A theoretical perspective for proof construction

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This theoretical paper suggests a perspective for understanding university students' proof construction based on the ideas of conceptual and procedural knowledge, explicit and implicit learning, behavioral schemas, automaticity, working memory, consciousness, and System 1 and System 2 cognition. In particular, we will discuss proving actions, such as the construction of proof frameworks, that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Keywords: University level, proving actions, behavioral schemas, System 1 and System 2 cognition, proof framework.

INTRODUCTION

This theoretical paper suggests a perspective for understanding university mathematics students' proof constructions and how the ability and skill to construct proofs might be learned and taught. We are interested both in how various types of knowledge (e.g., implicit, explicit, procedural, conceptual) are used during proof construction, and also in how such knowledge can be acquired. If that were better understood, then it might be possible to facilitate university students' learning through doing, that is, through proof construction experiences. Although one can learn some things from lectures, this is almost certainly not the most effective, or efficient, way to learn proof construction. Indeed, inquiry-based transition-to-proof courses seem more effective than lecture-based courses (e.g., Smith, 2006). Here we are referring just to inquiry into proof construction, not into theorem or definition generation. These ideas emerged from an ongoing sequence of design experiment courses meant to teach proof construction in a medium-sized U.S. PhD-granting university.

The Courses

There were two kinds of courses. One kind was for mid-level undergraduate mathematics students and was similar (in purpose) to transition-to-proof courses found in many U.S. university mathematics departments (Moore, 1994). In the U.S., such courses are often prerequisite for 3rd and 4th year courses in abstract algebra and real analysis. The other, somewhat unusual, kind of course was for beginning mathematics graduate students who felt that they needed help with writing proofs. The undergraduate course had from about 15 to about 30 students and the graduate course had between 4 and 10 students. Both kinds of course were taught from notes and devoted entirely to students attempting to construct proofs and to receiving feedback and advice on their work. Both courses included a little sets, functions, real analysis, and algebra. The graduate course also included some topology.

Psychological considerations

Much has been written in the psychological, neuropsychological, and neuroscience literature about ideas of conceptual and procedural knowledge, explicit and implicit learning, automaticity, working memory, consciousness, and System 1 (S1) and System 2 (S2) cognition (e.g., Bargh & Chartrand, 2000; Bargh & Morsella, 2008; Bor, 2012; Cleeremans, 1993; Hassin, Bargh, Engell, & McCulloch, 2009; Stanovich & West, 2000). In trying to relate these ideas to proof construction, we have discussed procedural knowledge, situation-action links, and behavioral schemas (Selden, McKee, & Selden, 2010; Selden & Selden, 2011). However, more remains to be done in order to weave these ideas into a coherent perspective. In doing this, two key ideas are working memory and the roles that S1 and S2 cognition can play in proof construction. Working memory includes the central executive and makes cognition possible. It is related to learning and attention and has a limited capacity which varies across individuals. When working memory capacity is exceeded, errors and oversights can occur. The idea behind S1 and S2 cognition is that there are two kinds of cognition that operate in parallel. S1 cognition is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast, S2 cognition is slow, conscious, effortful, evolutionarily recent, and puts considerable call on working memory (Stanovich & West, 2000). Of the several kinds of consciousness, we are referring to phenomenal consciousness—approximately, reportable experience. We turn now to the first of the two components of the proposed perspective.

THE PERSPECTIVE: MATHEMATICAL COMPONTENT

The genre of proofs

There are a number of characteristics that appear to commonly occur in published proofs. They tend to reduce unnecessary distractions to validation (reading for correctness) and raise the probability that any errors will be found, thereby increasing the reliability of the corresponding theorems. Proofs are not reports of the proving process, contain little redundancy, and contain minimal explanations of inferences. They contain only very short overviews or advance organizers and do not quote entire definitions that are available outside the proof. Symbols are generally introduced in one-to-one correspondence with objects. Finally, proofs are "logically concrete" in the sense that they avoid quantifiers, especially universal quantifiers, and their validity is independent of time, place, and author. (Selden & Selden, 2013).

