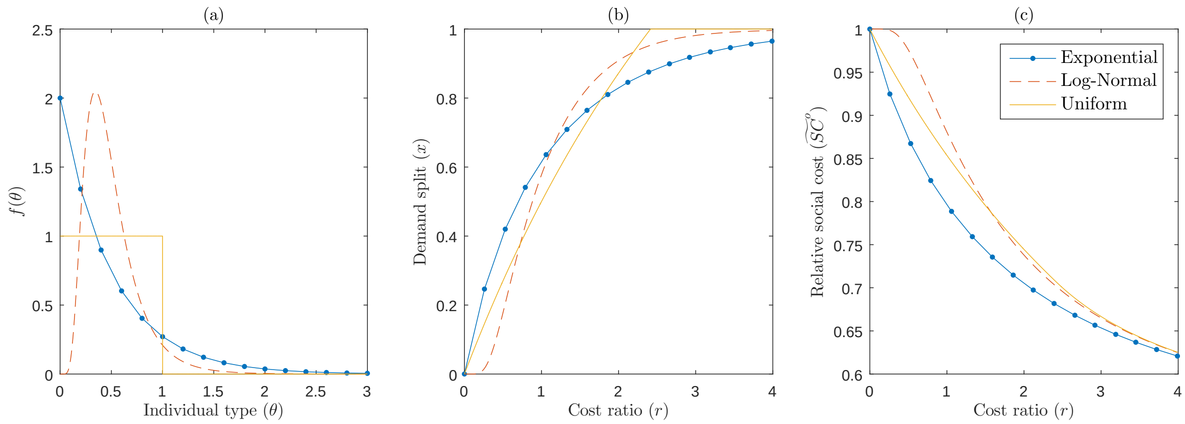

Figure 5: Comparison with socially optimal demand and capacity splits of the (a) probability density functions, (b) the demand split and (c) the relative social cost, for a uniform, an exponential and a log-normal distribution of the cooperation cost ($g = 1$).

in section 4.3, it is also possible to greatly reduce the social cost (although not as much) without sacrificing anyone.

3.2.2 Numerical application with other distributions

In order to obtain a closed-form expression of the demand split, it was assumed in the previous part that the cooperation cost was uniformly distributed. Nevertheless, common sense suggests that for any given cost, one can always find a user with a higher cooperation cost and that a more realistic distribution should have a long tail. Hence, two additional distributions with supports $[0, +\infty[$ are considered: the exponential and log-normal distributions. To allow for a fair comparison, the parameters of the log-normal distribution were set to $\mu = \ln(\frac{1}{2}) - \frac{1}{8}$ and $\sigma = 0.5$ (μ and σ are the expected value and standard deviation of the variable's natural logarithm) while the rate parameter λ of the exponential distribution was set to 2. Thus, the expected value of the type θ is equal to $\frac{1}{2}$ with both distributions, as for the uniform distribution. These distributions are represented in Fig. 5a. The demand split and the relative social cost were numerically evaluated with these distributions and the results are plotted in Fig. 5b and Fig. 5c, together with the uniform distribution case.

The analysis of Fig. 5 suggests that for our best guess estimate of the cost ratio ($r \sim 0.8$), the gains in terms of social cost depends on the exact distribution of the cooperation cost. While all the distributions used have the same expected value, the social cost reductions obtained vary from -8% to -17%, with the log-normal and exponential distributions respectively. Intuitively, only the cooperation costs of users who are likely to be cooperative do impact the benefits that can be obtained by allowing cooperation. Thus, a higher proportion of users with a low cooperation cost allows further

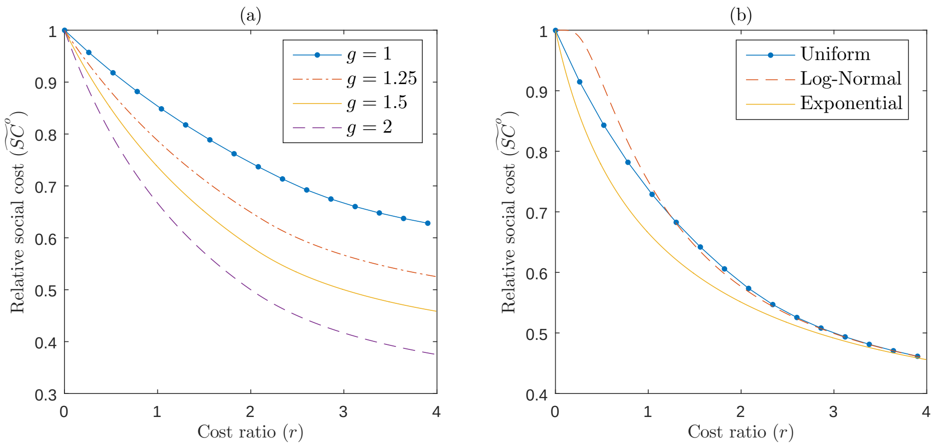

Figure 6: Comparison of the relative social cost obtained with socially optimal demand and capacity splits with (a) different values of g and with a uniform distribution, (b) $g = 1.5$ but with a uniform, a log-normal and an exponential distribution.

reductions of the social cost while users with a very high cooperation cost only have an impact for values of r that are unreasonably high (based on the considerations given in Section 2.7).

3.2.3 Impact of automation

In order to measure the impact of a more efficient use of the roadway by automation, the relative social cost with socially optimal capacity and demand splits was represented in Fig. 6a for different values of g (based on the literature reviewed in Section 2.1). This relative social cost is also represented in Fig. 6b for different distributions of the cooperation cost for $g = 1.5$. As expected, a more efficient use of the roadway by cooperative vehicles allows for further reductions in the social cost and this gain naturally increases with r, as low values of r imply a relatively large cooperation cost, which restricts the use of the roads benefiting from higher capacity. These figures show that overall a higher value of g has positive effects, and as it increases the proportion of users that are cooperative at equilibrium, it also tends to reduce the impact of the distribution chosen for the cooperation cost (as long as all distributions have the same expected value).

4 User equilibrium

In this section we show that many of the results obtained for the system optimum case, have similar properties under user equilibrium conditions. We first investigate (Section 4.1) the properties of the user equilibrium when the capacities for each group of users are set externally. Additionally, in Section 4.2 we derive the optimal tolling for the above case to reach System Optimum conditions. The findings of Section 4.3 are significant for the applicability of the general scheduling framework described in this paper. We show that unlike the social optimum, the user equilibrium for the optimal capacity split is Pareto-improving. Finally, in Section 4.4, we demonstrate that the price of anarchy is relatively small for different values of automation factor g.

As explained in the proof of Proposition 2, both sub-networks should be used in user equilibrium as long as some users have no cooperation cost (i.e. some users have $\theta = 0$, which has been assumed to be the case in Condition 1). Hence, the critical cooperation cost at equilibrium $\hat{\theta}(y)$ should satisfy the user equilibrium condition:

$$\kappa\hat{\theta}(y) + c_c = c_i + \tau, \tag{17}$$

where τ is the toll on the independent route (if the toll is on the cooperative route, then τ is simply negative).

