The role of mode of representation in students' argument constructions

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Research into students' understanding of proof has generally considered few of the factors that can mediate the relation between students' argument constructions or their evaluations of given arguments and conclusions about students' understanding of proof. This raises concern about the validity of research findings and creates difficulties in comparing findings from different studies. I summarize some of these factors and explore the role played by another one: the mode of representation used in students' argument constructions. In particular, I report and discuss findings from a classroom-based design experiment suggesting that the use of an oral mode of representation may be more likely, compared to a written mode, to support the construction of an argument that approximates or meets the standard of proof.

Keywords: Proof, argument construction, oral representation, written representation.

INTRODUCTION

An assumption (often tacit) underpinning the findings of many studies on students' understanding of proof is that students' argument constructions or students' evaluations of given (researcher-generated) arguments are indicative of their understanding of proof. While this is a sensible assumption to make, there exist some factors, often uncontrolled for in relevant studies, that may mediate the relation between students' argument constructions or evaluations and researchers' conclusions about students' understanding of proof based on that work. For example, there is evidence to suggest the following: it is easier for students to evaluate given arguments than it is for them to construct their own arguments (Reiss, Hellmich, & Reiss, 2002); it is easier for students to identify invalid arguments as invalid than it is for them to identify valid arguments as valid (ibid); students' constructions can be poor indicators of their understanding of proof, as students can be well aware of limitations of their non-proof constructions (Stylianides & Stylianides, 2009); and students can evaluate given arguments in different ways based on different perspectives, such as what would satisfy them personally or what would satisfy their teachers (Healy & Hoyles, 2000).

Studies in this area considered at most one or two of these factors that may mediate the relation between students' argument constructions or evaluations and conclusions about students' understanding of proof. Thus a concern is raised about validity of research findings regarding students' understanding of proof. Take, for example, a study that draws conclusions about students' understanding of proof based on students' argument constructions in response to a number of proving tasks. This study is likely to report a poorer picture of students' understanding of proof than another study that considered also students' evaluations of their own constructions, for relevant research (Stylianides & Stylianides, 2009) suggests some students are fully aware of the reasons for which their non-proof constructions are not proofs. In addition to the issue of validity of research findings, there is also the difficulty in comparing findings from different studies; this creates in turn an obstacle to the development of a cumulative and coherent body of research knowledge in this area.

In this paper, I explore another factor that is worth attention by future research in this area. The factor relates to the *mode of representation* (Stylianides, 2007) used in students' argument constructions, that this, the forms of expression (written, oral, pictorial, etc.) with which an argument is communicated (ibid). I focus on two main modes of representation – written and oral – and I address the following research question: How does the mathematical sophistication of a student's arguments, for the same claim, compare when the bulk of each argument is communicated with a different mode of representation – written versus oral?[1]

There is some evidence to suggest that the verbal mode of representation may be associated with arguments of higher level of mathematical sophistication than the written mode (Schoenfeld, 1985). Schoenfeld described an episode where two students produced a lucid verbal argument, essentially a proof, for a geometrical construction problem, but then the students put down their ideas in writing by producing a contorted argument following the strictly prescribed 'two-column' presentational form (statement; reason). If further evidence was found that oral modes of representation were generally associated with mathematical arguments of higher level of mathematical sophistication than written modes, an important methodological implication would follow: An interview study that examined orally students' argument constructions would likely report a better picture of students' understanding of proof than a survey study that examined in writing the argument constructions of the same group of students and using the same proving tasks.

RESEARCH CONTEXT

The data for the paper are derived from a design experiment, which examined what may be involved in engineering classroom instruction to support secondary students to learn about proof. The design experiment was carried out in an English state school with 165 Year 10 students (14–15 year olds) who were set in seven classes according to their performance in a national assessment at the end of Year 9. All 61 students from the two highest attaining Year 10 classes, and the two mathematics teachers of these classes, participated in the research over a period of two years.