Structure in proofs

A proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). Instead it depends on a kind of "technical skill". We call the remaining part of a proof the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved (Selden & Selden, 2009, 2011).

A major feature of the formal-rhetorical part is what we have called a *proof framework*, of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a proof framework would be "Let x be a real number. Suppose $P(x)$. ... Therefore $Q(x)$." A second-level framework would be obtained by "unpacking" the meaning of $Q(x)$ and putting the second-level framework for that between the beginning and end of the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in (Selden, Benkhalti, & Selden, 2014).

THE PERSPECTIVE: PSYCHOLOGICAL COMPONTENT

In this second, psychological component of the perspective, we view the proof construction process as a sequence of actions which can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one's focus from the hypothesis to the conclusion or trying to recall a relevant theorem). The sequence of all of the actions that eventually lead to a proof is usually considerably longer than the final written proof itself. This fine-grained approach appears to facilitate noticing which actions should be taken to write various parts of a proof correctly and how to encourage such actions on the part of students.

Situations and actions

We mean by an (inner) situation in proving, a portion of a partly completed proof construction, perhaps including an interpretation, drawn from long-term memory, that can suggest a further action. The interpretation is likely to depend on recognition of the situation, which is easier than recall, perhaps because fewer brain areas are involved (Cabeza, et al., 1997). An inner situation is unobservable. However, a teacher can often infer an inner situation from the corresponding outer situation, that is, from the, usually written, portion of a student's partly completed proof.

Here we are using the term, action, broadly, as a response to a situation. We include not only physical actions (e.g., writing a line of a proof), but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy (Selden & Selden, 2014). We also include "meta-actions" meant to alter one's own thinking, such as focusing on another part of a developing proof construction.

Situation-action links and behavioral schemas

If, in several proof constructions in the past, similar situations have corresponded to similar actions, then, just as in traditional associative learning, a link may be learned between them, so that another similar situation yields the corresponding action in future proof constructions without the earlier need for deliberate cognition. Using *situation-action links* strengthens them and after sufficient practice/experience, they can become overlearned, and thus, automated. A person executing an automated action tends to (1) be unaware of any needed mental processes, (2) be unaware of intentionally initiating the action, (3) put little load on working memory, and (4) find it difficult to stop or alter the action. We call automated situation-action links *behavioral schemas*. Morsella (2009) has pointed out

> Regarding skill learning and automaticity, it is known that the neural correlates of novel actions are distinct from those of actions that are overlearned, such as driving or tying one's shoes. Regions [of the brain] primarily responsible for the control of movements during the early stages of skill acquisition are different from the regions that are activated by overlearned actions. In essence, when an action becomes automatized, there is a 'gradual shift from cortical to subcortical involvement ...' (p. 13).

Because cognition often involves inner speech, which in turn is connected with the physical control of speech production, the above information on the brain regions involved in physical skill acquisition is at least a hint that forming behavioral schemas not only converts S2 cognition into S1 cognition, but also suggests that different parts of the brain are involved in access and retrieval. Something very similar to the above ideas on automaticity in proof construction has been investigated by social psychologists examining everyday life (e.g., Bargh & Chartrand, 2000).

We see behavioral schemas as partly conceptual knowledge (recognizing the situation) and partly procedural knowledge (the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We suggest that, in the use of a situation-action link or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious.

Here is an example of one such possible behavioral schema. One might be starting to prove a statement having a conclusion of the form *p* or *q*. This would be the situation at the beginning of the proof construction. If one had encountered this situation a number of times before, one might readily take an appropriate action, namely, in the written proof assume not *p* and prove *q* or vice versa. While this action can be warranted by logic (if not *p* then *q*, is equivalent to, *p* or *q*), there would no longer be a need to bring the warrant to mind.

It is our contention that large parts of proof construction skill can be automated, that is, that one can facilitate mid-level university students in turning parts of S2 cognition into S1 cognition, and that doing so would make more resources, such as working memory, available for the truly hard problems that need to be solved to complete many proofs.