4.1 User equilibrium with given capacities

Proposition 6. *For a given capacity split and with no toll, a demand split satisfies the user equilibrium equation* (17) *with a unit cooperation cost* $\kappa = K$ *if and only if it satisfies the social optimum equation* (11) *with a unit cooperation cost* $\kappa = 2K$*.*

Proof. Eq. (17) is equivalent to

$$\kappa\hat{\theta}(y) + \frac{\delta N}{S}\frac{x\left(\hat{\theta}(y)\right)}{2gy} = \frac{\delta N}{S}\frac{\left(1 - x\left(\hat{\theta}(y)\right)\right)}{1-y},$$

or

$$\frac{\delta N}{S}\left(\frac{1}{2gy} + \frac{1}{1-y}\right)x\left(\hat{\theta}(y)\right) = \frac{\delta N}{S}\frac{1}{1-x} - \kappa\hat{\theta}(y),$$

or

$$r\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)x\left(\hat{\theta}(y)\right) = r\frac{1}{1-y} - \hat{\theta}(y). \tag{18}$$

It suffices now to note that Eq. (18) with $\kappa = K$ is identical to the social optimum equation (11) for $\kappa = 2K$. □

A consequence of Proposition 6 is that most of the results obtained for the social optimum remains valid after multiplying κ by two. In particular:

Corollary 2. *For a given capacity split and with no toll, there exists a unique demand split satisfying the user equilibrium equation and it is a continuous function of the automation factor* g *(on* $[\frac{1}{2}, +\infty[$*), the cost ratio* r *(on* $]0, +\infty[$*), and of the capacity split* y *(on* $[0, 1]$*).*

In addition, one just has to notice that for a given capacity split, $\hat{\theta}^o$ is a decreasing function of κ to deduce from Proposition 6 this additional corollary:

Corollary 3. *For any interior capacity split, the social welfare can be improved by setting a positive toll on the independent route.*

If there is a non-negative toll, it can be proven that there still exists a unique demand split satisfying the user equilibrium condition but in this case, the solution is not necessarily interior. The proof is left to the reader as it is similar to the proof of Proposition 3.

4.1.1 Special case: uniform distribution

When the cooperation cost is uniformly distributed, Eq. (18) reduces to

$$[2gy(1-y) + r(1 + (2g-1)y)]\,\hat{\theta}(y) = 2rgy,$$

which has an explicit solution given by:

$$\hat{\theta}(y) = \frac{2rgy}{2gy(1-y) + r\,(1 + (2g-1)y)}. \tag{19}$$

The resulting social cost is studied in more details in Section 5.1.3, together with other situations involving private operators.

4.2 Decentralization of the social optimum

The objective of this part is to determine, given some capacity split, how the sub-networks should be tolled to shift the user equilibrium to socially optimal conditions. Corollary 3 implies that the independent network should be tolled and Proposition 6 can be used to determine the exact amount without additional calculations.

Indeed, if one solves the social optimum problem to find $\hat{\theta}^o$ and then sets the toll to $\tau = \frac{1}{2}\kappa\hat{\theta}^o$, Eq. (17) becomes:

$$\kappa\left(\hat{\theta}(y) - \frac{1}{2}\hat{\theta}^o\right) + c_c = c_i.$$

Proposition 6 implies that $\hat{\theta}^o$ is a solution and since the solution is unique, $\hat{\theta}(y) = \hat{\theta}^o$, i.e. the equilibrium is the social optimum.

Intuitively, this toll forces some naturally independent users that are close to being cooperative to become cooperative. Hence, it is natural that it should increase with the cooperation cost of these users ($\kappa\hat{\theta}^o$).

It is of practical interest to notice that this toll is only paid by independent users, is time-independent and is relatively small. A numerical application showed that for the range of κ considered, this toll is equal to approximately half the average toll required by Vickrey's time-dependent tolling strategy. Thus, the user acceptability of such a pricing strategy in a possible implementation is expected to be higher, as it may potentially resolve many of the issues highlighted in the introduction of the paper, that classical pricing schemes carry.

4.3 Optimal capacity split and Pareto-improvement

Even though Proposition 6 shows the existence of a strong relationship between the user equilibrium and the social optimum, the properties of these two situations are fundamentally different. In particular, while the user equilibrium assumes purely selfish users, we showed in Section 3.2 that the social optimum required the "sacrifice" of some cooperative users. We demonstrate in this section that unlike the social optimum, the user equilibrium is Pareto-improving for the socially optimal capacity split. This result is obtained without actually determining the socially optimal capacity split under user equilibrium as the calculations involved are particularly tedious even with simplistic assumptions[6].

Before actually demonstrating this result with Propositions 7 and 8, let us formulate the following lemma.

Lemma 3. *Assume that the distribution of the cooperation cost satisfies condition 1. Let y denote a capacity split and $x(y)$ denote the associated demand split at user equilibrium with no toll. Let us also define $y_e = F((1 - \frac{1}{2g})r)$.*

If $y_e < 1$, then $x(y) > y$ for all $y \in]0, y_e[$, $x(y) < y$ for all $y \in]y_e, 1[$ and $x(y) = y$ for $y \in \{0, y_e, 1\}$.

If $y_e \geq 1$, then $x(y) > y$ for all $y \in]0, 1[$ and $x(y) = y$ for $y \in \{0, 1\}$.

Lemma 3 states that the demand split is bigger than the capacity split if and only if the capacity split is smaller than some constant y_e. We argue below that this condition is also necessary and sufficient to have a Pareto improvement.

Proposition 7. *For any distribution of the cooperation cost verifying condition 1, the user equilibrium with no toll Pareto-dominates the user equilibrium with no cooperation if and only if the capacity split y is in the interval $]0, min(1, y_e)]$.*

Lemma 3 and the proof provided in A provides some intuition about this proposition. Intuitively, even though there may be a higher demand to capacity ratio on the cooperative route, at equilibrium cooperative users must always have a cost that is smaller than independent users (otherwise they would be independent). Thus, we should simply ensure that independent users are better-off, which requires that they have a demand to capacity ratio that is smaller than in the reference scenario, which, according to Lemma 3, happens if and only if $y \leq y_e$ (and $y > 0$).

Pareto-improvements are extremely important from a political point of view. While socially optimal policies should theoretically be sought, it happens that sub-optimal measures are implemented instead, simply because they are Pareto-improving, which is not necessarily the case of socially optimal

[6]Assuming a uniformly distributed cooperation cost, the differentiation of the function associating to a capacity split its social cost under user equilibrium leads to a rational function whose numerator is a 4^{th} order polynomial.

measures. In the situation at hand however, we argue below that the socially optimal capacity split exists and Pareto-dominates the user equilibrium with no cooperation.

Proposition 8. *For any distribution of the cooperation cost verifying condition 1, there exists a capacity split that minimizes the social cost under user equilibrium with no toll. The associated user equilibrium Pareto-dominates the user equilibrium with no cooperation.*

In other words, Proposition 8 states that the socially optimal capacity split exists and cannot be greater than y_e, i.e. that "sacrificing" the users with a high cooperation cost to reduce the travel time of more flexible users is overall detrimental. Note however that if we introduce a new source of heterogeneity by considering that users have different values of time (but the same relative value of earliness $\frac{\beta}{\alpha}$ and lateness $\frac{\gamma}{\alpha}$) and that the time of flexible users is more valuable (shift-workers usually have lower wages), Proposition 8 might not stand anymore. This is considered as a future research direction.