The focus of the study on high-attaining students was partly motivated by the findings of a prior large-scale longitudinal study in England (Küchemann & Hoyles, 2001–03) that showed (1) weak knowledge about proof amongst a national sample of high-attaining Year 8–10 students and (2) modest (if any) improvements in students' knowledge from Year 8 to Year 10. These findings raised concerns about English high-attaining secondary students' learning about proof, and suggested an even more pessimistic prospect for lower attaining or younger students.

The design experiment involved the development, implementation, and analysis of the effectiveness of six lesson sequences, each ranging from one to five 45-minute periods. Lesson sequences 1–4 were implemented when the students were in Year 10, while the rest in Year 11. At the beginning of the study I took the lead role in planning the lesson sequences, but over time the teachers felt more confident to take responsibility for planning the lesson sequences and this allowed me to assume more of a supportive role. All lessons were taught by the regular teacher of each class.

In this paper, I focus on lesson sequence 2, which lasted three 45-minute periods in one class and two in the other. It was implemented three months into Year 10 and capitalized on lesson sequence 1. Lesson sequence 1 lasted two 45-minute periods in each class, was implemented one month into Year 10, and had two main goals: (1) to help students begin to realize the limitations of empirical arguments as methods for validating mathematical generalizations and see a need to learn about more secure validation methods (i.e., proofs); and (2) to introduce students to the notion of proof in mathematics, including a list of criteria for deciding whether a mathematical argument met the standard of proof. The criteria were as follows.

> An argument that counts as *proof* [in our class] should satisfy the following criteria:
>
> 1. It can be used to convince not only myself or a friend but also a *sceptic*.
>
> — It should not require someone to make a leap of faith (e.g., "This is how it is" or "You need to believe me that this [pattern] will go on forever.")
>
> 2. It should help someone *understand why* a statement is true (e.g., why a pattern works the way it does).
>
> 3. It should use *ideas that our class knows already or is able to understand* (e.g., equations, pictures, diagrams).
>
> 4. It should contain *no errors* (e.g., in calculations).
>
> 5. It should be *clearly presented*. (PowerPoint slide used during Lesson Sequence 1)

The criteria were consistent with the following definition of proof, with care taken so that the phrasing of the criteria was suitable for secondary students.

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
>
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
>
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

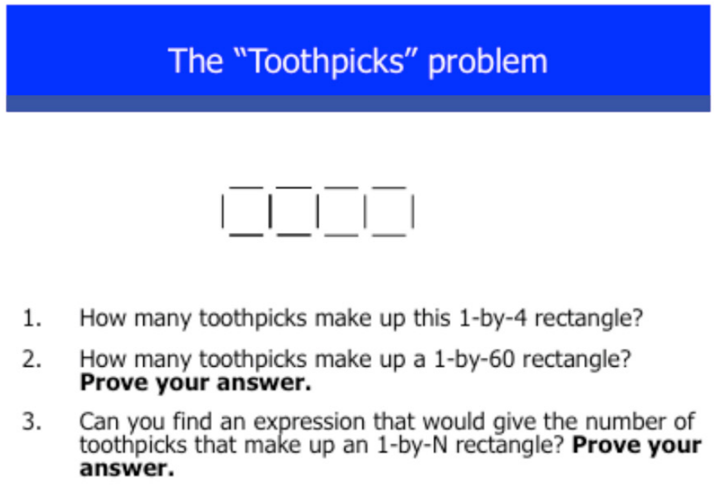

Figure 1: The first proving task

Criteria 1 and 4 correspond to the requirement in the definition for valid modes of argumentation; criterion 5 to the requirement for appropriate modes of argument representation; and criterion 3 to the requirement that all components of a mathematical argument (set of accepted statements, modes of argumentation, and modes of argument representation) be readily accepted, known to, or within the conceptual reach of the class. Furthermore, criteria 1 and 2 reflect, respectively, two important functions that the development of arguments and proofs intended to serve in the two classes: to promote conviction at both the individual and social levels (e.g., Mason, 1982) and understanding (e.g., Hanna, 1995).