The idea that much of the deductive reasoning that occurs during proof construction could become automated may be counterintuitive because many psychologists, and (given the terminology) probably many mathematicians, assume that deductive reasoning is largely S2.

Sequences of behavioral schemas

Behavioral schemas were once actions arising from situations through warrants, but that no longer need to be brought to mind. So one might reasonably ask, can several behavioral schemas be "chained together" outside of consciousness? For most persons, this seems not to be possible. If it were so, one would expect that a person familiar with solving linear equations could start with $3x + 5 = 14$, and *without bringing anything else to mind*, immediately say $x = 3$. We expect that very few (or no) people can do this, that is, consciousness is required.

Implicit learning of behavioral schemas

It appears that the entire process of learning a behavioral schema, as described above, can be implicit. That is, a person can acquire a behavioral schema without being aware that this is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (e.g., Cleeremans, 1993). In the case of proof construction, we suggest that with the experience of proving a considerable number of theorems in which similar situations occur, an individual might implicitly acquire a number of relevant behavioral schemas, and as a result, simply not have to think quite so deeply as before about certain portions of the proving process and might, as a consequence of having more working memory available, take fewer "wrong turns".

Something similar has been described in the psychology literature regarding the automated actions of everyday life. For example, an experienced driver can reliably stop at a traffic light while carrying on a conversation. But not all automated actions are positive. For example, a person can develop a prejudice without being aware of the acquisition process and can even be unaware of its triggering situations. This suggests that we should consider the possibility of mathematics students developing similarly unintended negative situation-action links, and behavioral schemas, implicitly during mathematics learning, and in particular, during proof construction.

Detrimental behavioral schemas

We begin with a simple and perhaps very familiar algebraic error. Many teachers can recall having a student write $\sqrt{(a^2 + b^2)} = a + b$, giving a counterexample to the student, and then having the student make the same error somewhat later. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned detrimental behavioral schema. If so, the student would not be thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation of the form $\sqrt{(a^2 + b^2)}$ should occur a number of times when the student can be prevented from automatically writing "$= a + b$" in response. However, this might require working with the student individually on a number of examples, mixed with nonexamples.

For another example of an apparently implicitly learned detrimental behavioral schema, we turn to Sofia, a first-year graduate student in one of the above mentioned graduate courses. Sofia was a diligent student, but as the course progressed what we came to call an "unreflective guess" schema emerged (Selden, McKee, & Selden, 2010, pp. 211–212). After completing just the formal-rhetorical part of a proof (essentially a proof framework) and realizing there was more to do, Sofia often offered a suggestion that we could not see as being remotely helpful. At first we thought she might be panicking, but on reviewing the videos there was no evidence of that. A first unreflective guess tended to lead to another, and another, and after a while, the proof would not be completed.

In tutoring sessions, instead of trying to understand, and work with, Sofia's unreflective guesses, we tried to prevent them. At what appeared to be the appropriate time, we offered an alternative suggestion, such as looking up a definition or reviewing the notes. Such positive suggestions eventually stopped the unreflective guesses, and Sofia was observed to have considerably improved in her proving ability by the end of the course (Selden, McKee, & Selden, 2010, p. 212).

USING THIS PERSPECTIVE

Decomposing the proving process

In order to facilitate students' automation of certain parts of the proving process by developing helpful behavioral schemas, we have been decomposing the reasoning parts of the proving process, and focusing on those that occur frequently. Such decompositions of parts of the proving process can be mainly mathematical in nature or mainly psychological in nature. We find the psychological decompositions to be more surprising because they include things one might expect university students to be able to do without instruction. Some more mathematical possibilities are: (1) writing the first- and second-level proof frameworks which themselves can have parts (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995); (2) noting when a conclusion is negatively phrased (e.g., a set is empty or a number is irrational) and early in the proving process attempting a proof by contradiction; and (3) noticing when the conclusion asserts the equivalence of two statements, "knowing" there are two implications to prove, and actually originating the two subproofs.