4.4 Price of anarchy

As explained in Section 4.3, the optimal capacity split under user equilibrium is particularly tedious to determine analytically. Thus, in order to gain some quantitative insight about the price of anarchy, the relative social cost was computed numerically for different situations[7]. Assuming a uniformly distributed cooperation cost, the relative social cost is given by:

$$\begin{aligned}\widetilde{SC}^e(y) &= \frac{S}{\delta N^2}\left[N\kappa\int_0^{\hat{\theta}(y)} uf(u)du + \frac{\delta N^2}{S}\frac{x^2\left(\hat{\theta}(y)\right)}{2gy} + \frac{\delta N^2}{S}\frac{\left(1-x\left(\hat{\theta}(y)\right)\right)^2}{1-y}\right] \\ &= \frac{1}{r}\frac{\left(\hat{\theta}(y)\right)^2}{2} + \frac{\left(\hat{\theta}(y)\right)^2}{2gy} + \frac{\left(1-\hat{\theta}(y)\right)^2}{1-y}, \qquad (20)\end{aligned}$$

where $\hat{\theta}(y)$ is given by Eq. (19).

The optimal capacity splits obtained for the user equilibrium are represented in Fig. 7a, together with those that were obtained analytically for the social optimum. While the overall shapes of these curves are very similar, it is striking that the demand and capacity splits are consistently closer under user equilibrium than under social optimum. This difference illustrates how the social optimum requires some naturally independent users to become cooperative for the greater good.

Despite these differences, it is of great practical interest to notice that the maximum social cost reductions that can be obtained under user equilibrium and social optimum are still relatively close. The price of anarchy is a measure of their difference, defined by $PoA = \widetilde{SC}^o - \widetilde{SC}^e$ and represented in

[7]Since the results obtained were very similar with different types of distributions (uniform, exponential and log-normal), we only present here the results obtained with a uniformly distributed cooperation cost.

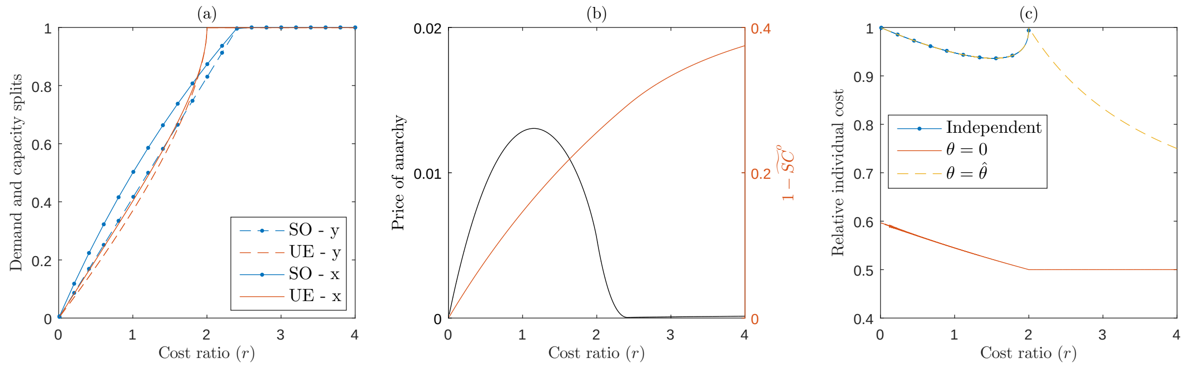

Figure 7: Representations as functions of r of (a): the demand and capacity splits under social optimum (SO) and under the user equilibrium (UE) with the optimal capacity split, (b): the reduction of the relative social cost obtained as a function of r with the socially optimal demand and capacity splits and the price of anarchy, (c): the relative individual cost under user equilibrium for independent users and for the two extreme cooperative users with the smallest and biggest cooperation cost. All figures are for $g = 1$.

Fig. 7b. Note that for sufficiently big values of r, the gain is sizeable ($\sim 10 - 40\%$) while the price of anarchy is relatively small ($\sim 1\%$). Thus, the scheduling system proposed could be very efficient even without tolling.

Finally, the Pareto-improvement property analytically obtained in Proposition 8 is illustrated in Fig. 7c, which represents the individual costs for some extreme users. It can be seen that the curves associated to the independent users and to the cooperative user with the maximum cooperation cost ($\theta = \hat{\theta}$) exactly overlap (this characterizes the user equilibrium) and are never strictly greater than 1. Fig. 7c shows that this scheme is actually extremely beneficial for the users with a low cooperation cost (θ close to 0) but still brings some (less substantial) benefits to all the other users. These findings, even obtained with idealized systems with simplified assumptions, highlight the potential of the proposed policy. It is also clear however that policies should be preceded by proper physical modeling and optimization of key parameters, in order to be beneficial for most users. Our work is a first step in this direction.

4.4.1 Impact of automation

Fig. 8 is the analogue of Fig. 6 for the user equilibrium. The social cost under user equilibrium with a socially optimal capacity split is represented in Fig. 8a, together with the social cost under social optimum, for different values of the automation factor g. In Fig. 8b this social cost under user equilibrium is computed again for two values of g but with different distributions for the cooperation

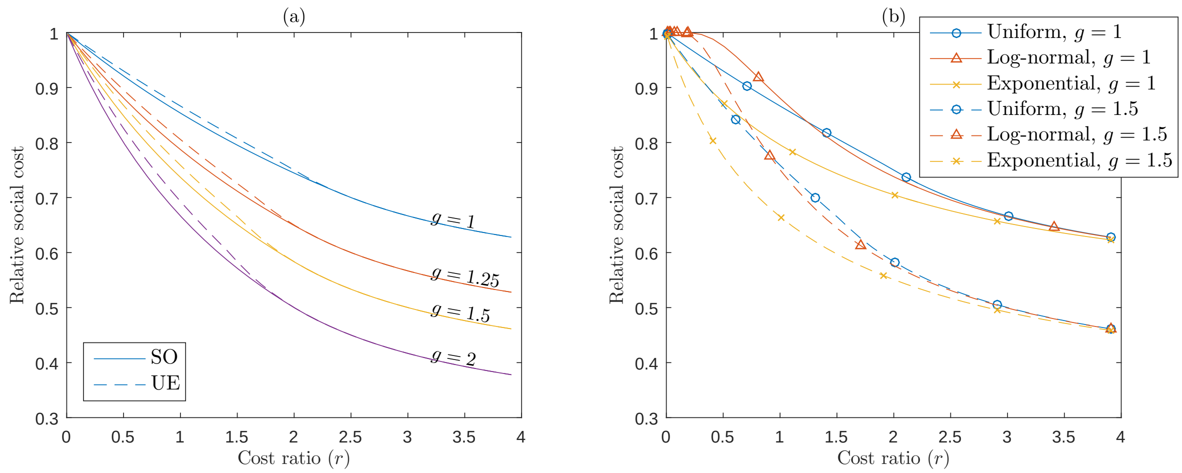

Figure 8: Comparison of the social cost obtained (a) with a socially optimal capacity split under user equilibrium and under social optimum with different values of the automation factor g and with a uniform distribution, (b) with a socially optimal capacity split under user equilibrium for $g = 1$ and $g = 1.5$ but with a uniform, a log-normal and an exponential distribution.

cost. The analysis of these two figures suggests that the trends observed for $g = 1$ exist also for $g > 1$ and that overall, bigger values of g amplify the differences (either between user equilibrium and social optimum or between different distributions).