The goals of lesson sequence 2 were (1) to help students further understand the proof criteria and (2) to offer students opportunities to apply these criteria in three proving tasks. The tasks were mathematically similar though not necessarily of the same level of difficulty; they all involved making and proving a generalization by reference to an underlying structure. Lesson sequence 2 started with review of the proof criteria, and then the teacher introduced task 1 (Figure 1). There was individual or small group work on the task, during which the teacher asked the students to write down their 'best' arguments. The students were given ample time to do that and were free to work in pairs or larger groups. While students were instructed to write down individually their arguments, few of them wrote arguments in pairs. Finally, there was a whole class discussion during which several students presented individually or in pairs their arguments. Similar procedure was followed for tasks 2 and 3.

METHOD

Data

The data for the paper are the written arguments of, and transcripts of the subsequent oral presentations of these arguments by, 17 students in the two classes. These were all the students who, in response to an open call by the teachers, offered to present their arguments for any of the three proving tasks during whole class discussions. Thirteen of these students presented arguments for only one task while two students presented arguments for two tasks. The distribution of student-presenters across the three tasks was 10 students for the first task, 4 for the second, and 3 for the third.

Analysis

A research assistant and myself coded independently all written and oral (transcribed) arguments of the 17 students. First, we used the coding scheme developed by Stylianides and Stylianides (2009) to code each argument into one of the following five categories according to the argument's level of mathematical sophistication. The codes are presented in decreasing level of mathematical sophistication. We compared our codes and discussed disagreements to reach consensus.

— code M1: proof

— code M2: valid general argument but not a proof

— code M3: unsuccessful attempt for a valid general argument

— code M4: empirical argument

— code M5: non-genuine argument

All three tasks required proving the truth of a generalization. The definition of code M1 was consistent with Stylianides' (2007) definition of proof, which underpinned in turn the criteria for proof used in the two classes. Specifically, code M1 was defined to be an argument that was *general* (i.e., it referred to all cases in the domain of the generalization), used *valid* modes of argumentation (i.e., it offered conclusive evidence for the truth of the generalization), and was accessible to the students in the class (i.e., it used statements that were readily acceptable by the class as well as modes of argumentation and modes of argument representation that were known to the students or within their conceptual reach at the particular time). Code M2 was used for arguments that approximated but not quite met the standard of proof, because, for example, of missing or inadequate justification of an assertion that could not be considered readily acceptable by the class. Code M3 was used for arguments that reflected an attempt to justify the generalization for all cases in its domain, but were either incomplete or used *invalid* modes of argumentation (i.e., they included a logical flaw). Code M4 was used for arguments that verified the truth of the generalization only in a proper subset of the cases in its domain but concluded it was true for all cases. Finally, code M5 was used for responses to the proving tasks that showed minimal engagement, were irrelevant to what was being asked, or were potentially relevant but the relevance was not made evident to the coder.

In addition to the above, for each argument we coded the following:

— Who wrote or orally presented the argument: an *individual student* or a *pair of students*;

— The kind of input from the teacher or the rest of the class during the oral presentation of the argument: *no input*, *some but not substantial input* (i.e., input that simply reiterated or briefly clarified a point mentioned by the student without influencing the presented argument), or *substantial input* (i.e., input that influenced the presented argument and possibly altered its level of mathematical sophistication).

Furthermore, we examined whether there was any evidence to suggest that preceding oral presentations of arguments for a proving task influenced subsequent presentations for the same task. We found no evidence of such influence: students' oral presentations were rather distinct from one another; students' orally presented arguments matched closely their written arguments (e.g., oral presentations tended to be based on the same figures or drawings as in students' written work); and students looked at or referred to their written work during their oral presentations.