Here are some decompositions that may be more psychological in nature. One can change one's focus, for example by deciding to unpack the conclusion of a theorem, by finding or recalling a relevant definition, or by applying a definition. Such actions are sometimes part of constructing a second-level proof framework (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995). Also developing a feeling of knowing or of self-efficacy can have a major effect (Selden & Selden, 2014). A student may develop and have for a time a feeling of not knowing what to do next, that is, the student might be at an impasse. Upon reaching such an impasse, the student might decide to do something else for a while, and coming back later, might hope to get a new idea. Many mathematicians have benefitted from this kind of "incubation". Nonemotional cognitive feelings (Selden, McKee, & Selden, 2010), such as those mentioned above can play a considerable role in proof construction, but we do not have space to elaborate on them here.

Proving activities that we have tried to help students automate include converting formal mathematical definitions into *operable interpretations*, which are similar to Bills and Tall's (1998) idea of operable definitions. For example, given $f: X \to Y$ and $A \subseteq Y$, we define $f^{-1}(A) = \{ x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa". One might think that this sort of translation into an operable form would be easy, but we have found that for some students it is not, even when the definition can be consulted. We have also noted instances in which students have had available both a definition and an operable interpretation, but still did not act appropriately. Thus, *actually* implementing the action is separate from *knowing* that one *can* implement it. We are not sure whether, in implementing such actions, automaticity is difficult to achieve, but not acting appropriately can clearly prevent a student from proving a theorem.

Seeing similarities, searching, and exploring

How does one recognize situations as similar? Different people see situations as similar depending both upon their past experiences and upon what they choose to, or happen to, focus on. While similarities can sometimes be extracted implicitly, teachers may occasionally need to direct students' attention to relevant proving similarities. On the other hand, such direction should probably be as little as possible because the ability to *autonomously* see similarities can be learned.

For example, it would be good to have general suggestions for helping students to "see", without being told, that the situations of a set being empty (i.e., having no elements), of a number being irrational (i.e., not rational), and of the primes being infinite (i.e., not finite) are similar. That is, the three situations—empty, irrational, and infinite—may not seem similar until one rephrases them to expose the existence of a negatively worded definition. Unless students rephrase these situations, it seems unlikely that they would see this similarity and link these situations (when they occur as conclusions of theorems to prove) to the action of beginning a proof by contradiction.

In addition to automating small portions of the proving process, we would also like to enhance students' searching skills (i.e., their tendency to look for helpful previously proved results) and to enhance students' tendency to "explore" possibilities when they don't know what to do next. In a previous paper (Selden & Selden, 2014, p. 250), we discussed the kind of exploring entailed in proving the rather difficult (for students) *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group.* Well before such a theorem appears in course notes, one can provide students with advice/experience showing the value of exploring what is not obviously useful (e.g., starting with $abba = e$ to show a semigroup with identity e, where for all $s \in S$, $s^2 = e$, is commutative, as discussed in Selden, Benkhalti, and Selden, 2014).

Understanding students' proof attempts

Here is a sample student's incorrect proof attempt of the *Theorem: Let S be a semigroup with identity e. If, for all s in S, ss = e, then S is commutative.* The student's accompanying scratchwork consisted of the definitions of identity and commutative. Here, the line numbers are for reference only.

1 Let S be a semigroup with an identity element, e.

2 Let $s \in S$ such that $ss = e$.

3 Because e is an identity element, $es = se = s$.

4 Now, $s = se = s(ss)$.

5 Since S is a semigroup, $(ss)s = es = s$.

6 Thus $es = se$.

7 Therefore, S is commutative. QED.

Line 2 only hypothesizes a single s and should have been, "Suppose for all $s \in$ S, $ss = e$." With this change, Lines 1, 2, and 7 are the correct first-level framework.