5 Private operator

This last section investigates the compatibility of the cost-reducing scheduling service introduced above with profit-maximizing objectives. Given the current global enthusiasm for privatization, it seems in fact very likely that if such a scheduling service were to be implemented, its operation would be left to some independent organization, as it is already the case for about one third of highways in Western Europe (Verhoef, 2007). If this independent operator is not subsidized, it would need to collect revenue, most likely via a toll. As discussed in Section 4.1 however, there are already fewer users that are cooperative at the equilibrium than at the social optimum. Thus, it seems a priori preferable that either the private operator manages the independent route (even though this does not address the issue of the scheduling service management), or that two tolls should are applied. Based on these first considerations, the exact impact of profit-maximizing strategies is studied hereafter, first with only one toll set by a private operator on one of the two routes, and then with two tolls defined in a Stackelberg setting in which the government is leader and the private operator adjusts its toll in function of the government's toll. Stackelberg competition is classically considered in games involving several heterogeneous players, where one player is able to implement its decision first, e.g. in a spatial market competition (Wang and Ouyang, 2013; Drezner et al., 2015), when building or expanding private transportation infrastructures (Xiao et al., 2007; van den Berg and Verhoef, 2012) or, in a case more similar to ours, when a central authority concerned with the social optimum delegates the network operations to independent organizations with different objectives (Zhang et al., 2011). Stackelberg competition is believed to be one of the prevailing strategic interactions in many market situations as it often allows all players to make more benefits than for instance in a Nash competition (Wang et al., 2014). In the case at hand, the heterogeneity of players and the strong grip of the government on public infrastructure are strong arguments for a Stackelberg framework. Note that since the amount tolled is only a transfer of money from some individuals to others, it is not taken into account in the calculation of the social cost. Therefore, profit-maximization and social cost minimization are two objectives that are not necessarily in opposition.

5.1 Profit-maximizing toll (one player only)

In this first sub-section, the private operator is the only player and it sets a toll to maximize its profit. Two cases are considered, depending on the route (with cooperative or independent users) that is managed by the private operator.

5.1.1 Cooperative service managed by a private operator

Consider that a private operator is given a proportion y of the capacity and manages the cooperative service. The toll should be set in order to maximize the profit, which is given by:

$$\Pi = xN\tau_{pc},$$

where τ_{pc} is the toll set by this private operator on the cooperative route (p stands for "private" and c for "cooperative"). The only user equilibria that bring some profit to the private operator are such that both networks are used (otherwise, the private operator would have either no customer, or it should pay them to use its network because the other one would have zero cost at least for some users). Thus:

$$\kappa\hat{\theta}(y) + c_c^e + \tau_{pc} = c_i^e, \tag{21}$$

or

$$\kappa\hat{\theta}(y) + \frac{\delta N}{S}\frac{x}{2gy} + \tau_{pc} = \frac{\delta N}{S}\frac{1-x}{1-y},$$

and therefore

$$\frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)x = \frac{\delta N}{S}\frac{1}{1-y} - \kappa\hat{\theta}(y) - \tau_{pc}. \tag{22}$$

Since given a capacity split y, the function $\hat{\theta} \to \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)x(\hat{\theta}) - \frac{\delta N}{S}\frac{1}{1-y} + \kappa\hat{\theta} + \tau_{pc}$ is strictly increasing, there can be only one equilibrium for a given toll but once again, there is no closed-form expression of this equilibrium demand split.

5.1.1.1 *Special case: uniform distribution*

With a uniform distribution of the cooperation cost, Eq. (22) is equivalent to

$$\hat{\theta}(y) = \frac{\frac{\delta N}{S}\frac{1}{1-y} - \tau_{pc}}{\kappa + \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)}.$$

The profit is now given by: $\Pi = N\hat{\theta}(y)\tau_{pc}$, and it is of the form $\Pi = -A\tau_{pc}^2 + B\tau_{pc}$ with

$$A = \frac{N}{\kappa + \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)}\ ;\ B = \frac{\frac{\delta N^2}{S}\frac{1}{1-y}}{\kappa + \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)}.$$

Thus, the toll which maximizes this 2^{nd} order polynomial is given by:

$$\tau_{pc} = \frac{\delta N}{S}\frac{1}{2(1-y)}. \tag{23}$$

Interestingly, this toll does not depend on κ and corresponds to the average cost of travel time if all users had to use the independent route. With such a toll, the profit-maximizing demand is:

$$\begin{aligned} N\hat{\theta}^{pc} &= N\frac{\frac{\delta N}{S}\frac{1}{1-y} - \frac{\delta N}{S}\frac{1}{2(1-y)}}{\kappa + \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)} \\ &= N\frac{rgy}{2gy(1-y) + r(1+(2g-1)y)}, \end{aligned} \tag{24}$$

which is exactly half the cooperative demand when there is no toll. Note that since there were already fewer cooperative users in user equilibrium than in social optimum, this profit-maximizing strategy moves the demand split in the “wrong” direction. Thus, such a strategy where the private operator is free to choose his optimal price is not recommended for an implementation.

5.1.2 Independent route managed by a private operator

The same approach can be used for this symmetric case, so we just provide the final results here. The profit-maximizing toll is:

$$\tau_{pi} = \frac{(2\kappa gyS + \delta N)}{4gyS}. \tag{25}$$

The profit-maximizing demand is:

$$\hat{\theta}_{pi} = \frac{2gy(1-y) + r(1+(4g-1)y)}{2[2gy(1-y) + r(1+(2g-1)y)]}.$$

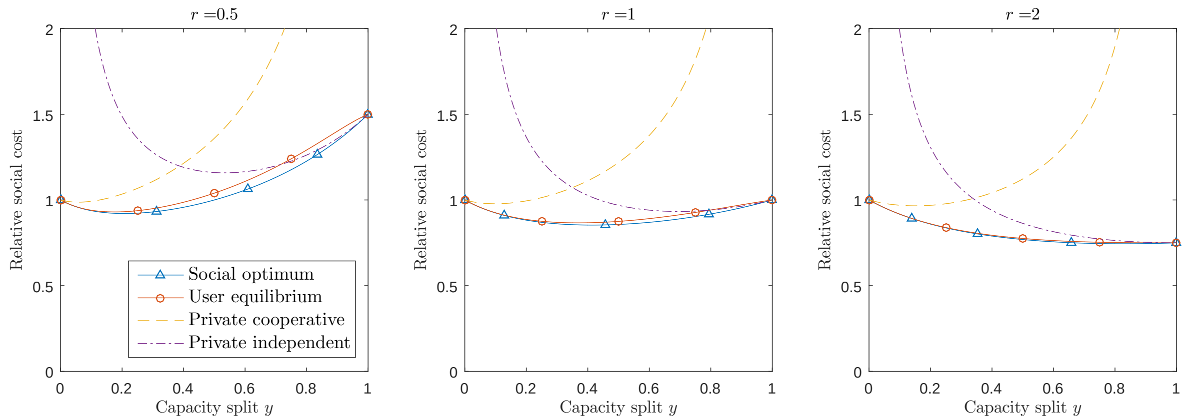

Figure 9: Comparison of the social costs obtained under social optimum, user equilibrium, if the independent-route is privately managed and if the cooperative route is privately managed, as functions of the capacity split and for cost ratios r equal to 0.5, 1 and 2. $g = 1$ for all figures.