Yet, the temporal sequencing of students' arguments (first written, then oral) was a factor we could not account for. It is possible that a student's original efforts to write an argument for a proving task helped the student build familiarity with the task and underlying concepts, thus placing the student in a position to orally present later on an argument of higher level of mathematical sophistication. Another factor we could not account for was whether the teacher offered any substantial input during students' written work in small groups. According to the plan that I had agreed with the teachers prior to the lessons, the teachers would ask students probing questions, but they would not directly influence students' argument constructions. There was no concrete evidence that the teachers deviated from the agreed plan. But even if they had done that, the result would have been better written arguments and, presumably, better oral presentations of those arguments, too. Thus there would likely be limited if any impact on the *comparison* between the levels of mathematical sophistication of the written and oral arguments, which is the issue examined in this article.

RESULTS

The results are summarized in Figures 2–4, which show the relationship between the level of mathematical sophistication of the written arguments produced

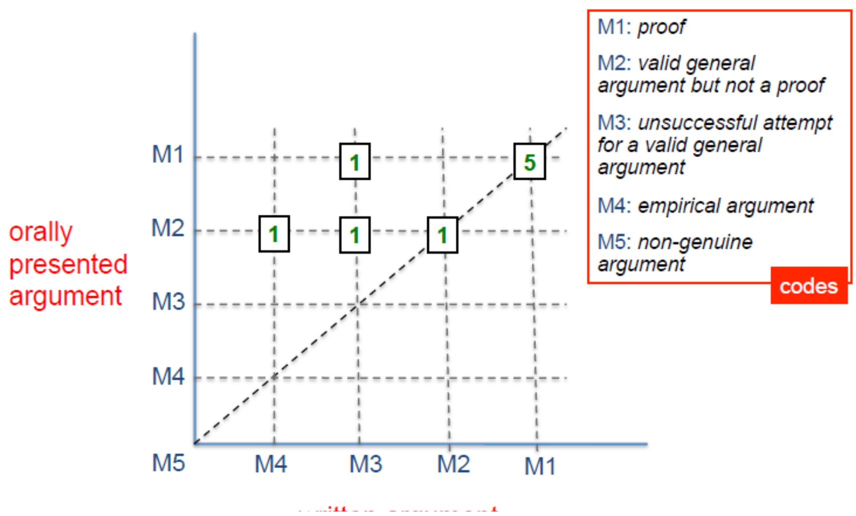

Figure 2: Students who wrote and presented their arguments individually, with no input (N=9)

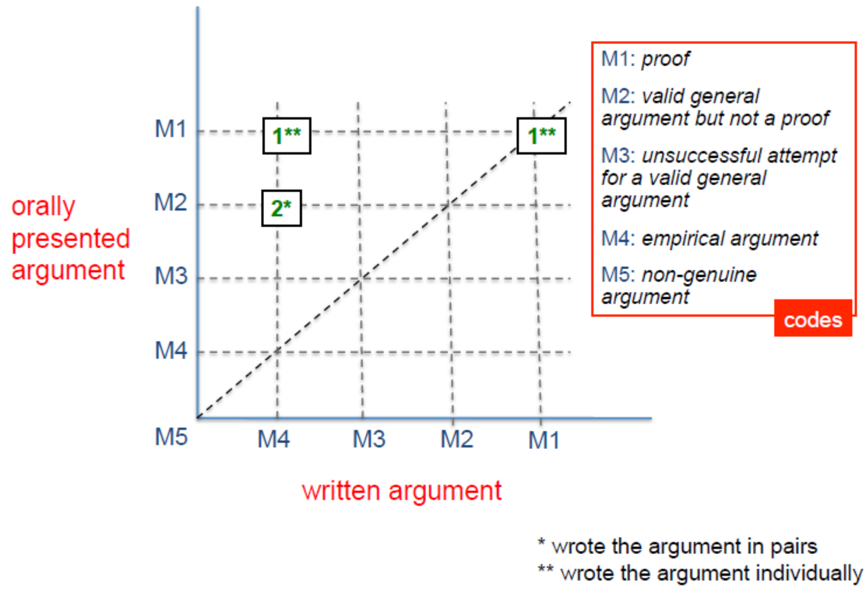

Figure 3: Students who presented their arguments in pairs, with no input (N=4)

by certain groups of students and the corresponding level of orally presented arguments by the same groups. The numbers in the figures represent frequencies of students.