There is no second-level framework between Lines 2 and 7. This was a beneficial action *not* taken and should have been: "Let $a \in S$ and $b \in S$. ... Then $ab = ba$." between Lines 2 and 7. Line 3 violates the genre of proof by including a definition easily available outside the proof. Lines 3, 4, 5, and 6 are not wrong, but do not move the proof forward. Writing these lines may have been detrimental actions that subconsciously primed the student's feeling that something useful had been accomplished, and thus, may have brought the proving process to a premature close.

TEACHING AND RESEARCH CONSIDERATIONS

The above considerations can lead to many possible teaching interventions. This then brings up the question of priorities. Which proving actions, of the kinds discussed above, are most useful for mid-level university mathematics students to automate, when they are learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions recently provided—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start.

Also, helping students interpret formal mathematical definitions so that these become operable might be another place to start. This would be helpful because one often needs to convert a definition into an operable form in order to use it to construct a second-level framework. However, eventually students should learn to make such interpretations themselves.

Finally, we believe this particular perspective on proving, using situation-action links and behavioral schemas, together with information from psychology and neuroscience, is mostly new to the field and is likely to lead to additional insights.

REFERENCES

Bargh, J. A., & Chartrand, T. L. (2000). Studying the mind in the middle: A practical guide to priming and automaticity research. In H. T. Reid & C. M. Judd (Eds.), *Handbook of research methods in social psychology* (pp. 253–285). New York: Cambridge University Press.

Bargh, J. A., & Morsella, E. (2008). The unconscious mind. *Perspectives on Psychological Science*, *3*(1), 73–79.

Bills, L., & Tall, D. (1998). Operable definitions in advanced mathematics: The case of the least upper bound. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 104–111). Stellenbosch, South Africa: University of Stellenbosch.

Bor, D. (2012). *The ravenous brain*. New York: Basic Books.

Cabeza, R., Kapur, S., Craik, F. I. M., McIntosh, A. R., Houle, S., & Tulving, E. (1997). Functional neuroanatomy of recall and recognition: A PET study of episodic memory. *Journal of Cognitive Neuroscience*, *9*, 254–265.

Cleeremans, A. (1993). *Mechanisms of implicit learning: Connectionist models of sequence processing*. Cambridge, MA: MIT Press.

Hassin, R. R., Bargh, J. A., Engell, A. D., & McCulloch, K. C. (2009). Implicit working memory. *Conscious Cognition*, *18*(3), 665–678.

Mason, J., & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to-act in the moment. *Educational Studies in Mathematics*, *28*(1–3), 135–161.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, *27*(3), 249–266.

Morsella, E. (2009). The mechanisms of human action: Introduction and background. In E. Morsella, J. A. Bargh, & P. M. Goldwitzer (Eds.), *Oxford handbook of human action* (pp. 1–34). Oxford: Oxford University Press.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas, and the proving process. *International Journal of Mathematical Education in Science and Technology*, *41*(2), 199–215.

Selden, A, & Selden, J. (2013). The genre of proof. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics and mathematics education: Searching for common ground* (pp. 248–251). New York: Springer.

Selden, A., & Selden, J. (2014). The roles of behavioral schemas, persistence, and self-efficacy in proof construction. In B. Ubuz, C. Hasar, & M. A. Mariotti (Eds.). *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* [CERME8] (pp. 246–255). Ankara, Turkey: Middle East Technical University.

Selden, J., Benkhalti, A., & Selden, A. (2014). An analysis of transition-to-proof course students' proof constructions with a view towards course redesign. In T. Fukawa-Connolly, G. Karakok, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 246–259). Denver, Colorado. Available online.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123–151.

Selden, J., & Selden, A. (2009). Teaching proving by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 339–354). New York/Washington, DC: Routledge/National Council of Teachers of Mathematics.

Selden, J., & Selden, A. (2011). The role of procedural knowledge in mathematical reasoning. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4 (pp. 145–152). Ankara, Turkey: Middle East Technical University.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior*, *25*, 73–90.

Stanovich, K. E., & West, R. F. (2000). Individual differences in reasoning: Implications for the rationality debate? *Behavioral and Brain Sciences*, *23*, 645–726.