5.1.3 Numerical applications

The social costs obtained with the profit-maximizing strategies studied in Sections 5.1.1 and 5.1.2 are represented in Fig. 9, together with the social costs associated to the user equilibrium with no toll (studied in Section 4.1.1) and to the social optimum. It can be noticed that when the private operator manages the independent route and has only a small percentage of the total capacity ($y \simeq 0.9$), the profit-maximizing strategy can slightly improve the user equilibrium, especially if r is small (e.g. for $r = 0.5$ and $g = 1$). Intuitively, the private operator forces a few users to become cooperative (which is overall beneficial for the society, as shown by Corollary 3) but not too many as it does not have control over a large part of the capacity and thus cannot force the users to pay extreme costs. However, this situation is not desirable since it leads to a social cost that is higher than the reference social cost. For more reasonable values of r (in the range of $0.5 - 1$), the social optimum is obtained for a capacity split that is quite balanced ($y \simeq 0.3$) and for this range of balanced capacity splits, the two profit-maximizing strategies studied always lead to poor social costs. Thus, it seems that profit-maximizing strategies are not compatible with the objective of minimizing the social cost if there is only one toll applied by the private operator. The same numerical applications were repeated with different values of g ranging between 1 and 2 but the results obtained (not shown here) were extremely similar. Note however that different governmental regulations (like taxing or price-caps) could guide such a system to better situations for the social good.

5.1.3.1 Relationship between capacity and profit

Let us now consider that the private operator has to pay a given amount c for each capacity unit that it rents from the government. Assuming that the private operator manages the cooperative route and that it always sets the toll to maximize its profit, the profit depends on the capacity split with the following function:

$$\Pi = N\hat{\theta}_{pc}\tau_{pc} - cyS.$$

Obviously, if the demand is not price-elastic and if all the capacity is given to the private operator, the users would be captive and the optimal toll would be infinite. However, for y small enough ($y \to 0$), $\Pi = \left(\frac{\delta N^2}{2S^2} - c\right) yS + o(yS)$. Thus, depending on the value of c, operating only a small part of the network might not be profitable. However, when the capacity that is privately operated gets closer to the full capacity, the situation becomes similar to a monopoly and profits dramatically increase. The situation is very similar if we consider that the private operator manages the independent route. Alternatively, one could consider that the government sets the capacity split y (or equivalently, the value of c) as a function of the toll proposed by the private operator. If the private operator manages the independent route and proposes a toll that is small enough (considering the population's cost ratio r), then the government could allocate to the private operator a capacity split that is such that the toll proposed is optimal.

5.2 Stackelberg equilibria

We now study the impact of a profit-maximizing strategy within a Stackelberg competition, where the government and a private company both impose a toll on one route. While the government aims at minimizing the social cost (i.e. the sum of the schedule delays, congestion and cooperation costs), the private company aims at maximizing its profit. Since the government has a dominant position, we will consider that the government sets its toll first, knowing how the private company will react (we will see however that the Stackelberg equilibria obtained are also Nash equilibria).

The fact that the leader is maximizing welfare should intuitively lead to a higher welfare in the society than in cases where all actors are profit-maximizing. In addition, the lack of flexibility of toll is indeed more credible for the public than for the private sector, although this does not prevent to consider some compensation for the potential profit loss of the private sector. Observe, however, that it is not necessarily the case that when the leader is maximizing the welfare, the welfare is higher in the society than when all actors are maximizing profit (this was shown, for example, by Anderson et al. (1997) in the context of mixed oligopoly with Logit demand function). Here, with congestion effect, the situation is substantially more complex. This question needs more study before the government should propose regulations.

Here, it is assumed that the capacity split is given. One could have considered another type of Stackelberg equilibrium in which there is only one toll set by a private operator and where the government sets the capacity split. However, the graphical representation of the social costs as functions of the capacity splits in Fig. 9 suggested that the gains that could be obtained with such a framework would be extremely small. The next two subsections investigate the cases where the cooperative and independent routes are managed by the government and the private operator and vice versa.

5.2.1 Cooperative service managed by a private company, independent route by the government

By including an additional toll set by the government τ_{gi} on the independent route in Eq. (21), the equilibrium equation becomes:

$$\kappa\hat{\theta}(y) + c_c^e + \tau_{pc} = c_i^e + \tau_{gi}.$$

In the Stackelberg framework, the leader takes his decision first and then does not react to the decision of the follower. Thus τ_{gi} is seen as a constant by the private operator and the calculations done with only one player (Eqs. (21) to (22)) remain valid after replacing τ_{pc} by $(\tau_{pc} - \tau_{gi})$. Assuming a uniform distribution for the cooperation cost, the equilibrium demand split is:

$$\hat{\theta}(y) = \frac{\frac{\delta N}{S}\frac{1}{1-y} - (\tau_{pc} - \tau_{gi})}{\kappa + \frac{\delta N}{S}\left(\frac{1+(2g-1)y}{2gy(1-y)}\right)}. \tag{26}$$

Proposition 9. *Assuming that the government is leader, for any capacity split y there exists a unique Stackelberg equilibrium and it is characterized by*

$$\tau_{gi} = \frac{\kappa r}{1-y}\frac{r(1+(2g-1)y) + 3gy(1-y)}{r(1+(2g-1)y) + gy(1-y)}. \tag{27}$$

This equilibrium is also the social optimum and a Nash equilibrium.

In the Stackelberg equilibrium studied, the government first sets its toll and the private operator reacts. As the demand is assumed inelastic, the equilibrium demand split is uniquely determined by the difference in the two tolls, independently of the level of these tolls. Thus, the objective of the government is to set its toll in such a way that the profit maximizing toll for the private operator minimizes the social cost, given this capacity split. We can actually demonstrate that by simply varying its own toll, the government can make any demand split profit-maximizing for the private operator and can therefore lead the system to the social optimum. Intuitively, when the private operator already has n customers, attracting the next one to its route implies reducing its toll by the sum of (i) a constant term resulting from the additional congestion cost imposed by one additional user (equal to $\frac{\delta}{2S}$) and (ii) the difference in cooperation cost between the previous critical user and the new one. This amount is independent of the current level of the toll. As it is profitable to attract this user if and only if the new amount of the toll is bigger than n times the toll reduction, the government can obtain the demand split desired simply by setting the level of toll. Note however that this mechanism can involve considerable financial transfers from the users to the private operator and the government.

In terms of comparative statics, Eq. (27) can be differentiated to show that the relative toll $\frac{S}{\delta N}\tau_{gi}$ increases with the capacity split y, with the automation factor g and decreases with the cost ratio r. Intuitively, if the private operator manages a greater proportion of the network, it is more powerful and can impose higher tolls. Similarly, if users have a lower cost ratio r, their cooperation cost is greater relatively to the congestion cost, so they are ready to pay higher tolls.