Nine of the 17 students wrote and presented their arguments individually, with no input from the teacher or the rest of the class during those presentations (Figure 2). Five of these students wrote and presented proofs (M1), one wrote and presented a valid general argument but not a proof (M2), and the other three presented more mathematically sophisticated arguments than they had written earlier: two wrote arguments that reflected an unsuccessful attempt for a valid general argument (M3) but presented M1 and M2 arguments respectively, while one wrote an empirical argument (M4) but presented an M2 argument.

Four other students presented their arguments in pairs and received no input from the teacher or the test of the class during those presentations (Figure 3). The students in one of the pairs wrote together an M4 argument but presented (again together) an M2 argument. The students in the other pair wrote different arguments but made a joint presentation, which was coded as M1; one student had written an M1 argument while the other had written an M4 argument.

The remaining four students wrote and presented their arguments individually, but during their presentations they received input from the teacher (Figure 4).

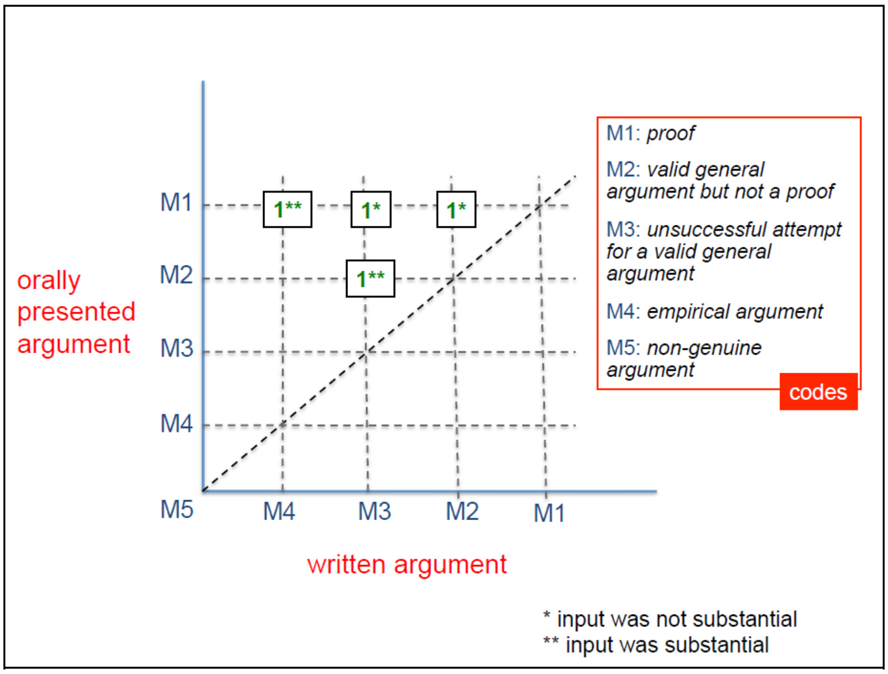

Figure 4: Students who wrote and presented their arguments individually, with input (N=4)

Yet, in two of the cases the input was not substantial: the teacher simply reiterated or clarified briefly a point mentioned by the students during their presentations, without influencing the presented argument. One student wrote an M3 argument and presented an M1 argument, and the other wrote an M2 argument and presented an M1 argument. In the remaining two cases the input from the teacher was substantial. For example, in one of the cases the teacher made a comment at a point during the student's oral presentation when the student paused and seemed to have difficulty articulating the general case; this comment might have helped the student articulate an argument of a higher level of mathematical sophistication than she would have presented otherwise. In both cases the presented arguments were of higher level of mathematical sophistication than the arguments the students had written in their papers: one was M4 and became M1, while the other was M3 and became M2.