5.2.2 Cooperative service managed by the government, independent route by a private company

Again, this problem is symmetrical to the previous one so the details of the calculations are left to the reader. Similarly to the previous case, it is possible to obtain a social optimum if the government sets the toll to:

$$\tau_{gc} = \kappa\left(1 + r\frac{r(1+(2g-1)y) - gy((1+4g)y-1)}{2gy[r(1+(2g-1)y+gy(1-y))]}\right). \tag{28}$$

The relative toll $\frac{S}{\delta N}\tau_{gc}$ decreases with the capacity split y and the cost ratio r.

5.2.3 Numerical applications

In order to limit the number of independent variables, note that all the optimal tolls obtained in the Eqs. (23), (25), (30), (27) and (28) can be expressed as κ times an expression that depends only on r, g and y. Thus, the relative toll, defined as the toll divided by the individual cost with no cooperation $\frac{\delta N}{S}$, depends only on r, g and y.

The optimal relative tolls obtained in the Sections 5.2.1 and 5.2.2 are plotted in Fig. 10, together with the profit-maximizing tolls obtained in Sections 5.1.1 and 5.1.2. Note that as r increases, the tolls imposed by the government and by the private operator in Stackelberg equilibria become almost identical, which is natural since the user equilibrium coincides with the social optimum for $\kappa = 0$, i.e. $r \to \infty$ (cf. Proposition 2). In addition, one can see that the toll set on the independent route is always slightly higher than the toll on the cooperative route. This was predictable since previous results showed that there are fewer cooperative users under user equilibrium (when both tolls are the same) than in social optimum. Thus, the government should always deter some users from using the independent route. Note also that as the private operator always sets its own toll relatively to the government's toll, the absolute value of the private toll is significantly greater in Stackelberg equilibrium than when the government does not set any toll.

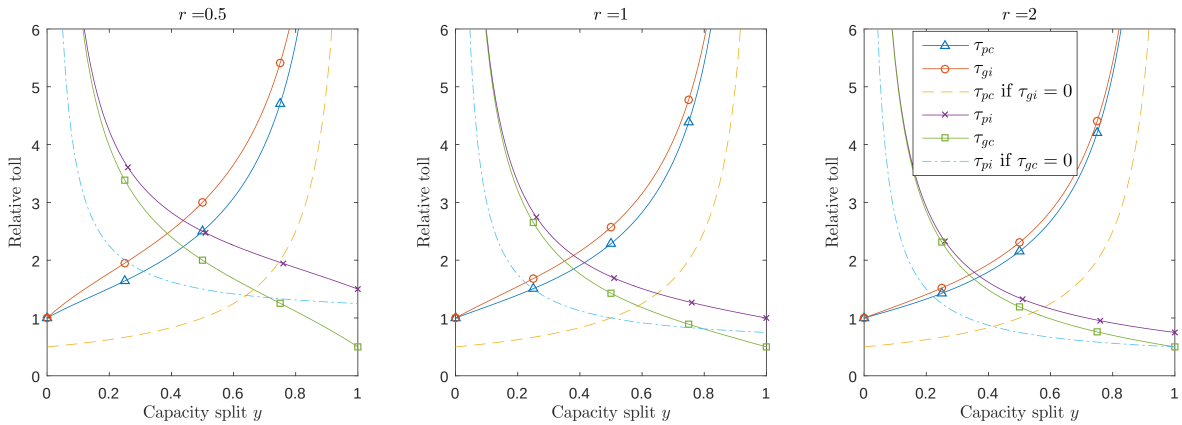

Figure 10: Comparison of the tolls set by the government and by the private operator when the private operator manages the cooperative route (τ_{pc} and τ_{gi}), or when it manages the independent route (τ_{pi} and τ_{gc}), with and without a toll set by the government ($\tau_{gi} = 0$ or $\tau_{gc} = 0$). $g = 1$ for all figures.

Finally, it is of practical interest to notice that the relative tolls applied under Stackelberg equilibrium are systematically greater than 0.5, and often much higher[8]. In comparison, an optimal time-varying toll applied on the entire capacity as described by Vickrey (1969) allows to reach the social optimum with a maximum relative toll of exactly 0.5. Thus, a Stackelberg equilibrium with cooperation requires more money transfers and leads to a greater social cost (because of cooperation costs) than a fine toll, which makes it less acceptable and less effective. Thus, a privatization of the proposed service should be followed by governmental regulations that would lead the system close to social optimum conditions without imposing very high tolls. In this case, the private operator will still obtain significant benefits, but restricting competition would result in much better conditions for the system. Again, different values of the automation factor g between 1 and 2 were tested but they led to very similar results (not included).

6 Conclusion and future work

This paper introduces a new approach of cooperation for Vickrey's bottleneck problem that can already be implemented with conventional vehicles but that could emerge naturally as the trends toward car-sharing and autonomous vehicles converge. This approach is based on a scheduling service that only manages a part of the capacity of a bottleneck. The choice of the sub-route was assumed to be a long-term decision, based on the congestion on the two sub-routes and on a new individual-specific

[8]The assumption of an inelastic demand becomes clearly unrealistic in these cases. In practice, the tolls would be limited to more reasonable levels, but would deter some users from traveling.

cooperation cost associated to using the scheduling service.

It was found that with a well-chosen capacity split, such a scheme allows for a reduction of the social cost, not only with a socially optimal demand split but also under user equilibrium. In addition, it was shown that under user equilibrium, all users are better-off (compared to the reference scenario with no cooperation). The social optimum can be obtained with a relatively small flat toll but in this case, users are not all better-off before redistributing the revenue of the toll. Finally, the possibility of delegating the management of a sub-route to a private operator was investigated but it was shown that if tolls are not restricted and if the private operator controls a significant proportion of the total capacity, this would lead to a dramatic increase in social cost. The social optimum can still be obtained if the government imposes another toll on the other sub-route but at the price of extremely high tolls. Thus, while the scheduling service described in this work has a great potential to reduce the cost of congestion, Section 5 showed that it is hardly compatible with profit-maximizing strategies and the design of suitable toll regulations should be one of the priority topics.

The cooperation cost was introduced in this work with classic simplifying assumptions and the relaxation of some of these assumptions should be investigated. In particular, different distributions of the desired arrival time should be considered to model longer peak hours. The assumption of a single long-term decision concerning the choice of the group could also be relaxed to allow for an easier implementation with privately-owned cooperative vehicles. Finally, more complex expressions of the cooperation cost might have to be considered with different time slot allocation mechanisms. For instance, if there is a variability in the allocated departure times, the cooperation cost might depend on the magnitude of this variability and therefore on the ratio of the demand to capacity. The potential implementation of such policies should be a priority as they are aimed at addressing the policy limitations of classical demand- and supply-oriented schemes.

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A Proofs

Proof of Lemma 1. Since $\underline{\theta} \leq 0$, $G(\underline{\theta}) = \underline{\theta} - b < 0$. Besides, $\lim_{\hat{\theta}\to\bar{\theta}} G(\hat{\theta}) = \bar{\theta} + a - b$. Thus, if $\bar{\theta} + a - b > 0$, the intermediate value theorem ensures the existence of a solution and since G is strictly increasing, this solution is unique. Else, $G(\hat{\theta}) < 0$ for all $\hat{\theta} < \bar{\theta}$ so there can be no interior solution.

Let us now see G as a function of $\hat{\theta}$, a and b. G is simply affine with a and b so if f is continuous, G is continuously differentiable with: $\frac{\partial G}{\partial \hat{\theta}} = 1 + af\left(\hat{\theta}\right)$, $\frac{\partial G}{\partial a} = F(\hat{\theta})$ and $\frac{\partial G}{\partial b} = -1$. As $\hat{\theta}$ is interior, all the previously mentioned derivatives are invertible so the implicit function theorem implies that $\hat{\theta}^{sol}$ is locally continuous and differentiable with:

$$\frac{\partial \hat{\theta}^{sol}}{\partial a}(a,b) = -\frac{F(\hat{\theta}^{sol})}{1 + af(\hat{\theta}^{sol})} < 0 \text{ and } \frac{\partial \hat{\theta}^{sol}}{\partial b}(a,b) = \frac{1}{1 + af(\hat{\theta}^{sol})} > 0.$$

□

Proof of Lemma 2. $\widetilde{SC}$ is trivially continuous on $]\underline{\theta}, \bar{\theta}[$. Let us now show that $\widetilde{SC}$ is continuous on the closed interval $[\underline{\theta}, \bar{\theta}]$. By combining Eq. (5) and Eq. (6) and after some manipulations, we obtain: $\forall \hat{\theta} \in]\underline{\theta}, \bar{\theta}[$,

$$\begin{aligned}
\widetilde{SC}\left(\hat{\theta}\right) &= \frac{1}{r}\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{x^2}{2g\frac{x}{\sqrt{2g}(1-x)+x}} + \frac{(1-x)^2}{1 - \frac{x}{\sqrt{2g}(1-x)+x}} \\
&= \frac{1}{r}\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{1}{2}\left(\sqrt{2g}(1-x) + x\right)\left[\frac{x}{g} + \frac{2(1-x)^2}{\sqrt{2g}(1-x)}\right] \\
&= \frac{1}{r}\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{1}{2g}\left(\sqrt{2g}(1-x) + x\right)^2.
\end{aligned}$$

We can now evaluate this expression at the boundaries of its domain:

$$\begin{aligned}
\lim_{\hat{\theta}\to\underline{\theta}}\left(\widetilde{SC}\left(\hat{\theta}\right)\right) &= \frac{1}{2g}\left(\sqrt{2g}\right)^2 = 1 = \widetilde{SC}\left(\underline{\theta}\right), \\
\lim_{\hat{\theta}\to\bar{\theta}}\left(\widetilde{SC}\left(\hat{\theta}\right)\right) &= N\kappa\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{1}{2g}\left(\sqrt{2g}(1-1) + 1\right)^2 \\
&= \frac{1}{r}\int_{\underline{\theta}}^{\bar{\theta}} uf(u)du + \frac{1}{2g} = \widetilde{SC}\left(\bar{\theta}\right).
\end{aligned}$$

Therefore, $\widetilde{SC}$ is continuous on $[\underline{\theta}, \bar{\theta}]$. □

Proof of Proposition 4. Since the expression of $\widetilde{SC}$ obtained in Eq. (13) is valid on $[\underline{\theta}, \bar{\theta}]$, we will now use it to avoid handling different cases. This function is continuous, differentiable and we have:

$$\begin{aligned}\frac{d\widetilde{SC}}{d\hat{\theta}}(\hat{\theta}) &= \frac{1}{r}\hat{\theta}f\left(\hat{\theta}\right) - \frac{1}{2g}\left[2\left(\sqrt{2g} - \left(\sqrt{2g}-1\right)x\left(\hat{\theta}\right)\right)(\sqrt{2g}-1)f\left(\hat{\theta}\right)\right] \\ &= \frac{1}{r}f\left(\hat{\theta}\right)H(\hat{\theta}),\end{aligned}$$

where $H(\hat{\theta}) \triangleq \hat{\theta} + \frac{r}{g}\left(\left(\sqrt{2g}-1\right)^2 x\left(\hat{\theta}\right) - \left(2g - \sqrt{2g}\right)\right)$.

Lemma 1 can be applied on function H with $\bar{\theta} + a - b = \bar{\theta} + \frac{r}{g}\left(1 + 2g - 2\sqrt{2g} - 2g + \sqrt{2g}\right) = \bar{\theta} - r\frac{\sqrt{2g}-1}{g}$. Thus, given g and $\bar{\theta}$, the existence of an interior solution depends on the cost ratio r.

If $r < \bar{\theta}\frac{g}{\sqrt{2g}-1}$, then $\bar{\theta} > r\frac{\sqrt{2g}-1}{g}$, so according to Lemma 1 there exists a unique $\hat{\theta}^o$ such that $H(\hat{\theta}^o) = 0$. Since $\frac{d\widetilde{SC}}{d\hat{\theta}}$ is negative on $[\underline{\theta}, \hat{\theta}^o[$, positive on $]\hat{\theta}^o, \bar{\theta}]$ and equal to zero at $\hat{\theta}^o$, $\hat{\theta}^o$ is the unique global minimum of $\widetilde{SC}$. Although the comparative statics of lemma 1 do not apply directly here, the implicit function theorem can be used in a very similar fashion to demonstrate that $\hat{\theta}^o$ increases with r and g. Since the demand split is an increasing function of $\hat{\theta}^o$ and the capacity split an increasing function of the demand split, both splits increase with r and g.

If $r \geq \bar{\theta}\frac{g}{\sqrt{2g}-1}$, then $\forall\hat{\theta} \in [\underline{\theta}, \bar{\theta}[$, $\frac{d\widetilde{SC}}{d\hat{\theta}} < 0$ and $\frac{d\widetilde{SC}}{d\hat{\theta}}(\bar{\theta}) \leq 0$. Thus, there is a unique global minimum and it is reached for $\hat{\theta} = \bar{\theta}$ (and $x = 1$). From a practical point of view, this means that if the cooperation cost is small enough for all the population, the social optimum is an entirely controlled infrastructure. □

Proof of Lemma 3. First, it is trivial that $x(0) = 0$ and $x(1) = 1$ as users have no choice in these conditions. Second, one can verify that that if $y_e < 1$, then $x = y_e$ is a user equilibrium. Indeed, if the critical user is of type $\hat{\theta} = (1 - \frac{1}{2g})r$, the total individual cost for the critical user is (following (2)) $\kappa\hat{\theta} + \frac{\delta Nx}{2gSy} = \frac{\delta N}{S}(1 - \frac{1}{2g}) + \frac{\delta N}{2gS} = \frac{\delta N}{S}$. As the demand split x is equal to the capacity split y, this is also the individual cost for independent users, so this is a user equilibrium. Corollary 2 ensures uniqueness so $x = y_e$ is the only user equilibrium for the capacity split $y = y_e$.

Now, let $y \in]0, min(1, y_e)[$. The reasoning above shows that $x = y$ is not a user equilibrium because the critical cooperation cost would be smaller than $(1 - \frac{1}{2g})r$, i.e. it would be in the interest of some independent users to become cooperative. Hence, as the individual costs for a given capacity split are monotonous functions of the demand split, the user equilibrium necessarily verifies $x > y$.