To sum up, the proportion of arguments for which the teacher's input was substantial was small (2 out of 17), while in all of the cases the orally presented arguments were of the same or of higher level of mathematical sophistication compared to the corresponding written arguments. All orally presented arguments were either proofs (M1) or valid general arguments but not proofs (M2). Eight of the written arguments were already at the M1 or M2 levels, but the remaining nine were either empirical (M4) or unsuccessful attempts for a valid general argument (M3) and yet, during the oral presentations, all of these arguments were elevated to the M1 or M2 levels.

DISCUSSION

The findings offer support to the hypothesis that the level of mathematical sophistication of students' arguments for the same claims may depend on the mode of representation (oral vs. written) students use to communicate these arguments. The students whose arguments I examined in this study tended to omit few essential steps or explanations in their written work, but they addressed most of these omissions during their oral presentations. All orally presented arguments were of the same or higher level of mathematical sophistication than their written counterparts. A methodological issue stands out from these findings: If a study had analyzed students' oral arguments only, it would have reported a better picture of students' ability to construct arguments than another study that analyzed students' written arguments only. Yet, limitations of the research design, notably the lack of control over the temporal sequencing of students' arguments (first written, then oral), do not warrant any definite statement that the oral mode of representation is generally advantageous over the written mode in the construction of arguments that approximate or meet the standard of proof.

Below I present four other possible and not necessarily competing reasons for which students' oral arguments tended to rank higher than their written arguments.[2] Reasons 1 and 2 reinforce, while reasons 3 and 4 weaken, the presumed role played by the mode of representation in the observed differences between students' written and oral arguments, thus highlighting the need for more research in this area.

1. *Relative difficulty of written versus oral arguments.* Writing mathematical arguments may be genuinely more difficult than presenting orally mathematical arguments, especially for students who were recently introduced to the concept of proof as were the students in this study.

2. *Relative preference for oral versus written expression.* Students may tend to prefer oral over written expression of their mathematical ideas, and so students in the study might have responded in a minimalistic way to the teacher's expectation to produce written arguments for the three tasks (indeed, a didactical contract regarding proof work was not yet established in the class).

3. *Possible role of the specific nature of proving tasks.* The three proving tasks all belonged to a special family of tasks and involved making and proving a generalization by reference to an underlying structure. High linguistic demands are imposed on solvers as they try to express a general argument by reference to a specific diagram that nevertheless exemplifies the underlying mathematical structure; students may be better able to cope with these demands when they express their ideas orally rather than in writing.

4. *Possible influence of the broader social context on students' oral presentations.* Even though almost all of the oral presentations were carried out with no verbal input from others in the class, extra-linguistic forms of expression, notably gestures, might have given to presenters some non-verbal cues (cf. Roth, 2001) that encouraged them to also evaluate or elaborate more on their arguments, thus addressing some of their limitations and elevating their status (cf. Stylianides & Stylianides, 2009).

To conclude, in this paper I have called, and reinforced the need, for more research into factors mediating the relation between students' argument constructions or evaluations and conclusions about students' understanding of proof. Do we, as a field, document an accurate picture of students' understanding of proof? Are findings from different studies in this area comparable with each other?

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ENDNOTES

1. While a written or an oral mode of representation may be used to communicate the bulk of a student's

argument, other modes of representation can also be used alongside this mode. For example, a student who presents orally an argument in front of a class can draw on the board a picture, or write on the board an algebraic expression, in order to supplement the verbal expression of the argument.

2. I do not mention time as a possible reason, for students were given ample time to produce their written arguments. Yet, one cannot completely exclude the possibility that few students recorded in writing their 'exploratory' work and then, during whole class discussion, shifted to a more 'deductive' form of presentation of their finished products.