Conversely, if $y_e < 1$ and $y \in]min(1, y_e), 1]$, $x = y$ is not a user equilibrium either because it would be in the interest of some cooperative users to become independent. Thus, the equilibrium verifies $x < y$. □

Proof of Proposition 7. By applying Lemma 3:

$$y \in]0, min(1, y_e)] \Leftrightarrow \begin{cases} x(y) \geq y \\ y > 0 \\ \text{if } y = 1, \text{ then } y_e \geq 1. \end{cases} \tag{29}$$

Then, in order to obtain a Pareto-improvement, three conditions should be verified: (i) the independent users are not worse-off, (ii) the user with the critical cooperation cost is not worse-off on the cooperative sub-route, and (iii) at least one user is better-off.

Let us first show that $y \in]0, min(1, y_e)]$ implies that there is a Pareto-improvement. Based on the equation of the individual cost for independent users (1), condition (i) is equivalent to $x \geq y$, i.e. the proportion of the demand that is cooperative should not be smaller than the proportion of the capacity they are allocated. Note then that if condition (i) is verified and $y < 1$, Condition (ii) must be verified as well since the critical user is indifferent at equilibrium. If $y = 1$, users have no choice, so we cannot use the indifference argument. However, as highlighted by Eq. (29), then we must have $y_e \geq 1$, so the cooperative user with the biggest cost is still not worse-off. In addition, if condition (ii) is verified and $x > 0$ (which is guaranteed by Condition 1 as long as $y > 0$), then there are some users with a cooperation cost strictly smaller than the critical one who are better-off. Thus, $y \in]0, min(1, y_e)]$ implies that there is a Pareto-improvement.

Conversely, $y = 0$ corresponds to the reference scenario, which violates condition (iii). If $y > min(1, y_e)$, then Lemma 3 implies that either $x(y) < y$ or $y = 1$ and $y_e < 1$. The first case clearly violates condition (i) while the second violates condition (ii). Indeed, in this last situation $x(y) = y$ so the congestion cost for cooperative users is equal to $\frac{\delta N}{2gS}$. The cooperation cost that makes a user indifferent between the reference scenario and cooperation in this scenario is reached for $x(y_e) = y_e$. As $y > y_e$, there are more cooperative users for $y = 1$ and these additional cooperative users are all worse-off. □

Proof of Proposition 8. By applying Corollary 2, the demand split is a continuous function of the capacity split for $y \in [0, 1]$. Thus, the social cost under user equilibrium is also a continuous function of the capacity split on $[0, 1]$ and the extreme value theorem guarantees the existence of a capacity split minimizing the social cost.

If $y_e \geq 1$, all capacity splits $y > 0$ are Pareto-improving so the result is trivial. Let us now assume that $y_e < 1$. Let $Y > y_e$ and X be a capacity split and its associated demand split at user equilibrium. We demonstrate hereafter that the social cost at user equilibrium for $y = Y$ is bigger than for $y = y_e$.

First, although the pair $(x = X, y = X)$ is not an equilibrium, we can show that $SC(X, Y) > SC(X, X)$. Indeed, note that the number of cooperative users is identical so the cooperation cost is exactly the same and we just have to compare the costs of congestion. We consider here the demand split as given so the total congestion cost is simply a function of the capacity split: $g(y) = \frac{X^2}{2gy} + \frac{(1-X)^2}{1-y}$. Differentiating this expression leads to $g'(y) = -\frac{X^2}{2gy^2} + \frac{(1-X)^2}{(1-y)^2}$, which is positive for all $y > X$. Thus, $SC(X, Y) > SC(X, X)$.

Second, since the demand split is a strictly increasing function of the capacity split, $X > x(y_e) = y_e$.

We can then show that $SC(X,X) > SC(y_e,y_e)$. In fact, the individual costs are the same in these two scenarios for all users that do not change their decision (i.e. those that are independent or cooperative in both scenarios). In addition, $x = y = y_e$ is an equilibrium so all users that are independent in these conditions would be worse-off if they were forced to be cooperative with the same congestion conditions, which is exactly what happens in the situation $(x = X, y = X)$.

Thus, $SC(X,Y) > SC(X,X) > SC(y_e,y_e)$ so the socially optimal capacity split verifies $y \le y_e$. To conclude, note that $y = 0$ is clearly not socially optimal as it is Pareto-dominated by any $y \in]0, y_e]$. $\square$

Proof of Proposition 9. The profit is now given by $\Pi = N\hat{\theta}(y)\tau_{pc}$, or

$$\Pi = -A\tau_{pc}^2 + B''\tau_{pc},$$

where A has the same expression as in Section 5.1.1.1 but

$$B'' = \frac{2\delta N^2 gy + 2NgyS(1-y)\tau_{gi}}{2\kappa gyS(1-y) + \delta N(1+(2g-1)y)}.$$

Thus, the maximum profit is obtained for

$$\tau_{pc} = \frac{B''}{2A} = \frac{\delta N^2 gy + NgyS(1-y)\tau_{gi}}{2NgyS(1-y)},$$

i.e

$$\tau_{pc} = \frac{\tau_{gi}}{2} + \frac{\delta N}{S}\frac{1}{2(1-y)}. \tag{30}$$

This is simply the arithmetic mean of the government toll and of the average congestion cost (schedule penalty and travel time cost) if all users had to use the independent route. By combining equations (26) and (30):

$$\begin{aligned}\hat{\theta}^{pc} &= \frac{2\delta Ngy + 2gyS(1-y)\tau_{gi}}{2\kappa gyS(1-y) + \delta N(1+(2g-1)y)} - \frac{2gyS(1-y)}{2\kappa gyS(1-y) + \delta N(1+(2g-1)y)}\frac{\delta N + (1-y)S\tau_{gi}}{2S(1-y)} \\ &= \frac{\kappa rgy + gy(1-y)\tau_{gi}}{2\kappa gy(1-y) + \kappa r(1+(2g-1)y)}.\end{aligned} \tag{31}$$

The only decision variable is now τ_{gi} and the government should set its value such that it minimizes the social cost. Although it is a priori not necessarily feasible, if the government can set a toll such that the demand found in Eq. (31) is equal to the demand found in Eq. (12), then this toll is optimal. Mathematically, this requires that $\hat{\theta}^o = \hat{\theta}^{pc}$, so

$$\frac{2rgy}{gy(1-y) + r(1+(2g-1)y)} = \frac{\kappa rgy + gy(1-y)\tau_{gi}}{2\kappa gy(1-y) + \kappa r(1+(2g-1)y)},$$

or

$$2r[2gy(1-y) + r(1+(2g-1)y)] = \left[r + (1-y)\frac{\tau_{gi}}{\kappa}\right][gy(1-y) + r(1+(2g-1)y)],$$

or

$$r(1+(2g-1)y)\left[r - (1-y)\frac{\tau_{gi}}{\kappa}\right] = gy(1-y)\left[(1-y)\frac{\tau_{gi}}{\kappa} - 3r\right],$$

or

$$(1-y)\frac{\tau_{gi}}{\kappa}[r(1+(2g-1)y)+gy(1-y)] = r[3gy(1-y)+r(1+(2g-1)y)],$$

which is equivalent to Eq. (27).

Thus, the social optimum can be obtained even when a private operator manages the cooperative service. Finally, note that since the government can impose the minimum social cost, in this case Stackelberg's equilibrium is also an equilibrium in the sense of Nash. $\square$